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To the blessed memory of Andrei Nikolaevich Tyurin

Derived categories of coherent sheaves and equivalences between them

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Abstract. This paper studies the derived categories of coherent sheaves on smooth complete algebraic varieties and equivalences between them. We prove that every equivalence of categories is represented by an object on the product of the varieties. This result is applied to describe the Abelian varieties and K3 surfaces that have equivalent derived categories of coherent sheaves.

Contents

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Introduction

The main objects of study in algebraic geometry are algebraic varieties (or schemes) and morphisms between them. Every algebraic variety X is a ringed topological space and thus has a topology (usually the Zariski topology) and a sheaf of rings of regular functions $\mathcal{O}_X$.

To a large extent, the study of an algebraic variety is the study of sheaves on it. Since the space is ringed, the natural sheaves are sheaves of $\mathcal{O}_X$-modules on it, among which the quasi-coherent and coherent sheaves are distinguished by their algebraic nature. Recall that a sheaf of $\mathcal{O}_X$-modules is quasi-coherent if it is locally representable as the cokernel of a homomorphism of free sheaves, and coherent if these free sheaves are of finite rank. (Locally free sheaves on a variety correspond one-to-one with vector bundles, and we therefore use these terms interchangeably in what follows.)

Thus, corresponding to every algebraic variety X we have the Abelian categories $\mathrm{coh}(X)$ of coherent sheaves and $\mathrm{Qcoh}(X)$ of quasi-coherent sheaves. Morphisms between varieties induce inverse image and direct image functors between these Abelian categories. However, these functors are not exact, that is, do not take exact sequences to exact sequences. This causes significant complications when working with Abelian categories and non-exact functors between them. To preserve functoriality, Cartan and Eilenberg [11] introduced the notion of derived functors which give necessary corrections to non-exact functors. This technique was developed by Grothendieck in [15], which subsequently led to the introduction of the new concepts: derived category and derived functors between them.

Derived categories, in contrast to Abelian categories, do not have short exact sequences, and the kernels and cokernels of morphisms are not defined. However, derived categories admit a certain internal structure, formalized by Verdier as the notion of triangulated category [44].

Passing from Abelian categories to their derived categories allows us to solve many problems related to difficulties arising in the study of natural functors. Among the first examples, we mention the creation of the global intersection theory and the proof of the Riemann–Roch theorem. These results, achieved by Grothendieck and his co-authors [41], were made possible by the introduction of the triangulated category of perfect complexes.

Another example relates to the introduction of perverse sheaves and to the establishment of the Riemann–Hilbert correspondence between holonomic modules with regular singularities and constructible sheaves (see [3], [23]); this correspondence only became possible on applying the notions and techniques of triangulated categories.

Many problems relating to the study of varieties require the study and description of the derived categories of coherent sheaves on them. In the simplest cases, when the variety is a point or a smooth curve, every object in the derived category of coherent sheaves is isomorphic to a direct sum of some family of coherent sheaves with suitable shifts; that is, every $A \in \mathbf{D}^b(\operatorname{coh} X)$ is isomorphic to $\bigoplus_{i=1}^{k} \mathcal{F}_i[n_i]$, where $\mathcal{F}_i$ are coherent sheaves. These examples reflect the fact that in these cases the Abelian category has homological dimension $\leqslant 1$. However, for higher dimensional varieties there are complexes that are not isomorphic in the derived category to the sum of their cohomology. Thus, describing the derived category for varieties of dimension greater than 1 is a difficult and interesting problem. The first steps in this direction were made in [4] and [2], which described the derived category of coherent sheaves on projective spaces, and subsequently allowed the technique to be applied to the study the moduli space of vector bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$. In particular, these papers showed that the derived category of coherent sheaves $\mathbf{D}^b(\operatorname{coh} \mathbb{P}^n)$ on projective space is equivalent to the derived category of finite-dimensional modules over the finite-dimensional algebra $A = \operatorname{End}\bigl(\bigoplus_{i=0}^{i=n} \mathcal{O}(i)\bigr)$. This approach has been perfected since then, and descriptions of the derived categories of coherent sheaves on quadrics and on flag varieties have also been obtained ([20]–[22]).

Introducing the notions of exceptional family and semi-orthogonal decomposition enabled one to formulate new principles for describing the derived categories of coherent sheaves [5], [6]. It turned out that the existence of a complete exceptional family always realizes an equivalence of the derived category of coherent sheaves with the derived category of finite-dimensional modules over the finite-dimensional algebra of endomorphisms of the given exceptional family [5]. The notion of semi-orthogonal decomposition allowed us to describe the derived category of a blowup in terms of the derived category of the variety that is blown up and that of the subvariety along which the blowup occurs [34].

However, for many types of varieties, no description of the derived category is possible. Nevertheless, the natural question can be posed roughly as follows: how much information is preserved on passing from a variety to its derived category of coherent sheaves? In fact it turns out that 'almost all' information is preserved under this correspondence. In many cases one can even recover the variety itself from its derived category of coherent sheaves, for example if the canonical (or anticanonical) sheaf is ample [8].

For certain types of varieties one nevertheless finds examples in which two distinct varieties have equivalent derived categories of coherent sheaves. The first example of two different varieties having equivalent derived categories of coherent sheaves was found by Mukai [29]. He showed that this happens for every Abelian variety and its dual variety. We generalized this construction in [38]: for any Abelian variety, we introduced an entire class of Abelian varieties, all of which have the same derived category of coherent sheaves. On the one hand, these examples show that there are varieties having equivalent derived category of coherent sheaves; on the other hand, every class of varieties with equivalent derived categories of coherent sheaves is 'small' (it is finite in all the examples).

To obtain a complete classification of varieties with equivalent derived categories of coherent sheaves, we need a description of the functors and equivalences between them. It turns out that equivalences are always geometric in nature, that is, they are represented by certain complexes of sheaves on the product of the varieties.

We explain what we mean. In what follows we write $\mathbf{D}^b(X)$ to denote the bounded derived category of coherent sheaves on X. Any morphism $f\colon X \to Y$ between smooth complete algebraic varieties induces two exact functors between their bounded derived categories of coherent sheaves: the direct image functor $\mathbf{R}f_*\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ and the inverse image functor $\mathbf{L}f^*\colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(X)$, which is left adjoint to $\mathbf{R}f_*$. Moreover, every object $\mathcal{E} \in \mathbf{D}^b(X)$ defines an exact tensor product functor $\otimes^{\mathbf{L}} \mathcal{E}\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X)$. We can use these standard derived functors, to introduce a new large class of exact functors between the derived categories $\mathbf{D}^b(X)$ and $\mathbf{D}^b(Y)$.

Let X and Y be two smooth complete varieties over a field k. Consider the Cartesian product $X \times Y$, and write

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y$$

for the projections of $X \times Y$ to X and Y respectively. Every object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ defines an exact functor $\Phi_{\mathcal{E}}$ from the derived category $\mathbf{D}^b(X)$ to the derived category $\mathbf{D}^b(Y)$, given by

$$\Phi_{\mathcal{E}}(\,\cdot\,) := \mathbf{R}^{\cdot} q_*(\mathcal{E} \otimes^{\mathbf{L}} p^*(\,\cdot\,)). \tag{1}$$

Every functor of this type has left and right adjoint functors.

Thus, to every smooth complete algebraic variety one can assign its derived category of coherent sheaves, and to every object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ on the product of two such varieties one can assign an exact functor $\Phi_{\mathcal{E}}$ from the triangulated category $\mathbf{D}^b(X)$ to the triangulated category $\mathbf{D}^b(Y)$. This paper is devoted to the study of this correspondence.

One of the first questions that arises in the study of derived categories of coherent sheaves is the following: can every functor between these categories be represented by an object on the product? that is, is it of the form (1)? In Chapter 3 we give an affirmative answer to this question if the functor is an equivalence.

Two other central questions here are as follows:

1) When are the derived categories of coherent sheaves on two different smooth complete varieties equivalent as triangulated categories?
2) What is the group of exact auto-equivalences of the derived category of coherent sheaves on a given variety X?

Some results in this direction were already known. Exhaustive answers to the above questions are known when the variety has ample canonical or anticanonical sheaf: in [8] we proved that a smooth projective variety X with ample canonical (or anticanonical) sheaf can be recovered from its derived category of coherent sheaves $\mathbf{D}^b(X)$; moreover, [8] also gives an explicit construction for recovering X. For varieties of this type, the group of exact auto-equivalences can also be described.

We now describe the contents and structure of this paper. Most of the results collected here can be found in some form in the papers [7], [8], [34], [35], and [37]. Chapter 1 collects material of a preliminary nature. We first give the definition of triangulated category and recall the notions of an exact functor between triangulated categories, the localization of a triangulated category with respect to a full subcategory, and the general definition of derived functor for localized triangulated categories. After this we define the homotopy category and the derived category of an Abelian category, and we also discuss the properties of derived categories of coherent and quasi-coherent sheaves on schemes and the functors between these categories.

In Chapter 2 we introduce the class of functors between the bounded derived categories of coherent sheaves on smooth complete algebraic varieties that are representable by objects on products, and describe their main properties. Using results from Chapter 3, we prove that, if two smooth projective varieties X and Y have equivalent derived categories, then there exists an isomorphism between the bigraded algebras $\mathrm{HA}(X)$ and $\mathrm{HA}(Y)$ defined by the following formula:

$$\mathrm{HA}(X)=\bigoplus_{i,k}\mathrm{HA}_{i,k}(X)=\bigoplus_{i,k}\bigoplus_{p+q=i}\mathrm{H}^p\Bigl(X,\bigwedge^q T_X\otimes\omega_X^k\Bigr),$$

where T_X is the tangent bundle and ω_X the canonical bundle of X (Theorem 2.1.8 and Corollary 2.1.10).

In the second section of Chapter 2 we present a whole class of pairs of varieties having equivalent derived categories of coherent sheaves. These examples are interesting in that the varieties that arise are birationally isomorphic (but not isomorphic in general) and are related by a birational transformation called a flop. In particular, these examples show that we cannot weaken the condition of ampleness of the canonical (or anticanonical) class in the theorem on recovering X from $\mathbf{D}(X)$.

Let Y be a smoothly embedded closed subvariety in a smooth complete algebraic variety X such that $Y\cong\mathbb{P}^k$ with normal bundle $N_{X/Y}\cong\mathcal{O}_Y(-1)^{\oplus(l+1)}$. We assume that $l\leqslant k$ and write $\widetilde{X}$ to denote the blowup of X with centre along Y. In this case the exceptional divisor $\widetilde{Y}$ is isomorphic to the product of projective spaces $\mathbb{P}^k\times\mathbb{P}^l$. There is a blowdown of $\widetilde{X}$ such that $\widetilde{Y}$ projects to the second factor $\mathbb{P}^l$. Consider the diagram of projections

$$X\xleftarrow{\ \pi\ }\widetilde{X}\xrightarrow{\ \pi^+\ }X^+.$$

The birational map $\mathrm{fl}\colon X\dashrightarrow X^+$ is the simplest example of a flip or flop; it is a flip for $l<k$ and a flop for $l=k$.

The main theorem of this section relates the derived categories of coherent sheaves on the varieties X and X^+. It asserts that for any line bundle $\mathcal{L}$ on $\widetilde{X}$, the functor

$$\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}(\,\cdot\,)\otimes\mathcal{L})\colon\mathbf{D}^b(X^+)\longrightarrow\mathbf{D}^b(X)$$

is fully faithful, and for $k=l$ this functor is an equivalence.

Chapter 3 is central. It is concerned with proving that every equivalence between derived categories of coherent sheaves on smooth projective varieties is represented by an object on the product. This assertion allows us to describe equivalences between derived categories of coherent sheaves and to answer the question of when two different varieties have equivalent derived categories of coherent sheaves.

In fact, in this chapter we prove a more general assertion: namely, that any functor between bounded derived categories of coherent sheaves on smooth projective varieties that is fully faithful and has an adjoint functor can be represented by an object $\mathcal{E}$ on the product of these varieties; that is, it is isomorphic to the functor $\Phi_{\mathcal{E}}$ defined by the rule (1). Moreover, the object $\mathcal{E}$ representing it is uniquely determined up to isomorphism (Theorem 3.2.1).

In Chapter 4 we study the derived categories of coherent sheaves on K3 surfaces. For any K3 surface S, the cohomology lattice $\mathrm{H}^*(S,\mathbb{Z})$ has a symmetric bilinear form defined by the rule

$$(u,u') = r\cdot s' + s\cdot r' - \alpha\cdot\alpha' \in \mathrm{H}^4(S,\mathbb{Z}) \cong \mathbb{Z}$$

for any pair $u=(r,\alpha,s)$, $u'=(r',\alpha',s') \in \mathrm{H}^0(S,\mathbb{Z})\oplus\mathrm{H}^2(S,\mathbb{Z})\oplus\mathrm{H}^4(S,\mathbb{Z})$. The cohomology lattice $\mathrm{H}^*(S,\mathbb{Z})$ with the bilinear form $(\,\cdot\,,\cdot\,)$ is called the Mukai lattice and denoted by $\widetilde{\mathrm{H}}(S,\mathbb{Z})$.

The lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ admits a natural Hodge structure. In the present case, by Hodge structure, we mean that we fix a distinguished one-dimensional subspace $\mathrm{H}^{2,0}(S)$ in the space $\widetilde{\mathrm{H}}(S,\mathbb{C})$. We say that the Mukai lattices of two K3 surfaces S_1 and S_2 are *Hodge isometric* if there is an isometry between them taking the one-dimensional subspace $\mathrm{H}^{2,0}(S_1)$ to $\mathrm{H}^{2,0}(S_2)$.

The main theorem of this chapter (Theorem 4.2.1) asserts that the derived categories $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ of coherent sheaves on two K3 surfaces over the field $\mathbb{C}$ are equivalent as triangulated categories if and only if there is a Hodge isometry $f\colon \widetilde{\mathrm{H}}(S_1,\mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S_2,\mathbb{Z})$ between their Mukai lattices. This theorem has another version in terms of lattices of transcendental cycles (Theorem 4.2.4).

In view of the Torelli theorem for K3 surfaces [39], [27], which says that a K3 surface can be recovered from the Hodge structure on its second cohomology, we obtain an answer in terms of Hodge structures to the question of when the derived categories of coherent sheaves on two K3 surfaces are equivalent.

In Chapter 5 we study the derived categories of coherent sheaves on Abelian varieties and their groups of auto-equivalences. Let A be an Abelian variety and $\widehat{A}$ the dual Abelian variety. As proved in [29], the derived categories of coherent sheaves $\mathbf{D}^b(A)$ and $\mathbf{D}^b(\widehat{A})$ are equivalent, and the equivalence, called the Fourier–Mukai transform, can be given by means of the Poincaré line bundle $\mathcal{P}_A$ on the product $A\times\widehat{A}$ by the rule (1): $F \mapsto \mathbf{R}^{\cdot}p_{2*}(\mathcal{P}_A\otimes p_1^*(F))$.

This construction of Mukai was generalized in [38] as follows. Consider two Abelian varieties A and B and an isomorphism f between the Abelian varieties $A\times\widehat{A}$ and $B\times\widehat{B}$. Write f in the matrix form

$$f = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

where x stands for a homomorphism from A to B, y from $\widehat{A}$ to B, and so on. We say that the isomorphism f is *isometric* if its inverse has the form

$$f^{-1} = \begin{pmatrix} \widehat{w} & -\widehat{y} \\ -\widehat{z} & \widehat{x} \end{pmatrix}.$$

We define $U(A\times\widehat{A}, B\times\widehat{B})$ to be the set of isometric isomorphisms f. If $B=A$, then we denote this set by $U(A\times\widehat{A})$; note that it is a subgroup of $\operatorname{Aut}(A\times\widehat{A})$.

We proved in [38] that if there is an isometric isomorphism between $A\times\widehat{A}$ and $B\times\widehat{B}$ for two Abelian varieties A and B over an algebraically closed field, then the derived categories of coherent sheaves $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ are equivalent. In Chapter 5 we prove that these conditions are equivalent over an algebraically closed field of characteristic zero; that is, the derived categories $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ are equivalent if and only if there is an isometric isomorphism from $A\times\widehat{A}$ to $B\times\widehat{B}$. In fact, the "only if" part holds over an arbitrary field (Theorem 5.1.16). As a corollary, we see that there are only finitely many non-isomorphic Abelian varieties whose derived categories are equivalent to $\mathbf{D}^b(A)$ for a given Abelian variety A (Corollary 5.1.17).

Representing equivalences by objects on the product, we construct a map from the set of all exact equivalences between $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ to the set of isometric isomorphisms from $A \times \widehat{A}$ to $B \times \widehat{B}$. We then prove that this map is functorial (Proposition 5.1.12). In particular, we obtain a homomorphism from the group of exact auto-equivalences of $\mathbf{D}^b(A)$ to the group $U(A\times\widehat{A})$ of isometric automorphisms of $A \times \widehat{A}$.

In § 5.2 we describe the kernel of this homomorphism, which turns out to be isomorphic to the direct sum of $\mathbb{Z}$ and the group of k-valued points of $A \times \widehat{A}$ (Proposition 5.2.3). Technically, this description is based on the fact that the object on the product of Abelian varieties that defines the equivalence is in fact a sheaf, up to a shift in the derived category (Proposition 5.2.2).

In the final § 5.3, under the assumption that the ground field is algebraically closed and $\mathrm{char}(k) = 0$, we give another proof of the assertion in [38]; this proof uses results in [30] describing semi-homogeneous bundles on Abelian varieties. In particular, we obtain a description of the group of auto-equivalences as an exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus (A \times \widehat{A})_k \longrightarrow \mathrm{Auteq}\,\mathbf{D}^b(A) \longrightarrow U(A \times \widehat{A}) \longrightarrow 1.$$

CHAPTER 1

Preliminaries

1.1. Triangulated categories and exact functors. A detailed treatment of the facts collected in this chapter may be found in [14], [24], [25], and [44]. The notion of triangulated category was first introduced by Verdier in [44]. Let $\mathcal{D}$ be some additive category. We define a structure of *triangulated category* on $\mathcal{D}$ by giving the following data:

a) an additive shift functor $[1]\colon \mathcal{D} \longrightarrow \mathcal{D}$ which is an auto-equivalence;
b) a class of *distinguished* (or *exact*) triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

that must satisfy the following set of axioms T1–T4.

T1. a) For any object X the triangle $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1]$ is distinguished.
b) If a triangle is distinguished, then any isomorphic triangle is also distinguished.

c) Any morphism $X \xrightarrow{u} Y$ in $\mathcal{D}$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.

T2. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle
$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$
is distinguished.

T3. Given two distinguished triangles and two morphisms between their first and second terms that form a commutative square, this diagram can be completed to a morphism of triangles:
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}.$$

T4. For any pair of morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z$ there is a commutative diagram
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "x"] \arrow[d, "v"'] & Z' \arrow[r] \arrow[d, "w"] & X[1] \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r, "y"] \arrow[d] & Y' \arrow[r, "w'"] \arrow[d, "t"] & X[1] \arrow[d, "u[1]"] \\
 & X' \arrow[r, equal] \arrow[d, "r"] & X' \arrow[r, "r"] \arrow[d] & Y[1] \\
 & Y[1] \arrow[r, "x[1]"] & Z'[1] &
\end{tikzcd}$$
in which the top two rows and the two central columns are distinguished triangles.

Let $\mathcal{D}$ be a triangulated category. We say that a full additive subcategory $\mathcal{N} \subset \mathcal{D}$ is a *triangulated subcategory* if it is closed under the shift functor and under taking the mapping cone of morphisms; that is, if two objects of some triangle
$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$
belong to $\mathcal{N}$, then so does the third object. We now describe the type of functors between triangulated categories that it makes sense to consider.

Definition 1.1.1. We say that an additive functor $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ between two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is *exact* if

a) F commutes with the shift functor, that is, there is a given isomorphism of functors
$$t_F\colon F \circ [1] \xrightarrow{\sim} [1] \circ F,$$

b) F takes each distinguished triangle in $\mathcal{D}$ to a distinguished triangle in $\mathcal{D}'$ (where we use the isomorphism t_F to replace $F(X[1])$ by $F(X)[1]$).

It follows at once from the definition that the composite of two exact functors is again exact. Another property we need concerns adjoint functors.

Lemma 1.1.2 ([6], [8]). *If a functor* $G\colon \mathcal{D}' \longrightarrow \mathcal{D}$ *is left* (*or right*) *adjoint to an exact functor* $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$, *then* G *is also exact.*

We define and describe the main properties of a Serre functor, the abstract definition of which was given in [6] (see also [8]).

Definition 1.1.3. Let $\mathcal{D}$ be a k-linear category with finite-dimensional Hom-spaces between objects. A covariant functor $S\colon \mathcal{D} \to \mathcal{D}$ is a *Serre functor* if it is an equivalence of categories, and there exists a bifunctorial isomorphism

$$\varphi_{A,B}\colon \operatorname{Hom}_{\mathcal{D}}(A,B) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(B,SA)^* \quad \text{for any objects } A,B \in \mathcal{D}.$$

Lemma 1.1.4 [8]. *Any equivalence of categories* $\Phi\colon \mathcal{D} \longrightarrow \mathcal{D}'$ *commutes with Serre functors*; *that is*, *there exists a natural isomorphism of functors* $\Phi \circ S \xrightarrow{\sim} S' \circ \Phi$, *where* S *and* S' *are Serre functors for the categories* $\mathcal{D}$ *and* $\mathcal{D}'$ *respectively.*

If we have two Serre functors for the same category, then they are isomorphic, and this isomorphism commutes with the bifunctorial isomorphisms $\varphi_{A,B}$ in the definition of Serre functor. Indeed, let S and S' be two Serre functors for the category $\mathcal{D}$. Then for any object A in $\mathcal{D}$ there is a natural isomorphism

$$\operatorname{Hom}(A,A) \cong \operatorname{Hom}(A,SA)^* \cong \operatorname{Hom}(SA,S'A).$$

Considering the image of the identity morphism id_A under this identification, we obtain a morphism $SA \longrightarrow S'A$, which gives an isomorphism $S \xrightarrow{\sim} S'$.

Thus, a Serre functor for a category $\mathcal{D}$ (if it exists) is uniquely determined (up to isomorphism). In what follows, we will be interested in Serre functors for triangulated categories.

Lemma 1.1.5 [6]. *A Serre functor for a triangulated category is exact.*

We recall the definition of localization of a category and, in particular, the localization of a triangulated category with respect to a full triangulated subcategory (see [13]). Let $\mathcal{C}$ be a category and Σ a class of morphisms in $\mathcal{C}$; the localization of $\mathcal{C}$ with respect to Σ has a good description if Σ admits a *calculus of left fractions*; that is, if the following properties hold:

L1. All the identity morphisms of the category belong to Σ.

L2. The composite of any two morphisms in Σ again belongs to Σ.

L3. Each diagram of the form $X' \xleftarrow{s} X \xrightarrow{u} Y$ with $s \in \Sigma$ can be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ X' \arrow[r, dashed, "u'"] & Y' \end{tikzcd},$$

with $t \in \Sigma$.

L4. If f and g are two morphisms, and there exists a morphism $s \in \Sigma$ satisfying $fs = gs$, then there also exists $t \in \Sigma$ such that $tf = tg$.

If Σ admits a calculus of left fractions, then the category $\mathcal{C}[\Sigma^{-1}]$ can be described as follows. The objects of $\mathcal{C}[\Sigma^{-1}]$ are just those of $\mathcal{C}$. The morphisms from X to Y are equivalence classes of diagrams (s, f) in $\mathcal{C}$ of the form

$$X \xrightarrow{f} Y' \xleftarrow{s} Y \quad \text{with } s \in \Sigma,$$

where two diagrams (f, s) and (g, t) are equivalent if they fit into a commutative diagram

$$\begin{tikzcd}
 & Y' \arrow[d] & \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, "r"'] \arrow[dl, "t"] \\
 & Y'' \arrow[u] &
\end{tikzcd}$$

with $r \in \Sigma$.

The composite of two morphisms (f, s) and (g, t) is the morphism $(g'f, s't)$ constructed using the square of axiom L3:

$$\begin{tikzcd}
 & & Z'' & & \\
 & Y' \arrow[ur, dashed, "g'"] & & Z' \arrow[ul, dashed, "s'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"']
\end{tikzcd}$$

One sees readily that $\mathcal{C}[\Sigma^{-1}]$ constructed in this way is indeed a category (with morphisms between objects forming a set), and that the canonical functor

$$Q\colon \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}] \qquad \text{defined by} \quad X \mapsto X, \quad f \mapsto (f, 1)$$

inverts all morphisms in Σ, and is universal in this sense (see [13]).

Consider a full subcategory $\mathcal{B} \subset \mathcal{C}$ and write $\Sigma \cap \mathcal{B}$ for the class of morphisms in $\mathcal{B}$ also belonging to Σ. We say that $\mathcal{B}$ is *right cofinal* in $\mathcal{C}$ with respect to Σ if for any $s\colon X \longrightarrow X'$ in Σ with $X \in \mathcal{B}$ there is a morphism $f\colon X' \longrightarrow Y$ such that $fs \in \Sigma \cap \mathcal{B}$.

Lemma 1.1.6 ([17], [25]). *The class $\Sigma \cap \mathcal{B}$ also admits a calculus of left fractions and, if $\mathcal{B}$ is right cofinal in $\mathcal{C}$ with respect to Σ, the canonical functor*

$$\mathcal{B}[(\Sigma \cap \mathcal{B})^{-1}] \longrightarrow \mathcal{C}[\Sigma^{-1}]$$

is fully faithful.

We recall the definition of fully faithful functor.

Definition 1.3. We say that a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is *fully faithful* if the natural map

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(FX, FY)$$

is a bijection for any two objects $X, Y \in \mathcal{C}$.

Now let $\mathcal{D}$ be a triangulated category and $\mathcal{N}$ a full triangulated subcategory. We write Σ for the class of morphisms s in $\mathcal{D}$ that fit in an exact triangle

$$N \longrightarrow X \xrightarrow{s} X' \longrightarrow N[1],$$

with $N \in \mathcal{N}$, and call Σ the *multiplicative system* associated with the subcategory $\mathcal{N}$. It follows from the general theory of localization that there exists an additive category $\mathcal{D}[\Sigma^{-1}]$ and an additive localization functor $Q\colon \mathcal{D} \longrightarrow \mathcal{D}[\Sigma^{-1}]$. We can give the category $\mathcal{D}[\Sigma^{-1}]$ the shift functor induced by $[1]\colon \mathcal{D} \longrightarrow \mathcal{D}$. Moreover, we define distinguished triangles in $\mathcal{D}[\Sigma^{-1}]$ to be the triangles isomorphic to the images of distinguished triangles in $\mathcal{D}$ under the localization. We set

$$\mathcal{D}/\mathcal{N} := \mathcal{D}[\Sigma^{-1}].$$

Proposition 1.1.8. *Giving $\mathcal{D}/\mathcal{N}$ the structure described above makes it into a triangulated category, and makes $Q\colon \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{N}$ into an exact functor.*

Note that in our situation the system Σ admits a calculus of left (and right) fractions, so that the category $\mathcal{D}/\mathcal{N}$ admits a good description as given above. Following Deligne [12] (see also [25]), we now describe the general construction of derived functors for the localizations of triangulated categories. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ an exact functor. Let $\mathcal{M} \subset \mathcal{C}$ and $\mathcal{N} \subset \mathcal{D}$ be full triangulated categories. Since we do not assume that $F\mathcal{M} \subset \mathcal{N}$, the functor F does not induce any functor from $\mathcal{C}/\mathcal{M}$ to $\mathcal{D}/\mathcal{N}$. However, there may exist a certain canonical approximation to an induced functor, namely, an exact functor $\mathbf{R}F\colon \mathcal{C}/\mathcal{M} \longrightarrow \mathcal{D}/\mathcal{N}$, and a morphism of exact functors can: $QF \longrightarrow (\mathbf{R}F)Q$. The construction proceeds as follows. Write Σ for the multiplicative system associated with the subcategory $\mathcal{M}$. Let Y be an object of $\mathcal{C}/\mathcal{M}$. We define a contravariant functor $\mathbf{r}FY$ from $\mathcal{D}/\mathcal{N}$ to the category of Abelian groups by the following rule: the value of $\mathbf{r}FY(X)$ at an object $X \in \mathcal{D}/\mathcal{N}$ is the equivalence classes of pairs (s, f)

$$Y \xrightarrow{s} Y', \qquad X \xrightarrow{f} FY',$$

with $s \in \Sigma$ and f a morphism in $\mathcal{D}/\mathcal{N}$. Two such pairs (s, f) and (t, g) are equivalent if there exist commutative diagrams in $\mathcal{C}$ and $\mathcal{D}/\mathcal{N}$ of the form

$$\begin{tikzcd} & Y' \arrow[d, "v"] \\ Y \arrow[ur, "s"] \arrow[r, "r"] \arrow[dr, "t"'] & Y''', \\ & Y'' \arrow[u, "w"'] \end{tikzcd} \qquad \begin{tikzcd} & FY' \arrow[d, "Fv"] \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & FY''', \\ & FY'' \arrow[u, "Fw"'] \end{tikzcd}$$

with $r \in \Sigma$. If the functor $\mathbf{r}FY$ is representable, we define $\mathbf{R}FY$ as the object that represents it, and say that the right derived functor $\mathbf{R}F$ is defined on Y. In this case we have an isomorphism

$$\mathrm{Hom}(X, \mathbf{R}FY) \cong \mathbf{r}FY(X).$$

One sees readily that a morphism of functors $\mathbf{r}F\alpha\colon \mathbf{r}FY \longrightarrow \mathbf{r}FZ$ is defined for any morphism $\alpha\colon Y \longrightarrow Z$ in $\mathcal{C}/\mathcal{M}$. Now if the derived functor $\mathbf{R}F$ is defined on both Y and Z, the morphism $\mathbf{R}F\alpha$ is also defined. This makes $\mathbf{R}F$ a functor $\mathcal{W} \longrightarrow \mathcal{D}/\mathcal{N}$ on some full subcategory $\mathcal{W} \subset \mathcal{C}/\mathcal{M}$, consisting of the objects on which $\mathbf{R}F$ is defined.

Proposition 1.1.9 [12]. *Suppose that*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

is a distinguished triangle in $\mathcal{C}/\mathcal{M}$ and $\mathbf{R}F$ is defined on X and Z. Then it is also defined on Y, and takes the given triangle into a distinguished triangle of $\mathcal{D}/\mathcal{N}$. Thus, $\mathcal{W}$ is a triangulated subcategory in $\mathcal{C}/\mathcal{M}$ and $\mathbf{R}F\colon \mathcal{W} \longrightarrow \mathcal{D}/\mathcal{N}$ is an exact functor.

It follows at once from the construction of the derived functor that there is a canonical morphism $\mathrm{can}\colon QFY \longrightarrow (\mathbf{R}F)QY$ for any object $Y \in \mathcal{C}$ (provided, of course, that $\mathbf{R}F$ is defined on $QY \in \mathcal{C}/\mathcal{M}$). All these morphisms define a natural transformation of triangulated functors $\mathrm{can}\colon QF_{|\mathcal{W}} \longrightarrow (\mathbf{R}F)Q_{|\mathcal{W}}$.

The left derived functor $\mathbf{L}F$ is defined in the dual way: for $Y \in \mathcal{C}/\mathcal{M}$, we define a covariant functor $\mathbf{l}FY$ whose value at $X \in \mathcal{D}/\mathcal{N}$ is the equivalence classes of pairs (s, f),

$$Y' \xrightarrow{s} Y, \qquad FX' \xrightarrow{f} Y,$$

with $s \in \Sigma$ and f a morphism in $\mathcal{D}/\mathcal{N}$. Then $\mathbf{L}FY$ (if it exists) is the object representing the functor $\mathbf{l}FY$; that is, $\mathrm{Hom}(\mathbf{L}FY, X) \cong \mathbf{l}FY(X)$. There is a canonical morphism $\mathrm{can}\colon \mathbf{L}FQY \longrightarrow QFY$.

Suppose that the functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ takes the subcategory $\mathcal{M}$ into $\mathcal{N}$. In this case the derived functors $\mathbf{R}F$ and $\mathbf{L}F$ are both isomorphic to the canonical functor $\mathcal{C}/\mathcal{M} \longrightarrow \mathcal{D}/\mathcal{N}$ induced by F.

Let $j\colon \mathcal{V} \hookrightarrow \mathcal{C}$ be the inclusion of a full triangulated subcategory which is right cofinal with respect to Σ. By Lemma 1.1.6 the induced functor $\mathcal{V}/(\mathcal{V}\cap\mathcal{M}) \longrightarrow \mathcal{C}/\mathcal{M}$ is fully faithful. We denote it by $\mathbf{R}j$.

Lemma 1.1.10. *For any object $V \in \mathcal{V}$ the functor $\mathbf{R}F$ is defined on V if and only if $\mathbf{R}(Fj)$ is defined on V, and there is an isomorphism of functors $\mathbf{R}(Fj)V \xrightarrow{\sim} \mathbf{R}F\mathbf{R}jV$.*

We now describe conditions under which the right derived functor is defined on the entire category $\mathcal{C}$.

Definition 1.1.11. An object $Y \in \mathcal{C}$ is said to be (right) *F-split* with respect to $\mathcal{M}$ and $\mathcal{N}$ if $\mathbf{R}F$ is defined on Y and the canonical morphism $QFY \longrightarrow (\mathbf{R}F)QY$ is an isomorphism.

The following lemma gives a characterization of F-split objects.

Lemma 1.1.12. *An object $Y \in \mathcal{C}$ is F-split if and only if for any morphism $s\colon Y \longrightarrow Y'$ in Σ the morphism QFs admits a retraction, that is, there exists a $p\colon QFY' \longrightarrow QFY$ such that $p \circ QFs = \mathrm{id}$.*

We say that $\mathcal{C}$ admits *enough F-split objects* (with respect to $\mathcal{M}$ and $\mathcal{N}$) if for any $Y \in \mathcal{C}$ there exists a morphism $s\colon Y \longrightarrow Y_0$ in Σ such that Y_0 is F-split. In this case $\mathbf{R}F$ is defined on the entire category $\mathcal{C}/\mathcal{M}$, and there are isomorphisms

$$\mathbf{R}FY \xrightarrow{\sim} \mathbf{R}FY_0 \xleftarrow{\sim} FY_0.$$

To conclude this section we say a few words on adjoint functors. Suppose that a functor F has a left adjoint $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ and assume that the derived functors $\mathbf{R}F$ and $\mathbf{L}G$ exist (that is, that they are everywhere defined). Then $\mathbf{L}G$ is again a left adjoint to $\mathbf{R}F$, and hence there are functorial isomorphisms

$$\mathrm{Hom}(\mathbf{L}GX, Y) \cong \mathrm{Hom}(X, \mathbf{R}FY) \quad \text{for } X \in \mathcal{D}/\mathcal{N} \text{ and } Y \in \mathcal{C}/\mathcal{M}. \tag{2}$$

1.2. Derived categories and derived functors. Let $\mathcal{A}$ be an additive category. We write $\mathbf{C}(\mathcal{A})$ to denote the category of differential complexes. Its objects are the complexes

$$M^\cdot = (\cdots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \longrightarrow \cdots) \qquad \text{with } M^p \in \mathcal{A} \text{ for } p \in \mathbb{Z}, \text{ and } d^2 = 0,$$

and the morphisms $f\colon M^\cdot \longrightarrow N^\cdot$ are families of morphisms $f^p\colon M^p \longrightarrow N^p$ in $\mathcal{A}$ that commute with the differentials; that is,

$$d_N f^p - f^{p+1} d_M = 0 \quad \text{for any} \quad p.$$

We write $\mathbf{C}^+(\mathcal{A})$, $\mathbf{C}^-(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ for the full subcategories of $\mathbf{C}(\mathcal{A})$ formed by complexes $M^\cdot$ for which $M^p = 0$ for all $p \ll 0$, respectively for all $p \gg 0$, respectively for all $p \gg 0$ and all $p \ll 0$.

We say that a morphism of complexes $f\colon M^\cdot \longrightarrow N^\cdot$ is *null-homotopic* if $f^p = d_N h^p + h^{p+1} d_M$ for all $p \in \mathbb{Z}$ for some family of morphisms $h^p\colon M^{p+1} \longrightarrow N^p$. We define the *homotopy category* $\mathbf{H}(\mathcal{A})$ to be the category having the same objects as $\mathbf{C}(\mathcal{A})$ and the morphisms in $\mathbf{H}(\mathcal{A})$ are classes $\overline{f}$ of morphisms f between complexes modulo null-homotopic morphisms.

We define the shift functor $[1]\colon \mathbf{H}(\mathcal{A}) \longrightarrow \mathbf{H}(\mathcal{A})$ by the rule

$$(M[1])^p = M^{p+1}, \qquad d_{M[1]} = -d_M.$$

We define a standard triangle in $\mathbf{H}(\mathcal{A})$ to be a sequence

$$L \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} Cf \xrightarrow{\overline{h}} L[1],$$

where $f\colon L \longrightarrow M$ is some morphism of complexes, $Cf = M \oplus L[1]$ is a graded object of $\mathbf{C}(\mathcal{A})$, with the differential

$$d_{Cf} = \begin{pmatrix} d_M & f \\ 0 & -d_L \end{pmatrix},$$

g is the canonical embedding $M \longrightarrow Cf$, and $-h$ the canonical projection. As usual, Cf is called the *mapping cone* of f.

Lemma 1.2.1. *The category* $\mathbf{H}(\mathcal{A})$ *with* $[1]$ *as shift functor and the class of triangles isomorphic to standard triangles as distinguished triangles is a triangulated category.*

We write $\mathbf{H}^+(\mathcal{A})$, $\mathbf{H}^-(\mathcal{A})$ and $\mathbf{H}^b(\mathcal{A})$ for the images in $\mathbf{H}(\mathcal{A})$ of the categories $\mathbf{C}^+(\mathcal{A})$, $\mathbf{C}^-(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ respectively. These categories are also triangulated. Suppose now that $\mathcal{A}$ is an Abelian category. To define the derived category of an Abelian category, we must recall the notions of acyclic complex and of quasi-isomorphism. For any complex $N^\cdot$ and each $p \in \mathbb{Z}$, the cohomology $H^p(N^\cdot) \in \mathcal{A}$ is defined as $\operatorname{Ker} d^p / \operatorname{Im} d^{p-1}$. Thus, for any integer p we have an additive functor $H^p \colon \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ taking a complex $N^\cdot$ to its pth cohomology $H^p(N^\cdot) \in \mathcal{A}$.

We say that a complex $N^\cdot \in \mathbf{C}(\mathcal{A})$ is *acyclic at the nth term* if $H^n(N^\cdot) = 0$, and simply *acyclic* if all its cohomology vanishes, $H^n(N^\cdot) = 0$ for $n \in \mathbb{Z}$. We denote by $\mathcal{N}$ the full subcategory of $\mathbf{H}(\mathcal{A})$ consisting of all acyclic complexes. The following lemma is practically obvious.

Lemma 1.2.2. *The subcategory* $\mathcal{N}$ *is a full triangulated subcategory of* $\mathbf{H}(\mathcal{A})$.

We say that a morphism $f \colon X \longrightarrow Y$ in $\mathbf{H}(\mathcal{A})$ is a *quasi-isomorphism* if its mapping cone is an acyclic complex. In other words, f is a quasi-isomorphism if the map it induces on cohomology is an isomorphism. Let Quis be the multiplicative system associated with $\mathcal{N}$, that is, the system consisting of all quasi-isomorphisms.

Definition 1.2.3. The *derived category* $\mathbf{D}(\mathcal{A})$ of an Abelian category $\mathcal{A}$ is defined as the localization of the homotopy category $\mathbf{H}(\mathcal{A})$ with respect to the subcategory of all acyclic complexes, that is,

$$\mathbf{D}(\mathcal{A}) := \mathbf{H}(\mathcal{A})/\mathcal{N} = \mathbf{H}(\mathcal{A})[\mathrm{Quis}^{-1}].$$

For $* \in \{+, -, b\}$, we define the corresponding derived category $\mathbf{D}^*(\mathcal{A})$ in the same way as the localization $\mathbf{H}^*(\mathcal{A})/(\mathbf{H}^*(\mathcal{A}) \cap \mathcal{N})$.

Lemma 1.2.4. *For* $* \in \{+, -, b\}$, *the canonical functors*

$$\mathbf{D}^*(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$$

define equivalences with the full subcategories of $\mathbf{D}(\mathcal{A})$ *formed by complexes that are acyclic respectively for* $n \ll 0$, *for* $n \gg 0$, *and for* $n \ll 0$ *and* $n \gg 0$. *The subcategory* $\mathbf{H}^+(\mathcal{A})$ *is right cofinal in* $\mathbf{H}(\mathcal{A})$ *with respect to the class of quasi-isomorphisms, and* $\mathbf{H}^-(\mathcal{A})$ *is left cofinal.*

Suppose that the Abelian category $\mathcal{A}$ has enough injective objects; that is, every object embeds in an injective. We denote by $\mathcal{I}$ the full subcategory of $\mathcal{A}$ consisting of the injective objects. In this case, one can prove that the composite functor

$$\mathbf{H}^+(\mathcal{I}) \longrightarrow \mathbf{H}^+(\mathcal{A}) \xrightarrow{Q} \mathbf{D}^+(\mathcal{A})$$

is an equivalence of categories (see [17]). Similarly, if an Abelian category $\mathcal{A}$ has enough projectives, then the composite functor

$$\mathbf{H}^-(\mathcal{P}) \longrightarrow \mathbf{H}^-(\mathcal{A}) \xrightarrow{Q} \mathbf{D}^-(\mathcal{A})$$

is an equivalence, where $\mathcal{P}$ in $\mathcal{A}$ is the full subcategory of projectives.

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor (not necessarily exact) between Abelian categories. Then F induces in an obvious way an exact functor $\mathbf{H}(\mathcal{A}) \longrightarrow \mathbf{H}(\mathcal{B})$, which we denote by the same symbol F. The general construction of (right) derived functor described in the previous section gives a functor $\mathbf{R}F$, defined on a certain full triangulated subcategory of $\mathbf{D}(\mathcal{A})$, and taking values in $\mathbf{D}(\mathcal{B})$. The same applies to the left derived functor. We define the nth right (respectively left) derived functor of F as the cohomology

$$\mathbf{R}^n FX = H^n(\mathbf{R}FX) \quad (\text{respectively } \mathbf{L}_n FX = H^{-n}(\mathbf{L}FX)) \quad \text{for } n \in \mathbb{Z}.$$

In applications, the right adjoint functor usually turns out to be well defined on the subcategory $\mathbf{D}^+(\mathcal{A})$. Using Lemmas 1.1.10 and 1.2.4, we can say that the restriction of the functor $\mathbf{R}F$ to $\mathbf{D}^+(\mathcal{A})$ coincides with the derived functor of the restriction of F to $\mathbf{H}^+(\mathcal{A}) \subset \mathbf{H}(\mathcal{A})$.

We now describe the conditions under which the right derived functor $\mathbf{R}F$ is defined on the entire category $\mathbf{D}^+(\mathcal{A})$. We say that a full additive subcategory $\mathcal{R} \subset \mathcal{A}$ is *right adapted* to a functor F if

a) F takes acyclic complexes in $\mathbf{C}^+(\mathcal{R})$ to acyclic ones;
b) every object of $\mathcal{A}$ embeds in some object of $\mathcal{R}$.

We say that the objects of $\mathcal{R}$ are right F-acyclic. If there exists a subcategory $\mathcal{R}$ right adapted to F, one often says that $\mathcal{A}$ has enough (right) F-acyclic objects.

Suppose that $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a functor for which a right adapted subcategory $\mathcal{R} \subset \mathcal{A}$ exists. Applying Lemma 1.1.12, one checks readily that every right bounded complex of objects in $\mathcal{R}$ is right F-split. From condition b) one deduces that for each object $X \in \mathbf{H}^+(\mathcal{A})$ there is a quasi-isomorphism $X \longrightarrow X'$ with $X' \in \mathbf{H}^+(\mathcal{R})$. As a corollary, we see that the canonical functor

$$\mathbf{H}^+(\mathcal{R})[\mathrm{Quis}^{-1}] \longrightarrow \mathbf{D}^+(\mathcal{A})$$

is an equivalence of triangulated categories.

Lemma 1.2.5. *Suppose that F is a functor for which a right adapted subcategory $\mathcal{R} \subset \mathcal{A}$ exists. Then the functor $\mathbf{R}F$ is defined on $\mathbf{D}^+(\mathcal{A})$, and for any left bounded complex X there is an isomorphism $\mathbf{R}FX \xrightarrow{\sim} X'$, where $X \longrightarrow X'$ is a quasi-isomorphism with $X' \in \mathbf{H}^+(\mathcal{A})$.*

If $\mathcal{A}$ has enough injectives, then the full subcategory $\mathcal{I} \subset \mathcal{A}$ consisting of all injectives is right adapted to every additive functor. In this case we can compute the right derived functor $\mathbf{R}FX$ by applying F to an injective resolution X' of the complex X.

Dually, one can introduce the notion of subcategory left adapted to a functor F. If such a subcategory exists, the left derived functor $\mathbf{L}F\colon \mathbf{D}^-(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B})$ is defined.

1.3. Derived categories of sheaves on schemes. Several Abelian categories of sheaves can be assigned to any scheme. Let X be a scheme over a field k, with structure sheaf $\mathcal{O}_X$. We denote by $\mathcal{O}_X$-Mod the Abelian category of all sheaves of $\mathcal{O}_X$-modules in the Zariski topology. The category $\mathcal{O}_X$-Mod has all limits and colimits, and has a set of generators. Direct colimits are exact. For this reason, the category $\mathcal{O}_X$-Mod is a Grothendieck Abelian category, and has enough injectives (see [15], [42], Exp. IV).

From now on, we consider only Noetherian schemes (although many of the facts treated below also hold in the more general situation). We denote by $\mathrm{Qcoh}(X)$ the full Abelian subcategory of $\mathcal{O}_X$-Mod consisting of quasi-coherent sheaves. On a Noetherian scheme X every quasi-coherent sheaf is the direct colimit of its subsheaves of finite type (see [16], EGA1, 9.4). In this case the category $\mathrm{Qcoh}(X)$ has a set of generators and is a Grothendieck Abelian category, and thus has enough injectives.

The third category that we can assign to a scheme X is the category of coherent sheaves $\mathrm{coh}(X)$; it is a full Abelian subcategory of $\mathrm{Qcoh}(X)$. Although the definition of (quasi-)coherent sheaves is local, in fact they do not depend on the topology. We could, for example, consider not just the Zariski topology but also, say, the etale or flat topology. In this case, although the notion of sheaf of $\mathcal{O}_X$-modules depends on the choice of topology, (quasi-)coherent sheaves do not (see [36]). In particular, for an affine scheme X, the category $\mathrm{Qcoh}(X)$ is equivalent to the category of modules over the algebra corresponding to X.

In what follows we will focus on the category of coherent sheaves, and, more precisely, on the derived category of coherent sheaves. However, since $\mathrm{coh}(X)$ does not have enough injectives, in constructing derived functors we make use of the categories $\mathrm{Qcoh}(X)$ and $\mathcal{O}_X$-Mod.

For a Noetherian scheme X, the full embedding of Abelian categories $\mathrm{Qcoh}(X) \hookrightarrow \mathcal{O}_X$-Mod takes injectives to injectives. From this, we can deduce by a simple procedure (see [17], I.4.6, [43], Appendix B) that the triangulated subcategory $\mathbf{H}^+(\mathrm{Qcoh})$ is right cofinal in the triangulated category $\mathbf{H}^+(\mathcal{O}_X\text{-Mod})$. Thus, applying Lemma 1.1.6, we obtain the following assertion.

Proposition 1.3.1 ([17], [41], Exp. II). *If X is a Noetherian scheme, the canonical functor*

$$\mathbf{D}^+(\mathrm{Qcoh}(X)) \longrightarrow \mathbf{D}^+(\mathcal{O}_X\text{-Mod})$$

is fully faithful and defines an equivalence with the full subcategory

$$\mathbf{D}^+(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}} \subset \mathbf{D}^+(\mathcal{O}_X\text{-Mod})$$

consisting of complexes with quasi-coherent cohomology.

Under additional conditions on the scheme we can also prove the analogous assertion for unbounded derived categories.

Proposition 1.3.2 ([41], Exp. II). *If X is a finite-dimensional Noetherian scheme, then the canonical functor*

$$\mathbf{D}(\mathrm{Qcoh}(X)) \longrightarrow \mathbf{D}(\mathcal{O}_X\text{-Mod})$$

is fully faithful and defines an equivalence with the full subcategory

$$\mathbf{D}(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}} \subset \mathbf{D}(\mathcal{O}_X\text{-Mod}),$$

which consists of complexes with quasi-coherent cohomology.

The proof makes use of the fact that the embedding functor has a right adjoint $Q\colon \mathcal{O}_X\text{-Mod} \longrightarrow \mathrm{Qcoh}(X)$, and for finite-dimensional schemes this functor has finite cohomological dimension (see [41], II.3.7).

We now consider the embedding of Abelian categories $\mathrm{coh}(X) \subset \mathrm{Qcoh}(X)$. Assertions similar to those just described are also known for the canonical functor between derived categories; however, these assertions relate only to right bounded derived categories.

Proposition 1.3.3 ([41], Exp. II). *For a Noetherian scheme X, the canonical functor*

$$\mathbf{D}^-(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^-(\mathrm{Qcoh}(X))$$

is fully faithful and gives an equivalence with the full subcategory $\mathbf{D}^-(\mathrm{Qcoh}(X))_{\mathrm{coh}}$.

Combining this proposition with Propositions 1.3.1 and 1.3.2, we obtain the following corollary.

Corollary 1.3.4 ([41], Exp. II). *Let X be a Noetherian scheme* (*respectively, a finite-dimensional Noetherian scheme*). *Then the canonical functor*

$$\mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\mathcal{O}_X\text{-Mod}) \quad (\textit{respectively } \mathbf{D}^-(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}))$$

is fully faithful and defines an equivalence with the full subcategory

$$\mathbf{D}^b(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}} \quad (\textit{respectively } \mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}).$$

We now describe the main derived functors between the derived categories of sheaves on schemes. Let $f\colon X \longrightarrow Y$ be a morphism of Noetherian schemes. There exists an inverse image functor

$$f^*\colon \mathcal{O}_Y\text{-Mod} \longrightarrow \mathcal{O}_X\text{-Mod},$$

which is right exact. Since $\mathcal{O}_Y$-Mod has enough flat $\mathcal{O}_Y$-modules and they are f^*-acyclic, it follows that the left derived functor

$$\mathbf{L}f^*\colon \mathbf{D}^-(\mathcal{O}_Y\text{-Mod}) \longrightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod})$$

is defined. One proves readily that $\mathbf{L}f^*$ takes the categories $\mathbf{D}^-(\mathcal{O}_Y\text{-Mod})_{\mathrm{Qcoh}}$ and $\mathbf{D}^-(\mathcal{O}_Y\text{-Mod})_{\mathrm{coh}}$ to $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}}$ and $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$ respectively. Thus, for finite-dimensional Noetherian schemes we obtain a derived inverse image functor $\mathbf{L}f^*$ on right bounded derived categories of quasi-coherent and coherent sheaves.

If f^* has finite cohomological dimension (in which case we say that f has finite Tor-dimension), we can extend the derived functor $\mathbf{L}f^*$ to the unbounded derived categories. Moreover, if f has finite Tor-dimension, the derived inverse image functor takes the bounded derived category to the bounded derived category. In particular, we have the functor

$$\mathbf{L}f^*\colon \mathbf{D}^b(\mathcal{O}_Y\text{-Mod})_{\mathrm{coh}} \longrightarrow \mathbf{D}^b(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}.$$

Let $\mathcal{E}, \mathcal{F} \in \mathbf{C}(\mathcal{O}_X\text{-Mod})$ be two complexes of $\mathcal{O}_X$-modules. We define the tensor product $\mathcal{E} \otimes \mathcal{F}$ as the complex associated to the double complex $\mathcal{E}^p \otimes \mathcal{F}^q$, that is,

$$(\mathcal{E} \otimes \mathcal{F})^n = \sum_{p+q=n} \mathcal{E}^p \otimes \mathcal{F}^q,$$

with the differential $d = d_{\mathcal{E}} + (-1)^n d_{\mathcal{F}}$. A homotopy between morphisms of complexes extends to the tensor product, and we obtain a functor

$$\mathcal{E} \otimes : \mathbf{H}(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{H}(\mathcal{O}_X\text{-Mod}).$$

Suppose now that $\mathcal{E} \in \mathbf{C}^-(\mathcal{O}_X\text{-Mod})$. The category $\mathbf{H}^-(\mathcal{O}_X\text{-Mod})$ has enough objects that are left split with respect to the functor $\mathcal{E} \otimes$; indeed, right bounded complexes of flat $\mathcal{O}_X$-modules have this property. Therefore, there exists a left derived functor

$$\mathcal{E} \otimes^{\mathbf{L}} : \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}).$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are quasi-isomorphic, then $\mathcal{E}_1 \otimes^{\mathbf{L}}$ and $\mathcal{E}_2 \otimes^{\mathbf{L}}$ are isomorphic. In fact, we obtain a functor in two variables

$$\otimes^{\mathbf{L}} \colon \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \times \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}),$$

which is exact with respect to both arguments. The derived functor of the tensor product is obviously associative and symmetric.

Suppose that an object $\mathcal{E}$ has finite Tor-dimension, that is, $\mathcal{E}$ is quasi-isomorphic to a bounded complex of flat $\mathcal{O}_X$-modules. Then, on the one hand, $\mathcal{E} \otimes^{\mathbf{L}}$ extends to the unbounded derived category, and on the other, by restriction we obtain a functor from the bounded derived category to itself. We obtain the functors

$$\mathcal{E} \otimes^{\mathbf{L}} : \mathbf{D}(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}(\mathcal{O}_X\text{-Mod}), \qquad \mathcal{E} \otimes^{\mathbf{L}} : \mathbf{D}^b(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^b(\mathcal{O}_X\text{-Mod}).$$

Note that if $\mathcal{E}$ is in $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$ (respectively $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}}$), then $\mathcal{E} \otimes^{\mathbf{L}}$ takes objects with (quasi-)coherent cohomology to objects with (quasi-)coherent cohomology.

Let $f \colon X \longrightarrow Y$ be a morphism of Noetherian schemes. The direct image functor

$$f_* \colon \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_Y\text{-Mod}$$

is left exact. Since the category of $\mathcal{O}_X$-modules has enough injectives, it follows that the right derived functor

$$\mathbf{R}f_* \colon \mathbf{D}^+(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^+(\mathcal{O}_X\text{-Mod})$$

exists. Moreover, in this case, $\mathbf{R}f_*$ takes the subcategory $\mathbf{D}^+(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}}$ to the subcategory $\mathbf{D}^+(\mathcal{O}_Y\text{-Mod})_{\mathrm{Qcoh}}$.

If in addition, f_* has finite cohomological dimension, then $\mathbf{R}f_*$ can be extended to the category of unbounded complexes. This holds, for example, if X is a finite-dimensional Noetherian scheme. On the other hand, in this case (that is, when f_* has finite cohomological dimension), the right derived functor between the bounded derived categories

$$\mathbf{R}f_* \colon \mathbf{D}^b(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^b(\mathcal{O}_X\text{-Mod})$$

exists.

For the right derived functor to be defined between derived categories of coherent sheaves, we need additional conditions on the morphism.

Proposition 1.3.5 ([16], III, 3.2.1, [17]). *Suppose that $f\colon X \longrightarrow Y$ is a proper morphism of Noetherian schemes. Then the functor $\mathbf{R}f_*$ takes the subcategory $\mathbf{D}^+(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$ to the subcategory $\mathbf{D}^+(\mathcal{O}_Y\text{-Mod})_{\mathrm{coh}}$. If in addition, X is finite-dimensional, then the analogous assertion holds for the bounded and unbounded derived categories.*

Let $\mathcal{E},\mathcal{F}\in\mathbf{C}(\mathcal{O}_X\text{-Mod})$ be two complexes of $\mathcal{O}_X$-modules. We define a complex $\underline{\mathcal{H}om}^{\cdot}(\mathcal{E},\mathcal{F})$ by the rule

$$\underline{\mathcal{H}om}^n(\mathcal{E},\mathcal{F})=\prod_p\underline{\mathcal{H}om}(\mathcal{E}^p,\mathcal{F}^{p+n})$$

with the differential $d=d_{\mathcal{E}}+(-1)^{n+1}d_{\mathcal{F}}$. A homotopy between morphisms of complexes extends to the local $\underline{\mathcal{H}om}$, and we obtain a bifunctor

$$\underline{\mathcal{H}om}\colon \mathbf{H}(\mathcal{O}_X\text{-Mod})^{\mathrm{op}}\times\mathbf{H}(\mathcal{O}_X\text{-Mod})\longrightarrow\mathbf{H}(\mathcal{O}_X\text{-Mod}).$$

Since every left bounded complex has an injective resolution, we obtain a derived bifunctor

$$\mathbf{R}\underline{\mathcal{H}om}\colon \mathbf{D}(\mathcal{O}_X\text{-Mod})^{\mathrm{op}}\times\mathbf{D}^+(\mathcal{O}_X\text{-Mod})\longrightarrow\mathbf{D}(\mathcal{O}_X\text{-Mod}).$$

In this situation we define the local hyper-Ext

$$\underline{\mathcal{E}xt}^i(\mathcal{E},\mathcal{F}):=H^i(\mathbf{R}\underline{\mathcal{H}om}(\mathcal{E},\mathcal{F})).$$

For a Noetherian scheme X, if $\mathcal{E}$ and $\mathcal{F}$ are (quasi-)coherent $\mathcal{O}_X$-modules, then the sheaves $\underline{\mathcal{E}xt}^i(\mathcal{E},\mathcal{F})$ are also (quasi-)coherent for any $i\geqslant 0$.

Now if $\mathcal{E}\in\mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$ and $\mathcal{F}\in\mathbf{D}^+(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$, then $\mathbf{R}\underline{\mathcal{H}om}(\mathcal{E},\mathcal{F})$ belongs to $\mathbf{D}(\mathcal{O}_X\text{-Mod})_{\mathrm{coh}}$.

We describe the main properties and relations between the derived functors introduced in this section. Consider two morphisms $f\colon X\to Y$ and $g\colon Y\to Z$. In this situation we have two functors $\mathbf{L}(gf)^*$ and $\mathbf{L}f^*\mathbf{L}g^*$ from $\mathbf{D}^-(\mathcal{O}_Z\text{-Mod})$ to $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})$. Then the natural transformation

$$\mathbf{L}(gf)^*\xrightarrow{\sim}\mathbf{L}f^*\mathbf{L}g^*$$

is an isomorphism. The proof of this assertion follows from the fact that the functor g^* takes flat $\mathcal{O}_Z$-modules to flat $\mathcal{O}_Y$-modules (see, for example, [17]).

In the same way, we have an isomorphism

$$\mathbf{R}(gf)_*\longrightarrow\mathbf{R}g_*\mathbf{R}f_*$$

of functors from $\mathbf{D}^+(\mathcal{O}_X\text{-Mod})$ to $\mathbf{D}^+(\mathcal{O}_Z\text{-Mod})$. This assertion follows from the fact that f_* takes injective sheaves to flabby sheaves on Y, which in turn are g_*-acyclic (see [17]).

The other relations that we use fairly frequently are called the projection formula and flat base change.

Proposition 1.3.6 ([17], II.5.6). *Let $f\colon X \to Y$ be a morphism between finite-dimensional Noetherian schemes. Then for any objects $\mathcal{E} \in \mathbf{D}^-(\mathcal{O}_Z\text{-Mod})$ and $\mathcal{F} \in \mathbf{D}^-(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}}$ there is a natural isomorphism of functors*

$$\mathbf{R}f_*\mathcal{E} \otimes^{\mathbf{L}} \mathcal{F} \xrightarrow{\sim} \mathbf{R}f_*(\mathcal{E} \otimes^{\mathbf{L}} \mathbf{L}f^*\mathcal{F}). \tag{3}$$

Proposition 1.3.7 ([17], II.5.12). *Let $f\colon X \to Y$ be a morphism of finite type between finite-dimensional Noetherian schemes and $g\colon Y' \to Y$ a flat morphism. We consider the Cartesian square*

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}.$$

In this situation there is a natural isomorphism of functors

$$\mathbf{L}g^*\mathbf{R}f_*\mathcal{E} \xrightarrow{\sim} \mathbf{R}f'_*\mathbf{L}g'^*\mathcal{E} \quad \textit{for any } \mathcal{E} \in \mathbf{D}(\mathcal{O}_X\text{-Mod})_{\mathrm{Qcoh}}. \tag{4}$$

We state another relation that we need.

Proposition 1.3.8 ([17], II.5.16). *Let $\mathcal{E}$ be a bounded complex of locally free sheaves of finite rank on a Noetherian scheme X. Then the following natural isomorphisms of functors*

$$\mathbf{R}\underline{\mathcal{H}om}(\mathcal{F},\ \mathcal{G}) \otimes^{\mathbf{L}} \mathcal{E} \xrightarrow{\sim} \mathbf{R}\underline{\mathcal{H}om}(\mathcal{F}, \mathcal{G} \otimes^{\mathbf{L}} \mathcal{E}) \xrightarrow{\sim} \mathbf{R}\underline{\mathcal{H}om}(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{E}^{\vee}, \mathcal{G}) \tag{5}$$

hold for any $\mathcal{F} \in \mathbf{D}^-(\mathcal{O}_X\text{-Mod})$, $\mathcal{G} \in \mathbf{D}^+(\mathcal{O}_X\text{-Mod})$, where $\mathcal{E}^{\vee} := \mathbf{R}\underline{\mathcal{H}om}(\mathcal{E}, \mathcal{O}_X)$.

CHAPTER 2

Categories of coherent sheaves and functors between them

2.1. Basic properties of categories of coherent sheaves. From now on, we consider only bounded derived categories of coherent sheaves on smooth complete algebraic varieties. For brevity, we always write simply $\mathbf{D}^b(X)$ instead of $\mathbf{D}^b(\mathrm{coh}(X))$. Moreover, we omit the symbol of derived functor if the functor is exact, for example, for inverse image under a flat morphism or for tensor product by a locally free sheaf.

For a smooth complete variety X of dimension n the bounded derived category of coherent sheaves admits a Serre functor (see Definition 1.1.3), given by $(\,\cdot\,)\otimes\omega_X[n]$, where ω_X is the canonical sheaf (see [6]). Thus, we have an isomorphism

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \omega_X[n])^* \tag{6}$$

for any pair of objects $\mathcal{E}, \mathcal{F} \in \mathbf{D}^b(X)$.

As shown in the previous section, every morphism $f\colon X \to Y$ between smooth complete algebraic varieties induces two exact functors, the direct image functor

$\mathbf{R}f_*\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ and inverse image functor $\mathbf{L}f^*\colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(X)$, and these functors are mutually adjoint. Moreover, each object $\mathcal{E} \in \mathbf{D}^b(X)$ defines the exact tensor product functor $\otimes^{\mathbf{L}} \mathcal{E}\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X)$.

We can use these standard derived functors to introduce a large new class of exact functors between the derived categories $\mathbf{D}^b(X)$ and $\mathbf{D}^b(Y)$.

Let X and Y be two smooth complete varieties over a field k, of dimension n and m respectively. Consider the Cartesian product $X \times Y$ and write p and q for the projections of $X \times Y$ to X and Y respectively:

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y.$$

Every object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ determines an exact functor $\Phi_{\mathcal{E}}$ from the derived category $\mathbf{D}^b(X)$ to the derived category $\mathbf{D}^b(Y)$, defined by the formula

$$\Phi_{\mathcal{E}}(\,\cdot\,) := \mathbf{R}^{\cdot} q_*(\mathcal{E} \otimes^{\mathbf{L}} p^*(\,\cdot\,)). \tag{7}$$

Moreover, to the same object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ one can assign another functor $\Psi_{\mathcal{E}}$ from the derived category $\mathbf{D}^b(Y)$ to the derived category $\mathbf{D}^b(X)$, defined by a rule similar to (7):

$$\Psi_{\mathcal{E}}(\,\cdot\,) := \mathbf{R}p_*(\mathcal{E} \otimes^{\mathbf{L}} q^*(\,\cdot\,)).$$

One checks readily that the functor $\Phi_{\mathcal{E}}$ has both left and right adjoint functors.

Lemma 2.1.1. *The functor $\Phi_{\mathcal{E}}$ has left and right adjoint functors $\Phi_{\mathcal{E}}^*$ and $\Phi_{\mathcal{E}}^!$ respectively, defined by the formulae*

$$\Phi_{\mathcal{E}}^* \cong \Psi_{\mathcal{E}^\vee \otimes q^*\omega_Y[m]} \qquad \textit{and} \qquad \Phi_{\mathcal{E}}^! \cong \Psi_{\mathcal{E}^\vee \otimes p^*\omega_X[n]}. \tag{8}$$

Here ω_X and ω_Y are the canonical sheaves on X and Y respectively, and $\mathcal{E}^\vee$ is a convenient notation for $\mathbf{R}\underline{\mathcal{H}om}(\mathcal{E}, \mathcal{O}_{X\times Y})$.

Proof. We give the proof for the left adjoint functor. It comes from the following sequence of isomorphisms:

$$\begin{aligned}
\operatorname{Hom}(A, \mathbf{R}q_*(\mathcal{E} \otimes^{\mathbf{L}} p^*B)) &\cong \operatorname{Hom}(q^*A, \mathcal{E} \otimes^{\mathbf{L}} p^*B) \\
&\cong \operatorname{Hom}(p^*B, \mathcal{E}^\vee \otimes^{\mathbf{L}} q^*A \otimes \omega_{X\times Y}[n+m])^* \\
&\cong \operatorname{Hom}(B, \mathbf{R}p_*(\mathcal{E}^\vee \otimes^{\mathbf{L}} q^*(A \otimes \omega_Y[m])) \otimes \omega_X[n])^* \\
&\cong \operatorname{Hom}(\mathbf{R}p_*(\mathcal{E}^\vee \otimes^{\mathbf{L}} q^*(A \otimes \omega_Y[m])), B).
\end{aligned}$$

Here we have used the adjunction between direct and inverse image functors, Serre duality (6) (twice), and also formula (5).

We note that, of course, any diagram of the form

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

and any object $\mathcal{E} \in \mathbf{D}^b(Z)$ can be assigned a functor from the derived category of coherent sheaves on X to the derived category of coherent sheaves on Y, given by a formula similar to (7). However, any functor of this kind is isomorphic to a functor of the form (7), with the object $\mathbf{R}(p,q)_*\mathcal{E}$ on $X\times Y$, where (p,q) is the canonical morphism from Z to the direct product $X\times Y$.

Now let X, Y and Z be three smooth complete varieties and $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ objects of the derived categories $\mathbf{D}^b(X\times Y)$, $\mathbf{D}^b(Y\times Z)$ and $\mathbf{D}^b(X\times Z)$ respectively. Consider the following diagram of projections:

$$\begin{tikzcd}
 & X\times Y\times Z \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
X\times Y \arrow[d, "\pi^1_{12}"'] \arrow[dr, "\pi^2_{12}"] & X\times Z \arrow[dl, "\pi^1_{13}"'] \arrow[dr, "\pi^3_{13}"] & Y\times Z \arrow[dl, "\pi^2_{23}"'] \arrow[d, "\pi^3_{23}"] \\
X & Y & Z
\end{tikzcd}.$$

The objects $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ define three functors,

$$\Phi_{\mathcal{E}}\colon \mathbf{D}^b(X)\longrightarrow \mathbf{D}^b(Y),\quad \Phi_{\mathcal{F}}\colon \mathbf{D}^b(Y)\longrightarrow \mathbf{D}^b(Z),\quad \Phi_{\mathcal{G}}\colon \mathbf{D}^b(X)\longrightarrow \mathbf{D}^b(Z),$$

given by formula (7), that is,

$$\Phi_{\mathcal{E}}:=\mathbf{R}\pi^2_{12*}(\mathcal{E}\otimes^{\mathbf{L}}{\pi^1_{12}}^*(\,\cdot\,)),\qquad \Phi_{\mathcal{F}}:=\mathbf{R}\pi^3_{23*}(\mathcal{F}\otimes^{\mathbf{L}}{\pi^2_{23}}^*(\,\cdot\,))$$
$$\text{and}\qquad \Phi_{\mathcal{G}}:=\mathbf{R}\pi^3_{13*}(\mathcal{G}\otimes^{\mathbf{L}}{\pi^1_{13}}^*(\,\cdot\,)).$$

We consider the object $p_{12}^*\mathcal{E}\otimes^{\mathbf{L}}p_{23}^*\mathcal{F}\in\mathbf{D}^b(X\times Y\times Z)$, which we always denote by $\mathcal{E}\underset{Y}{\boxtimes}\mathcal{F}$ in what follows. The following assertion gives the composition rule for the exact functors between derived categories represented by objects on the product.

Proposition 2.1.2. *The composite of functors $\Phi_{\mathcal{F}}\circ\Phi_{\mathcal{E}}$ is isomorphic to the functor $\Phi_{\mathcal{G}}$ represented by*

$$\mathcal{G}=\mathbf{R}p_{13*}\Bigl(\mathcal{E}\underset{Y}{\boxtimes}\mathcal{F}\Bigr). \tag{9}$$

The proof is a direct verification.

Thus, to each smooth complete algebraic variety we assign its derived category of coherent sheaves, and to every object $\mathcal{E}\in\mathbf{D}^b(X\times Y)$ on the product of two varieties we assign an exact functor $\Phi_{\mathcal{E}}$ from the triangulated category $\mathbf{D}^b(X)$ to the triangulated category $\mathbf{D}^b(Y)$, with the composition law give just described.

The following problems are fundamental to understanding this correspondence:

1) When are the derived categories of coherent sheaves on two different smooth complete algebraic varieties equivalent as triangulated categories?
2) What is the group of exact auto-equivalences of the derived category of coherent sheaves for a given variety X? (By this we mean the group of isomorphism classes of exact auto-equivalences.)
3) Is every exact functor between derived categories of coherent sheaves represented by an object on the product, that is, of the form (7)?

Some results in this direction are already known. For example, one can give definitive answers to the first two questions when the variety has ample canonical or anticanonical sheaf.

Theorem 2.1.3 [8]. *Let X be a smooth projective variety whose canonical* (*or anticanonical*) *sheaf is ample. Suppose that the category $\mathbf{D}^b(X)$ is equivalent as a triangulated category to the derived category $\mathbf{D}^b(X')$ for some smooth algebraic variety X'. Then X' is isomorphic to X.*

The proof of this theorem given in [8] is constructive, and gives a method for recovering a variety from its derived category of coherent sheaves. Moreover, in the assumptions of the theorem one can assume that the derived categories are equivalent only as graded categories rather than as triangulated categories (see [8]).

In this situation one can also describe the group of exact auto-equivalences.

Theorem 2.1.4 [8]. *Let X be a smooth projective variety whose canonical* (*or anticanonical*) *sheaf is ample. Then the group of isomorphism classes of exact auto-equivalences of the category $\mathbf{D}^b(X)$ is generated by automorphisms of the variety, twists by line bundles, and shifts in the derived category.*

For any variety X the group $\operatorname{Auteq}\mathbf{D}^b(X)$ of exact auto-equivalences always contains the subgroup $G(X)$ which is the semidirect product of the normal subgroup $G_1 = \operatorname{Pic}(X)\oplus\mathbb{Z}$ and the subgroup $G_2 = \operatorname{Aut} X$ acting naturally on G_1. Under this inclusion $G(X) \subset \operatorname{Auteq}\mathbf{D}^b(X)$, the generator of $\mathbb{Z}$ goes to the shift functor [1], a line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ goes to the functor $\otimes\,\mathcal{L}$, and an automorphism $f\colon X \to X$ induces the auto-equivalence $\mathbf{R}f_*$. We proved in [8] that, under the assumption of Theorem 2.1.4, the group $\operatorname{Auteq}\mathbf{D}^b(X)$ of exact auto-equivalences equals $G(X)$; that is, in this case

$$\operatorname{Auteq}\mathbf{D}^b(X) \cong \operatorname{Aut} X \ltimes (\operatorname{Pic}(X) \oplus \mathbb{Z}).$$

To study the problem of when two varieties have equivalent derived categories of coherent sheaves and to describe their groups of auto-equivalences, it is desirable to have explicit formulae for all exact functors. There is a conjecture that they are always representable by objects on the product, that is, are of the form (7). In the next chapter we give the proof of this conjecture for fully faithful functors and, in particular, for equivalences. The whole of the next chapter is taken up with the proof of this result. This will thus allow us to consider only functors of the form (7) in studying equivalences between derived categories of coherent sheaves on smooth projective varieties. Another problem that arises in connection with the solution of these questions is the need for a criterion to determine whether a given functor is an equivalence. To prove that a functor F is an equivalence, it is enough to show that both F and its right (or left) adjoint are fully faithful functors (see Definition 1.1.7).

There is a method to decide whether a functor $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ is fully faithful.

Theorem 2.1.5 [7]. *Let M and X be smooth projective varieties over an algebraically closed field of characteristic* 0 *and $\mathcal{E} \in \mathbf{D}^b(M \times X)$. In this case the functor $\Phi_{\mathcal{E}}$ is fully faithful if and only if the following orthogonality conditions hold*:

1) $\operatorname{Hom}^i_X(\Phi_{\mathcal{E}}(\mathcal{O}_{t_1}), \Phi_{\mathcal{E}}(\mathcal{O}_{t_2})) = 0$ *for all i and all $t_1 \neq t_2$;*

2) $\operatorname{Hom}^0_X(\Phi_{\mathcal{E}}(\mathcal{O}_t), \Phi_{\mathcal{E}}(\mathcal{O}_t)) = k$ *and* $\operatorname{Hom}^i_X(\Phi_{\mathcal{E}}(\mathcal{O}_t), \Phi_{\mathcal{E}}(\mathcal{O}_t)) = 0$ *for any* $i \notin \{0, \dim M\}$.

Here t, t_1 *and* t_2 *are points of* M, *and* $\mathcal{O}_{t_i}$ *the corresponding skyscraper sheaves.*

The assumptions of this theorem are in general rather difficult to verify; however, the criterion works rather well when the object $\mathcal{E}$ on the product is a vector bundle. Consider four smooth complete algebraic varieties X_1, X_2, Y_1 and Y_2. We take two objects $\mathcal{E}_1$ and $\mathcal{E}_2$ belonging to the categories $\mathbf{D}^b(X_1 \times Y_1)$ and $\mathbf{D}^b(X_2 \times Y_2)$ respectively, and consider the object

$$\mathcal{E}_1 \boxtimes \mathcal{E}_2 \in \mathbf{D}^b((X_1 \times X_2) \times (Y_1 \times Y_2)),$$

which is $p_{13}^*(\mathcal{E}_1) \otimes^{\mathbf{L}} p_{24}^*(\mathcal{E}_2)$ by definition. As above (see (7)), the objects $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ define functors

$$\Phi_{\mathcal{E}_1} \colon \mathbf{D}^b(X_1) \longrightarrow \mathbf{D}^b(Y_1), \qquad \Phi_{\mathcal{E}_2} \colon \mathbf{D}^b(X_2) \longrightarrow \mathbf{D}^b(Y_2),$$
$$\text{and} \quad \Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \colon \mathbf{D}^b(X_1 \times X_2) \longrightarrow \mathbf{D}^b(Y_1 \times Y_2).$$

We consider an object $\mathcal{G} \in \mathbf{D}^b(X_1 \times X_2)$ and write $\mathcal{H}$ to denote the object $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}(\mathcal{G}) \in \mathbf{D}^b(Y_1 \times Y_2)$. To each of these two objects one can assign functors by the rule (7):

$$\Phi_{\mathcal{G}} \colon \mathbf{D}^b(X_1) \longrightarrow \mathbf{D}^b(X_2) \quad \text{and} \quad \Phi_{\mathcal{H}} \colon \mathbf{D}^b(Y_1) \longrightarrow \mathbf{D}^b(Y_2).$$

Proposition 2.1.6. *In the above notation there is an isomorphism of functors*

$$\Phi_{\mathcal{H}} \cong \Phi_{\mathcal{E}_2} \circ \Phi_{\mathcal{G}} \circ \Psi_{\mathcal{E}_1}.$$

The proof follows at once from Proposition 2.1.2.

Now if Z_1 and Z_2 are two other smooth complete varieties and $\mathcal{F}_1$ and $\mathcal{F}_2$ objects of $\mathbf{D}^b(Y_1 \times Z_1)$ and $\mathbf{D}^b(Y_2 \times Z_2)$ respectively, then there are also functors $\Phi_{\mathcal{F}_1}$, $\Phi_{\mathcal{F}_2}$, and $\Phi_{\mathcal{F}_1 \boxtimes \mathcal{F}_2}$. By the rule (9) we can find objects $\mathcal{G}_1$ and $\mathcal{G}_2$ belonging to $\mathbf{D}^b(X_1 \times Z_1)$ and $\mathbf{D}^b(X_2 \times Z_2)$ such that

$$\Phi_{\mathcal{G}_1} \cong \Phi_{\mathcal{F}_1} \circ \Phi_{\mathcal{E}_1} \quad \text{and} \quad \Phi_{\mathcal{G}_2} \cong \Phi_{\mathcal{F}_2} \circ \Phi_{\mathcal{E}_2}.$$

A direct check shows that there is a natural relation

$$\Phi_{\mathcal{F}_1 \boxtimes \mathcal{F}_2} \circ \Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \cong \Phi_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}. \tag{10}$$

Using this, one readily proves the following assertion.

Proposition 2.1.7. *Under the above conditions, assume that* $\Phi_{\mathcal{E}_1}$ *and* $\Phi_{\mathcal{E}_2}$ *are fully faithful* (*respectively, are equivalences*). *Then the functor*

$$\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \colon \mathbf{D}^b(X_1 \times X_2) \longrightarrow \mathbf{D}^b(Y_1 \times Y_2)$$

is also fully faithful (*respectively, an equivalence of categories*).

Proof. If F has a left adjoint F^* (say), then it is fully faithful if and only if the composite F^*F is isomorphic to the identity functor. The functors $\Phi_{\mathcal{E}_i}$ have left adjoints $\Phi^*_{\mathcal{E}_i}$ defined by (8). Since they are fully faithful, it follows that the composites $\Phi^*_{\mathcal{E}_i} \circ \Phi_{\mathcal{E}_i}$ are isomorphic to the identity functors, which are representable by the structure sheaves of the diagonals $\Delta_i \in X_i \times X_i$. One sees readily that the sheaf $\mathcal{O}_{\Delta_1} \boxtimes \mathcal{O}_{\Delta_2}$ is isomorphic to the structure sheaf of the diagonal $\mathcal{O}_\Delta$, where Δ is the diagonal in $(X_1 \times X_2) \times (X_1 \times X_2)$. Using formula (10), we see that the composite $\Phi^*_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \circ \Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ is represented by the structure sheaf of the diagonal Δ, and is thus isomorphic to the identity functor. Thus, $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ is fully faithful. The assertion concerning equivalences can be proved in a similar way.

Now assume that the functor $\Phi_{\mathcal{E}} \colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ is an equivalence and that $\mathcal{F} \in \mathbf{D}^b(X \times Y)$ is an object such that $\Psi_{\mathcal{F}} \cong \Phi_{\mathcal{E}}^{-1}$. By (8), we have isomorphisms

$$\mathcal{F} \cong \mathcal{E}^\vee \otimes p^*\omega_X[n] \cong \mathcal{E}^\vee \otimes q^*\omega_Y[m],$$

which imply at once that the dimensions n and m of the varieties X and Y are equal.

Consider the functor

$$\Phi_{\mathcal{F} \boxtimes \mathcal{E}} \colon \mathbf{D}^b(X \times X) \longrightarrow \mathbf{D}^b(Y \times Y) \tag{11}$$

and denote it by $Ad_{\mathcal{E}}$. By Proposition 2.1.7 it is also an equivalence. Moreover, by Proposition 2.1.6, for any object $\mathcal{G} \in \mathbf{D}^b(X \times X)$ there is an isomorphism of functors

$$\Phi_{Ad_{\mathcal{E}}(\mathcal{G})} \cong \Phi_{\mathcal{E}} \circ \Phi_{\mathcal{G}} \circ \Phi_{\mathcal{E}}^{-1}. \tag{12}$$

Consider the special case when $\mathcal{G}$ is the structure sheaf of the diagonal $\mathcal{O}_{\Delta_X}$, representing the identity functor. Thus, applying (12), we see that the functor $Ad_{\mathcal{E}}$ takes the structure sheaf of the diagonal $\mathcal{O}_{\Delta_X}$ to the structure sheaf of the diagonal $\mathcal{O}_{\Delta_Y}$.

Consider the more general situation. We denote by i_X and i_Y the embeddings of the diagonals in $X \times X$ and $Y \times Y$ respectively. We apply the functor $Ad_{\mathcal{E}}$ to the object $i_{X*}\omega_X^k$, where ω_X is the canonical sheaf of X (as above). The object $i_{X*}\omega_X^k$ represents the functor $S^k[-nk]$, where S is the Serre functor of $\mathbf{D}^b(X)$. Since every equivalence commutes with Serre functors by Lemma 1.1.4, we see that

$$Ad_{\mathcal{E}}(i_{X*}\omega_X^k) \cong i_{Y*}\omega_Y^k. \tag{13}$$

Now for every variety X we define the bigraded algebra

$$\mathrm{HA}(X) = \bigoplus_{i,k} \mathrm{HA}_{i,k}(X) := \bigoplus_{i,k} \mathrm{Ext}^i_{X \times X}(\mathcal{O}_{\Delta_X}, i_{X*}\omega_X^k).$$

The algebra structure is defined here by composition of Ext's, bearing in mind the canonical identification

$$\mathrm{Ext}^i_{X \times X}(\mathcal{O}_{\Delta_X}, i_{X*}\omega_X^k) \cong \mathrm{Ext}^i_{X \times X}(i_{X*}\omega^m, i_{X*}\omega_X^{m+k}).$$

To prove the next theorem, we need to apply the main result of Chapter 3, which states that every equivalence is represented by an object on the product.

Theorem 2.1.8. *Let X and Y be smooth projective varieties whose derived categories of coherent sheaves are equivalent as triangulated categories. Then the bigraded algebras* $\mathrm{HA}(X)$ *and* $\mathrm{HA}(Y)$ *are isomorphic.*

Proof. By Theorem 3.2.2, every equivalence $F\colon \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ is represented by some object on the product, and is thus isomorphic to a functor of the form $\Phi_{\mathcal{E}}$ for some $\mathcal{E} \in \mathbf{D}^b(X \times Y)$. Each equivalence of this kind defines an equivalence

$$Ad_{\mathcal{E}}\colon \mathbf{D}^b(X \times X) \longrightarrow \mathbf{D}^b(Y \times Y),$$

taking $i_{X*}\omega_X^k$ to $i_{Y*}\omega_Y^k$. The equivalence $Ad_{\mathcal{E}}$ induces isomorphisms

$$\mathrm{Ext}^i_{X\times X}(\mathcal{O}_{\Delta_X}, i_{X*}\omega_X^k) \cong \mathrm{Ext}^i_{Y\times Y}(\mathcal{O}_{\Delta_Y}, i_{Y*}\omega_Y^k),$$

and hence an isomorphism of the bigraded algebras $\mathrm{HA}(X)$ and $\mathrm{HA}(Y)$.

We note that one can obtain both the canonical and anticanonical algebras of X from the bigraded algebra $\mathrm{HA}(X)$. Indeed,

$$\bigoplus_{k\geqslant 0} \mathrm{H}^0(X, \omega_X^k) = \bigoplus_{k\geqslant 0} \mathrm{HA}_{0,k}(X) \quad \text{and} \quad \bigoplus_{k\leqslant 0} \mathrm{H}^0(X, \omega_X^k) = \bigoplus_{k\leqslant 0} \mathrm{HA}_{0,k}(X).$$

Thus, Theorem 2.1.8 implies the following corollary.

Corollary 2.1.9. *If the derived categories of coherent sheaves on two smooth projective varieties X and Y are equivalent, then the canonical* (*and anticanonical*) *algebras of X and Y are isomorphic.*

The statement of this corollary is very close to Theorem 2.1.3. However, we should note that the proof of Theorem 2.1.3 given in [8] does not depend on the main result of the next chapter and, moreover, is constructive. Also note that in Theorem 2.1.3, we do not assume that the canonical (or anticanonical) sheaf of the second variety X' is ample; this follows from the proof of the theorem.

We can also describe all the other spaces $\mathrm{HA}_{i,k}(X)$. In [40] it is proved that the spectral sequence that computes

$$\mathrm{HA}_{i,k}(X) = \mathrm{Ext}^i(\mathcal{O}_{\Delta_X}, i_{X*}\omega_X)$$

in terms of the cohomology of $\mathcal{O}_{\Delta_X}$ restricted to the diagonal degenerates at the term E_2. In particular, there are isomorphisms

$$\mathrm{HA}_{i,k}(X) \cong \bigoplus_{p+q=i} \mathrm{H}^p(X, \bigwedge^q T_X \otimes \omega_X^k), \tag{14}$$

where T_X is the tangent bundle to X. Moreover, this isomorphism turns into an algebra isomorphism, that is,

$$\mathrm{HA}(X) \cong \bigoplus_{i,k} \bigoplus_{p+q=i} \mathrm{H}^p(X, \bigwedge^q T_X \otimes \omega_X^k)$$

as bigraded algebras. This relation and Theorem 2.1.8 imply the following corollary.

Corollary 2.1.10. *If the derived categories of coherent sheaves on two smooth projective varieties X and Y are equivalent, then there are vector space isomorphisms*

$$\bigoplus_{p+q=i} \mathrm{H}^p(X, \bigwedge^q T_X \otimes \omega^k) \cong \bigoplus_{p+q=i} \mathrm{H}^p(Y, \bigwedge^q T_Y \otimes \omega^k). \tag{15}$$

In particular, we obtain isomorphisms between the verticals of the Hodge diamond:

$$\bigoplus_{p-q=i} \mathrm{H}^p(X, \Omega_X^q) \cong \bigoplus_{p-q=i} \mathrm{H}^p(Y, \Omega_Y^q). \tag{16}$$

Proof. The isomorphisms (15) follow at once from Theorem 2.1.8 and the equality (14). The isomorphisms (16) are the special case $k = 1$ of (15).

The isomorphisms between the verticals of the Hodge diamond can also be obtained in another way. Suppose that the ground field k is $\mathbb{C}$.

For any element $\xi \in \mathrm{H}^*(X \times Y, \mathbb{Q})$ we can define linear maps

$$v_\xi \colon \mathrm{H}^*(X, \mathbb{Q}) \longrightarrow \mathrm{H}^*(Y, \mathbb{Q}) \quad \text{and} \quad w_\xi \colon \mathrm{H}^*(Y, \mathbb{Q}) \longrightarrow \mathrm{H}^*(X, \mathbb{Q})$$

by the formulae

$$v_\xi(-) = q_*(\xi \cdot p^*(-)) \quad \text{and} \quad w_\xi(-) = p_*(\xi \cdot q^*(-)). \tag{17}$$

For these maps one can write out a composition formula similar to formula (9) for the composition of functors. Let X, Y, and Z be three smooth complete varieties and ξ and η elements of $\mathrm{H}^*(X \times Y, \mathbb{Q})$ and $\mathrm{H}^*(Y \times Z, \mathbb{Q})$ respectively. Then the composite $v_\eta \circ v_\xi$ coincides with the map v_ζ, where $\zeta \in \mathrm{H}^*(X \times Z, \mathbb{Q})$ is given by the formula

$$\zeta = p_{XZ*}\big(p_{YZ}^*(\eta) \cup p_{XY}^*(\xi)\big).$$

To any functor of the form $\Phi_{\mathcal{E}} \colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ we can assign a linear map $\varphi_{\mathcal{E}} \colon \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q})$. For this, define an element $\varepsilon \in \mathrm{H}(X \times Y, \mathbb{Q})$ by the rule

$$\varepsilon = p^* \sqrt{\mathrm{td}_X} \cdot \mathrm{ch}(\mathcal{E}) \cdot q^* \sqrt{\mathrm{td}_Y}, \tag{18}$$

where td_X and td_Y are the Todd classes of X and Y respectively, and $\mathrm{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$. We define the maps

$$\begin{aligned} \varphi_{\mathcal{E}}(-) &:= v_\varepsilon(-) = q_*(\varepsilon \cdot p^*(-)), \\ \psi_{\mathcal{E}}(-) &:= w_\varepsilon(-) = p_*(\varepsilon \cdot q^*(-)). \end{aligned} \tag{19}$$

The next proposition follows immediately from the Grothendieck form of the Riemann–Roch theorem.

Proposition 2.1.11. *Suppose that $\Phi_{\mathcal{E}} \colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Z)$ is a composite $\Phi_{\mathcal{G}} \circ \Phi_{\mathcal{F}}$ for some*

$$\Phi_{\mathcal{F}} \colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y), \qquad \Phi_{\mathcal{G}} \colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(Z).$$

Then $\varphi_{\mathcal{E}} = \varphi_{\mathcal{G}} \circ \varphi_{\mathcal{F}}$.

This implies at once the following corollary.

Corollary 2.1.12. *If the functor* $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Z)$ *is an equivalence, then the map* $\varphi_{\mathcal{E}}\colon \mathrm{H}^*(X,\mathbb{Q}) \to \mathrm{H}^*(Y,\mathbb{Q})$ *is an isomorphism, and its complexification induces the isomorphisms* (16) *between the verticals of the Hodge diamond.*

Proof. It follows from Proposition 2.1.11 that the quasi-inverse functor to $\Phi_{\mathcal{E}}$ induces the inverse map of $\varphi_{\mathcal{E}}$. Moreover, since the element $\varepsilon \in \mathrm{H}^*(X \times Y, \mathbb{Q})$ corresponds to an algebraic cycle by (18), one checks readily that the complexification of $\varphi_{\mathcal{E}}$ preserves the verticals of the Hodge diamond.

In conclusion we observe also that every functor $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ induces a map $\overline{\Phi_{\mathcal{E}}}\colon K(X) \longrightarrow K(Y)$ between the Grothendieck groups $K(X)$ and $K(Y)$ of the categories $\mathbf{D}^b(X)$ and $\mathbf{D}^b(Y)$. Consider the map

$$\mathrm{ch}\cdot\sqrt{\mathrm{td}_X}\colon K(X) \longrightarrow \mathrm{H}^*(X,\mathbb{Q})$$

that takes an element of $K(X)$ to its Chern character times the square root of the Todd class. Using the Riemann–Roch theorem, one can show that the diagram

$$\begin{tikzcd} K(X) \arrow[r, "\overline{\Phi}_{\mathcal{E}}"] \arrow[d, "\mathrm{ch}\cdot\sqrt{\mathrm{td}_X}"'] & K(Y) \arrow[d, "\mathrm{ch}\cdot\sqrt{\mathrm{td}_Y}"] \\ \mathrm{H}^*(X,\mathbb{Q}) \arrow[r, "\varphi_{\mathcal{E}}"] & \mathrm{H}^*(Y,\mathbb{Q}) \end{tikzcd}$$

is commutative.

2.2. Examples of equivalences: flopping birational transformations. In this section we present an entire class of examples of pairs of smooth varieties for which the derived categories of coherent sheaves are equivalent. Examples of such varieties were of course already known (the first example is an Abelian variety and its dual, considered by Mukai [29]). The principal difference with the examples treated in this section is that here we obtain pairs of (in general non-isomorphic) varieties related by a birational transformation which is a flop. It also follows from our examples that the conditions on the (anti-)canonical sheaf in Theorem 2.1.3 cannot be weakened.

To start this section we recall the definitions of admissible subcategories and semi-orthogonal decompositions (see [5], [6]).

Definition 2.2.1. Let $\mathcal{B}$ be a full additive subcategory of an additive category $\mathcal{A}$. By the *right orthogonal* to $\mathcal{B}$ in $\mathcal{A}$ we mean the full subcategory $\mathcal{B}^\perp \subset \mathcal{A}$ consisting of all objects C such that $\mathrm{Hom}(B,C)=0$ for any $B \in \mathcal{B}$. The left orthogonal ${}^\perp\mathcal{B}$ is defined dually.

Note that, if $\mathcal{B}$ is a triangulated subcategory in a triangulated category $\mathcal{A}$, then ${}^\perp\mathcal{B}$ and $\mathcal{B}^\perp$ are also triangulated subcategories.

Definition 2.2.2. Let $I\colon \mathcal{N} \longrightarrow \mathcal{D}$ be an embedding of a full triangulated subcategory in a triangulated category $\mathcal{D}$. We say that $\mathcal{N}$ is *right admissible* (or left admissible) if the embedding functor I has a right adjoint $P\colon \mathcal{D} \longrightarrow \mathcal{N}$ (respectively left adjoint).

For a subcategory $\mathcal{N}$, the property of being right admissible (or left admissible) is equivalent to the following property, stated in terms of orthogonals: for any object $X \in \mathcal{D}$, there is a distinguished triangle $N \to X \to M$ with $N \in \mathcal{N}$ and $M \in \mathcal{N}^{\perp}$ (respectively, $M \to X \to N$ with $M \in {}^{\perp}\mathcal{N}$ and $N \in \mathcal{N}$). We say simply that a subcategory is *admissible* if it is both right and left admissible.

If $\mathcal{N} \subset \mathcal{D}$ is an admissible subcategory, we say that $\mathcal{D}$ admits a *semi-orthogonal decomposition* of the form $\langle \mathcal{N}^{\perp}, \mathcal{N} \rangle$ or $\langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle$. This process of decomposition can sometimes be extended further, decomposing the subcategory $\mathcal{N}$ or its orthogonals. We give the general definition of semi-orthogonal decomposition.

Definition 2.2.3. A sequence $(\mathcal{N}_0, \dots, \mathcal{N}_n)$ of admissible subcategories of a triangulated category $\mathcal{D}$ is said to be *semi-orthogonal* if $\mathcal{N}_j \subset \mathcal{N}_i^{\perp}$ for all $0 \leqslant j < i \leqslant n$. We say that a semi-orthogonal sequence is *complete* if it generates the category $\mathcal{D}$, that is, the minimal triangulated subcategory in $\mathcal{D}$ containing all the $\mathcal{N}_i$ coincides with $\mathcal{D}$. In this case this sequence is called a *semi-orthogonal decomposition* of the category $\mathcal{D}$ and is represented as follows:

$$\mathcal{D} = \langle \mathcal{N}_0, \dots, \mathcal{N}_n \rangle.$$

The simplest example of a semi-orthogonal decomposition is when $\mathcal{D}$ has a complete exceptional family.

Definition 2.2.4. We say that an object E in a triangulated category $\mathcal{D}$ is *exceptional* if $\operatorname{Hom}^i(E,E) = 0$ for $i \neq 0$ and $\operatorname{Hom}(E,E) = k$. An ordered family $(E_0, \dots, E_n)$ of exceptional objects is called a *complete exceptional family* if it generates $\mathcal{D}$ and $\operatorname{Hom}^{\cdot}(E_i, E_j) = 0$ for $i > j$.

The best-known example of a complete exceptional family is provided by projective space.

Example 2.2.5 [2]. On projective space $\mathbb{P}^N$, given any $i \in \mathbb{Z}$, the family

$$(\mathcal{O}(i), \dots, \mathcal{O}(i+N))$$

is exceptional and complete. In particular, we obtain that the derived category of coherent sheaves $\mathbf{D}^b(\mathbb{P}^N)$ is equivalent to the derived category of finite-dimensional modules over the finite-dimensional algebra $\operatorname{End}\bigl(\bigoplus_{j=0}^N \mathcal{O}(j)\bigr)$ of endomorphisms of the exceptional family.

Similar decompositions exist for some other varieties, for example, for quadrics and flag varieties [20]–[22].

We now present some facts we need on blowups and the behaviour of the derived categories of coherent sheaves under blowups. All these results are contained in the paper [34] (see also [7]). Let X be a smooth complete algebraic variety and $Y \subset X$ a smoothly embedded closed subvariety of codimension r. We denote by $\widetilde{X}$ the blowup of X with centre along Y. The variety $\widetilde{X}$ is also smooth, and there is a commutative diagram:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{i} & X \end{array}$$

with i and j closed embeddings and $p\colon \widetilde{Y} \to Y$ the projective bundle of the exceptional divisor of $\widetilde{Y}$ over the centre Y of the blowup; in particular, p is a flat morphism. We recall that $\widetilde{Y} \cong \mathbb{P}(N_{X/Y})$, where $N_{X/Y}$ is the normal bundle to Y in X. We denote by $\mathcal{O}_{\widetilde{Y}}(1)$ the canonical relatively ample line bundle on $\widetilde{Y} = \mathbb{P}(N_{X/Y})$. It is well known that this bundle is isomorphic to the restriction of the line bundle $\mathcal{O}(-\widetilde{Y})$ to $\widetilde{Y}$.

Proposition 2.2.6 [34]. *The derived inverse image functors*

$$\mathbf{L}\pi^*\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(\widetilde{X}) \quad \textit{and} \quad p^*\colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(\widetilde{Y})$$

are fully faithful.

Proof. The projection formula (3) gives an isomorphism

$$\mathrm{Hom}(\mathbf{L}\pi^*F, \mathbf{L}\pi^*G) \cong \mathrm{Hom}(F, \mathbf{R}\pi_*\mathbf{L}\pi^*G) \cong \mathrm{Hom}(F, \mathbf{R}\pi_*\mathcal{O}_{\widetilde{Y}} \otimes^{\mathbf{L}} G)$$

for $F, G \in \mathbf{D}^b(X)$. Similarly for p^*. Combining these with $\mathbf{R}\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ and $\mathbf{R}p_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ gives the proof.

Proposition 2.2.7 ([34], [7]). *For any invertible sheaf $\mathcal{L}$ on $\widetilde{Y}$, the functor*

$$\mathbf{R}j_*(\mathcal{L} \otimes p^*(\,\cdot\,))\colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(\widetilde{X})$$

is fully faithful.

Proof. To prove that the functor is fully faithful, it is enough to show that conditions 1)–2) of Theorem 2.1.5 hold. For any closed point $y \in Y$ the image $\Phi(\mathcal{O}_y)$ is the structure sheaf of the corresponding fibre of the map p, viewed as a sheaf on $\widetilde{X}$. Since the fibres over distinct points are disjoint, the orthogonality condition 1) of Theorem 2.1.5 is satisfied.

Consider the structure sheaf $\mathcal{O}_F$ of some p-fibre $F \subset \widetilde{Y}$. We have an isomorphism

$$\mathrm{Hom}^i(j_*\mathcal{O}_F, j_*\mathcal{O}_F) \cong \mathrm{Hom}^i(\mathbf{L}j^*j_*\mathcal{O}_F, \mathcal{O}_F).$$

In the derived category $\mathbf{D}^b(\widetilde{Y})$ we have a distinguished triangle

$$\mathcal{O}_F \otimes \mathcal{O}_{\widetilde{Y}}(1)[1] \longrightarrow \mathbf{L}j^*j_*\mathcal{O}_F \longrightarrow \mathcal{O}_F,$$

where $\mathcal{O}_{\widetilde{Y}}(1)$ is the relatively ample line bundle on $\widetilde{Y}$, isomorphic to $\mathcal{O}(-\widetilde{Y})_{|\widetilde{Y}}$. The fibre of F is a projective space, and the restriction of $\mathcal{O}_{\widetilde{Y}}(1)$ to F is isomorphic to $\mathcal{O}(1)$. Thus,

$$\mathrm{Hom}^i(\mathcal{O}_F \otimes \mathcal{O}_{\widetilde{Y}}(1), \mathcal{O}_F) = 0$$

for all i. Hence,

$$\mathrm{Hom}^i(j_*\mathcal{O}_F, j_*\mathcal{O}_F) \cong \mathrm{Hom}^i(\mathcal{O}_F, \mathcal{O}_F).$$

Therefore, condition 2) of Theorem 2.1.5 also holds.

We write $D(X)$ for the full triangulated subcategory of $\mathbf{D}^b(\widetilde{X})$ which is the image of $\mathbf{D}^b(X)$ under $\mathbf{L}\pi^*$, and $D(Y)_k$ for the full subcategory in $\mathbf{D}^b(\widetilde{X})$ which is the image of $\mathbf{D}^b(Y)$ under $\mathbf{R}j_*(\mathcal{O}_{\widetilde{Y}}(k) \otimes p^*(\,\cdot\,))$, where $\mathcal{O}_{\widetilde{Y}}(k) = \mathcal{O}_{\widetilde{Y}}(1)^{\otimes k}$ and $\mathcal{O}_{\widetilde{Y}}(1)$ is the canonical relatively ample line bundle on $\widetilde{Y} = \mathbb{P}(N_{X/Y})$. It follows from Propositions 2.2.6 and 2.2.7 that $D(X) \cong \mathbf{D}^b(X)$ and $D(Y)_k \cong \mathbf{D}^b(Y)$.

Theorem 2.2.8 [34]. *The sequence of admissible subcategories*

$$\big\langle D(Y)_{-r+1}, \dots, D(Y)_{-1}, D(X)\big\rangle$$

is semi-orthogonal, and it gives a semi-orthogonal decomposition of the category $\mathbf{D}^b(\widetilde{X})$.

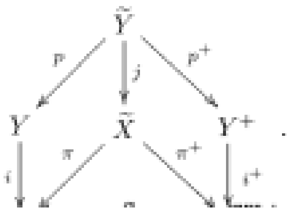
This theorem provides a description of the derived category of the blowup $\widetilde{X}$ in terms of the blown up variety X and the centre of the blowup Y. Using this description of the derived category of a blowup, we now study the behaviour of the derived category under the simplest flipping and flopping transformations. Consider the following example.

Let Y be a smoothly embedded closed subvariety in a smooth complete algebraic variety X such that $Y \cong \mathbb{P}^k$ with the normal bundle $N_{X/Y} \cong \mathcal{O}_Y(-1)^{\oplus(l+1)}$. We suppose that $l \leqslant k$.

Write $\widetilde{X}$ for the blowup of X along the centre Y. In this case the exceptional divisor $\widetilde{Y}$ is isomorphic to the product of projective spaces $\mathbb{P}^k \times \mathbb{P}^l$. Moreover, in this situation we have the following description of the normal sheaf to $\widetilde{Y}$ in $\widetilde{X}$:

$$N_{\widetilde{X}/\widetilde{Y}} = \mathcal{O}_{\widetilde{X}}(\widetilde{Y})_{|\widetilde{Y}} \cong \mathcal{O}(-1;-1),$$

where $\mathcal{O}(-1;-1) := p_1^*\mathcal{O}_{\mathbb{P}^k}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(-1)$. These facts allow us to assert that there is a blowdown of $\widetilde{X}$ under which $\widetilde{Y}$ projects to the second factor $\mathbb{P}^l$. This blowdown exists in the analytic category, and its result is a smooth variety X^+ which in general may not be algebraic. We assume that X^+ is algebraic. All the geometry described above is contained in the following diagram:



(20)

The birational map fl: $X \longrightarrow X^+$ is the simplest example of a flip or flop. It is a flip for $l < k$ and a flop for $l = k$. In what follows, we need a formula for the restriction of the canonical sheaf $\omega_{\widetilde{X}}$ to the divisor $\widetilde{Y}$. For the blowup of a smooth subvariety we obtain

$$\omega_{\widetilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(l\widetilde{Y}).$$

The adjunction formula gives

$$\omega_{X\,|Y} \cong \omega_Y \otimes \bigwedge^{l+1} N^*_{X/Y} \cong \mathcal{O}_Y(l-k).$$

Combining these facts together, we obtain the isomorphism

$$\omega_{\widetilde{X}\,|\widetilde{Y}} \cong (\pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(l\widetilde{Y}))_{|\widetilde{Y}} \cong p^*(\omega_{X\,|Y}) \otimes \mathcal{O}_{\widetilde{X}}(l\widetilde{Y})_{|\widetilde{Y}} \cong \mathcal{O}(-k;-l). \qquad (21)$$

The main theorem of this section relates the derived categories of coherent sheaves on X and X^+.

Theorem 2.2.9. *Let $\mathcal{L}$ be a line bundle on $\widetilde{X}$. In the above notation, the functor*

$$\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}(\,\cdot\,)\otimes\mathcal{L})\colon \mathbf{D}^b(X^+)\longrightarrow \mathbf{D}^b(X)$$

is fully faithful.

Proof. We first consider the restriction of $\mathcal{L}$ to $\widetilde{Y}$. Since $\widetilde{Y}=\mathbb{P}^k\times\mathbb{P}^l$, it follows that $\mathcal{L}_{|\widetilde{Y}}\cong\mathcal{O}(a;b)$ for some integers a and b.

We must show that for any pair $A,B\subset\mathbf{D}^b(X^+)$ the composite map

$$\begin{aligned}\operatorname{Hom}(A,B)&\xrightarrow{\sim}\operatorname{Hom}(\mathbf{L}\pi^{+*}A,\mathbf{L}\pi^{+*}B)\\&\longrightarrow\operatorname{Hom}(\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}A\otimes\mathcal{L}),\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}B\otimes\mathcal{L}))\end{aligned}\qquad(22)$$

is an isomorphism. Using adjunction of the functors, we obtain an isomorphism

$$\begin{aligned}&\operatorname{Hom}(\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}A\otimes\mathcal{L}),\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}B\otimes\mathcal{L}))\\&\qquad\cong\operatorname{Hom}(\mathbf{L}\pi^*\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}A\otimes\mathcal{L}),\mathbf{L}\pi^{+*}B\otimes\mathcal{L}).\end{aligned}$$

Consider the distinguished triangle

$$\mathbf{L}\pi^*\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}A\otimes\mathcal{L})\longrightarrow\mathbf{L}\pi^{+*}A\otimes\mathcal{L}\longrightarrow\overline{A}.\qquad(23)$$

Thus, to prove that the composite (22) is an isomorphism, it is necessary and sufficient to show that

$$\operatorname{Hom}(\overline{A},\mathbf{L}\pi^{+*}B\otimes\mathcal{L})=0.\qquad(24)$$

Since by Proposition 2.2.6 the composite $\mathbf{R}\pi_*\mathbf{L}\pi^*$ is isomorphic to the identity functor, by applying the functor $\mathbf{R}\pi_*$ to the distinguished triangle (23) we see that $\mathbf{R}\pi_*\overline{A}=0$. Thus, $\operatorname{Hom}(\mathbf{L}\pi^*C,\overline{A})=0$ for any object $C\in\mathbf{D}^b(X^+)$. Hence, $\overline{A}$ belongs to the subcategory $D(X)^\perp$.

Theorem 2.2.8 implies the semi-orthogonal decomposition

$$D(X)^\perp=\langle D(Y)_{-l},\dots,D(Y)_{-1}\rangle.$$

Since Y is a projective space, it follows from Example 2.2.5 that each $D(Y)_{-i}$ admits a complete exceptional family. Collecting these families, we obtain a complete exceptional family in $D(X)^\perp$. The following family will be convenient for our purposes:

$$\begin{aligned}D(X)^\perp=\Big\langle&\mathbf{R}j_*\mathcal{O}(a-k;-l), &&\dots\quad\dots &&\mathbf{R}j_*\mathcal{O}(a;-l),\\&\mathbf{R}j_*\mathcal{O}(a-k+1;-l+1), &&\dots\quad\dots &&\mathbf{R}j_*\mathcal{O}(a+1;-l+1),\\&\qquad\vdots && &&\qquad\vdots\\&\mathbf{R}j_*\mathcal{O}(a-k+l-1;-1), &&\dots\quad\dots &&\mathbf{R}j_*\mathcal{O}(a+l-1;-1)\Big\rangle.\end{aligned}$$

We can now regroup this exceptional sequence to obtain a semi-orthogonal decomposition of $D(X)^\perp$ of the form

$$D(X)^\perp = \langle \mathcal{B}, \mathcal{A} \rangle,$$

where $\mathcal{A}$ and $\mathcal{B}$ are the subcategories generated by $\mathbf{R}j_*\mathcal{O}(i;s)$ with $i \geqslant a$ and with $i < a$ respectively. For $1 \leqslant i \leqslant k$ and $1 \leqslant s \leqslant l$, the objects $\mathbf{R}j_*\mathcal{O}(a-i;-s)$ belong simultaneously to the subcategories $D(X)^\perp$ and $D(X^+)^\perp \otimes \mathcal{L}$. In particular, $\mathcal{B} \subset D(X)^\perp \cap (D(X^+)^\perp \otimes \mathcal{L})$. Applying Hom to the distinguished triangle (23), we obtain

$$\operatorname{Hom}(\overline{A}, \mathbf{R}j_*\mathcal{O}(a-i;-s)) = 0 \quad \text{for } 1 \leqslant i \leqslant k \text{ and } 1 \leqslant s \leqslant l.$$

Since $\overline{A} \in D(X)^\perp$ and $\overline{A}$ is orthogonal to the subcategory $\mathcal{B}$, it follows at once that $\overline{A} \in \mathcal{A}$. Now note that if the object $\mathbf{R}j_*\mathcal{O}(a+i;s)$ belongs to the subcategory $\mathcal{A}$, then i satisfies the inequalities $0 \leqslant i < l$. Taking account of the formula (21) for the canonical class $\omega_{\widetilde{X}|\widetilde{Y}} \cong \mathcal{O}(-k;-l)$ and of the condition $l \leqslant k$, we see that $\mathcal{A} \otimes \omega_{\widetilde{X}} \subset D(X^+)^\perp \otimes \mathcal{L}$. Hence, for any object $B \in \mathbf{D}^b(X^+)$ we have

$$\operatorname{Hom}(\mathbf{L}\pi^{+*}B \otimes \mathcal{L}, \overline{A} \otimes \omega_{\widetilde{X}}) = 0.$$

Applying Serre duality (6) gives the desired equality $\operatorname{Hom}(\overline{A}, \mathbf{L}\pi^{+*}B \otimes \mathcal{L}) = 0$ immediately.

Theorem 2.2.10. *In the above notation, if $l = k$ (and thus* fl *is a flop), the functor $\mathbf{R}\pi_*(\mathbf{L}\pi^{+*}(\,\cdot\,) \otimes \mathcal{L})$ is an equivalence of triangulated categories.*

Proof. By the previous theorem, the functor in question is fully faithful. Its left adjoint is of the form $\mathbf{R}\pi^+_*(\mathbf{L}\pi^*(\,\cdot\,) \otimes \mathcal{L}')$, where $\mathcal{L}' = \mathcal{L}^{-1} \otimes \omega_{\widetilde{X}} \otimes \pi^{+*}\omega^{-1}_{X^+}$. Thus, it is also fully faithful by the previous theorem. This proves that both functors are equivalences.

We note that the proof of the above assertions remains valid if a flop is carried out simultaneously in some finite set $Y_1, \dots, Y_s$ of disjoint subvarieties, each satisfying the condition of the theorems. This simple remark is essential in connection with our assumption that the variety X^+ we obtain is algebraic. The point is that there are many examples in which birational transformations of the above kind carried out in just one of the subvarieties Y_i lead to non-algebraic varieties, whereas a flip (or flop) carried out simultaneously in the whole set gives a variety which is algebraic.

A second remark is that the flopped varieties X and X^+ of Theorem 2.2.10 are of course not isomorphic in general; flops often occur in birational geometry, for example, in the construction used to describe Fano threefolds by the method known as double projection from a line (see [19], § 8). Suppose that we have a Fano threefold V of index 1 with $\operatorname{Pic} V = \mathbb{Z}$ embedded in projective space by its anticanonical system. Then the blowup of this variety in a line gives a variety X whose anticanonical class is 'almost' ample; that is, the map defined by its anticanonical system contracts a certain set of curves on this variety, namely the proper transforms of the lines on V meeting the blown up line. In many examples these curves have the normal sheaf $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, which puts us in the situation of our theorem. Making simultaneous flops in these curves gives a variety X^+ that is not isomorphic to X; however, by Theorem 2.2.10, it has the same derived category of coherent sheaves. In particular, this example shows that the ampleness condition on the anticanonical class in Theorem 2.1.3 on recovering X from $\mathbf{D}^b(X)$ cannot be weakened. There are similar examples for varieties of general type arising in the minimal model programme.

These results have another natural generalization. Suppose that a smooth subvariety Y in a smooth complete algebraic variety X is the projectivization of a vector bundle E of rank $k+1$ over a smooth variety Z, that is, $Y\cong\mathbb{P}(E)\to Z$. We also assume that the normal bundle $N_{X/Y}$ when restricted to the fibre of the map $Y\to Z$ is isomorphic to $\mathcal{O}_{\mathbb{P}^k}(-1)^{\oplus(l+1)}$. We again assume that $l\leqslant k$. Denoting by $\widetilde{X}$ the blowup of X with centre along Y, we again obtain a diagram of the form (20), where Y^+ is the projectivization of a bundle of rank $l+1$ over Z. In this situation we can assert that the analogues of Theorems 2.2.9 and 2.2.10 remain valid.

Other similar examples arise when X is a threefold and Y is a rational curve satisfying $Y\cdot K_X=0$. In this case the normal bundle on Y can be of the form $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, $\mathcal{O}\oplus\mathcal{O}(-2)$ or $\mathcal{O}(1)\oplus O(-3)$. In each of these cases there exists a flopping birational transformation, $\mathrm{fl}\colon X\dashrightarrow X^+$. Moreover, in each of these cases the derived categories of coherent sheaves of X and X^+ are equivalent. The first case is a special case of Theorem 2.2.10. The second case was treated in [7]. More recently, the equivalence of categories was proved in all these cases together in [10].

CHAPTER 3

Fully faithful functors between derived categories

3.1. Postnikov diagrams and their convolutions. In this section we consider Postnikov diagrams in triangulated categories and find conditions under which a Postnikov diagram admits a convolution and this convolution is uniquely determined.

Let $X^\cdot=\{X^c\xrightarrow{d^c}X^{c+1}\xrightarrow{d^{c+1}}\cdots\longrightarrow X^0\}$ with $c<0$ be a bounded complex of objects in a triangulated category $\mathcal{D}$. This means that all the composites $d^{i+1}\circ d^i$ vanish.

By definition, a *left Postnikov system* associated with $X^\cdot$ is a diagram of the form

$$\begin{tikzcd}
X^c \arrow[rr, "d^c"] \arrow[dr, "i_c=\mathrm{id}"'] & \arrow[d, phantom, "\circlearrowleft"] & X^{c+1} \arrow[rr, "d^{c+1}"] \arrow[dr, "i_{c+1}"] & \arrow[d, phantom, "\circlearrowleft"] & X^{c+2} \arrow[dr, "i_{c+2}"] & & & & X^0 \arrow[dr, "i_0"] & \\
& Y^c=X^c \arrow[ur, "j_c"] \arrow[r, phantom, "\star"] & & Y^{c+1} \arrow[ll, "{[1]}"] \arrow[ur, "j_{c+1}"] \arrow[r, phantom, "\star"] & & Y^{c+2} \arrow[ll, "{[1]}"] & \cdots \arrow[l] & Y^{-1} \arrow[l] \arrow[ur, "j_{-1}"] \arrow[r, phantom, "\star"] & & Y^0 \arrow[ll, "{[1]}"]
\end{tikzcd}$$

in which the triangles marked with $\star$ are all distinguished, and those marked with $\circlearrowleft$ are all commutative (that is, $j_k \circ i_k = d^k$). An object $E \in \operatorname{Ob} \mathcal{D}$ is called a *left convolution* of the complex $X^\cdot$ if there is a left Postnikov system associated with $X^\cdot$ such that $E = Y^0$. We denote by $\operatorname{Tot}(X^\cdot)$ the class of all convolutions of the complex $X^\cdot$. Postnikov systems and their convolutions are obviously stable under exact functors between triangulated categories.

Note that the class $\operatorname{Tot}(X^\cdot)$ may contain many non-isomorphic objects, or may also be empty. In what follows we shall describe a sufficient condition for the class $\operatorname{Tot}(X^\cdot)$ to consist of a single object up to isomorphism. The following lemma was proved in [3].

Lemma 3.1.1. *Let g be a morphism between objects Y and Y' that are in turn included into distinguished triangles*:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, dashed, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & X[1] \arrow[d, dashed, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}.$$

If $v'gu = 0$, then there exist morphisms $f\colon X \to X'$ and $h\colon Z \to Z'$ such that the triple (f, g, h) is a morphism of triangles.

Suppose in addition that $\operatorname{Hom}(X[1], Z') = 0$. *Then the morphisms f and h (respectively making the first and second squares of the diagram commute) are uniquely determined by these conditions.*

We now prove two lemmas that generalize the previous lemma to Postnikov diagrams.

Lemma 3.1.2. *Let $X^\cdot = \{X^c \xrightarrow{d^c} X^{c+1} \xrightarrow{d^{c+1}} \cdots \longrightarrow X^0\}$ be a bounded complex of objects in a triangulated category $\mathcal{D}$. Suppose that it satisfies*

$$\operatorname{Hom}^i(X^a, X^b) = 0 \quad \text{for } i < 0 \text{ and for all } a < b. \tag{25}$$

Then a convolution of $X^\cdot$ exists, and all convolutions are (non-canonically) isomorphic.

Suppose in addition that

$$\operatorname{Hom}^i(X^a, Y^0) = 0 \quad \text{for } i < 0 \text{ and for all } a \tag{26}$$

holds for some convolution Y^0 (and therefore for any convolution). Then all convolutions of $X^\cdot$ are canonically isomorphic.

Lemma 3.1.3. *Let $X_1^\cdot$ and $X_2^\cdot$ be bounded complexes satisfying condition* (25) *and $(f_c, \dots, f_0)$ a morphism between these complexes*:

$$\begin{tikzcd}
X_1^c \arrow[r, "d_1^c"] \arrow[d, "f_c"] & X_1^{c+1} \arrow[r] \arrow[d, "f_{c+1}"] & \cdots \arrow[r] & X_1^0 \arrow[d, "f_0"] \\
X_2^c \arrow[r, "d_2^c"] & X_2^{c+1} \arrow[r] & \cdots \arrow[r] & X_2^0
\end{tikzcd}.$$

Suppose that

$$\operatorname{Hom}^i(X_1^a, X_2^b) = 0 \quad \textit{for } i < 0 \textit{ and for } a < b. \tag{27}$$

Then for each convolution Y_1^0 *of* $X_1^{\cdot}$ *and for each convolution* Y_2^0 *of* $X_2^{\cdot}$ *there is a morphism* $f\colon Y_1^0 \to Y_2^0$ *that commutes with the morphism* f_0. *If in addition we have*

$$\operatorname{Hom}^i(X_1^a, Y_2^0) = 0 \quad \textit{for } i < 0 \textit{ and for any } a, \tag{28}$$

then this morphism is uniquely determined.

Proof. We prove both lemmas at the same time by induction based on Lemma 3.1.1. Let Y^{c+1} be the mapping cone of the morphism d^c,

$$X^c \xrightarrow{d^c} X^{c+1} \xrightarrow{\alpha} Y^{c+1} \longrightarrow X^c[1]. \tag{29}$$

By assumption, $d^{c+1} \circ d^c = 0$ and $\operatorname{Hom}(X^c[1], X^{c+2}) = 0$. Thus, there is a unique morphism $\overline{d}^{c+1}\colon Y^{c+1} \to X^{c+2}$ such that $\overline{d}^{c+1} \circ \alpha = d^{c+1}$. Consider the composite

$$d^{c+2} \circ \overline{d}^{c+1}\colon Y^{c+1} \longrightarrow X^{c+3}.$$

It is known that $d^{c+2} \circ \overline{d}^{c+1} \circ \alpha = d^{c+2} \circ d^{c+1} = 0$; moreover, we have the equality $\operatorname{Hom}(X^c[1], X^{c+3}) = 0$. This immediately implies that the composite $d^{c+2} \circ \overline{d}^{c+1}$ also vanishes.

Considering the distinguished triangle (29), we see that

$$\operatorname{Hom}^i(Y^{c+1}, X^b) = 0$$

for $i < 0$ and $b > c+1$. Thus, the complex $Y^{c+1} \longrightarrow X^{c+2} \longrightarrow \cdots \longrightarrow X^0$ also satisfies (25). This complex has a convolution by induction. Thus, $X^{\cdot}$ also has a convolution, and hence the class $\operatorname{Tot}(X^{\cdot})$ is not empty.

We now show that, under condition (27), every morphism of complexes extends to a morphism of Postnikov systems. Consider the mapping cones Y_1^{c+1} and Y_2^{c+1} of the morphisms d_1^c and d_2^c. There is a morphism $g_{c+1}\colon Y_1^{c+1} \to Y_2^{c+1}$ completing the pair (f_c, f_{c+1}) to a morphism of triangles,

$$\begin{array}{ccccccc}
X_1^c & \xrightarrow{d_1^c} & X_1^{c+1} & \xrightarrow{\alpha} & Y_1^{c+1} & \longrightarrow & X_1^c[1] \\
\Big\downarrow{\scriptstyle f_c} & & \Big\downarrow{\scriptstyle f_{c+1}} & & \Big\downarrow{\scriptstyle g_{c+1}} & & \Big\downarrow{\scriptstyle f_c[1]} \\
X_2^c & \xrightarrow{d_2^c} & X_2^{c+1} & \xrightarrow{\beta} & Y_2^{c+1} & \longrightarrow & X_2^c[1]
\end{array}\quad.$$

As already shown above, there exist morphisms $\overline{d}_i^{c+1}\colon Y_i^{c+1} \to X_i^{c+2}$ for $i = 1, 2$, which are uniquely determined. Consider the diagram

$$\begin{array}{ccc}
Y_1^{c+1} & \xrightarrow{\overline{d}_1^{c+1}} & X_1^{c+2} \\
\Big\downarrow{\scriptstyle g_{c+1}} & & \Big\downarrow{\scriptstyle f_{c+2}} \\
Y_2^{c+1} & \xrightarrow{\overline{d}_2^{c+1}} & X_2^{c+2}
\end{array}\quad.$$

We prove that the square is commutative. Indeed, write $h = f_{c+2}\circ\overline{d}_1^{c+1} - \overline{d}_2^{c+1}\circ g_{c+1}$ for the difference. We have the equality $h\circ\alpha = f_{c+2}\circ d_1^{c+1} - d_2^{c+1}\circ f_{c+1} = 0$. And by the assumption of the lemma, $\operatorname{Hom}(X_1^c[1], X_2^{c+2}) = 0$. This implies immediately that $h = 0$.

Thus, we obtain a morphism of complexes

$$\begin{array}{ccccccc}
Y_1^{c+1} & \xrightarrow{\overline{d}_1^{c+1}} & X_1^{c+2} & \longrightarrow & \cdots & \longrightarrow & X_1^0 \\
\Big\downarrow{\scriptstyle g_{c+1}} & & \Big\downarrow{\scriptstyle f_{c+2}} & & & & \Big\downarrow{\scriptstyle f_0} \\
Y_2^{c+1} & \xrightarrow{\overline{d}_2^{c+1}} & X_2^{c+2} & \longrightarrow & \cdots & \longrightarrow & X_2^0
\end{array}.$$

These complexes satisfy conditions (25) and (27). By the induction assumption, a morphism between these complexes extends to a morphism between the Postnikov systems. We thus obtain a morphism between the Postnikov systems associated with $X_1^{\cdot}$ and $X_2^{\cdot}$.

Moreover, one sees that, if all morphisms f_i are isomorphisms, then the morphism between the Postnikov systems is also an isomorphism. Hence, if condition (25) holds, all objects in $\operatorname{Tot}(X^{\cdot})$ are isomorphic.

In conclusion, consider a morphism between the distinguished triangles taking part in the Postnikov diagrams,

$$\begin{array}{ccccccc}
Y_1^{-1} & \xrightarrow{j_{1,-1}} & X_1^0 & \xrightarrow{i_{1,0}} & Y_1^0 & \longrightarrow & Y_1^{-1}[1] \\
\Big\downarrow{\scriptstyle g_{-1}} & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle g_{-1}[1]} \\
Y_2^{-1} & \xrightarrow{j_{2,-1}} & X_2^0 & \xrightarrow{i_{2,0}} & Y_2^0 & \longrightarrow & Y_2^{-1}[1]
\end{array}.$$

If the complexes $X_i^{\cdot}$ satisfy condition (28) (that is, $\operatorname{Hom}^i(X_1^a, Y_2^0) = 0$ for $i < 0$ and for all a), then $\operatorname{Hom}(Y_1^{-1}[1], Y_2^0) = 0$. By Lemma 3.1.1, the morphism g_0 is defined uniquely. This completes the proof of the lemmas.

3.2. Fully faithful functors between derived categories of coherent sheaves. Let X and M be two smooth complete varieties over some field k. As before, we denote by $\mathbf{D}^b(X)$ and $\mathbf{D}^b(M)$ the bounded derived categories of coherent sheaves on X and M respectively. We proved above that these categories have the structure of triangulated categories.

Consider the product $M\times X$ and write p and π for the projections of $M\times X$ to M and X respectively:

$$M \xleftarrow{p} M\times X \xrightarrow{\pi} X.$$

For every object $\mathcal{E}\in\mathbf{D}^b(M\times X)$ we defined an exact functor $\Phi_{\mathcal{E}}$ from $\mathbf{D}^b(M)$ to $\mathbf{D}^b(X)$ by (7):

$$\Phi_{\mathcal{E}}(\,\cdot\,) := \mathbf{R}\pi_*(\mathcal{E}\otimes^{\mathbf{L}} p^*(\,\cdot\,)). \tag{30}$$

The functor $\Phi_{\mathcal{E}}$ has left and right adjoint functors $\Phi_{\mathcal{E}}^*$ and $\Phi_{\mathcal{E}}^!$ respectively, given by the formulae (8),

$$\Phi_{\mathcal{E}}^*(\,\cdot\,) = \mathbf{R}p_*(\mathcal{E}^\vee\otimes^{\mathbf{L}}\pi^*(\omega_X[\dim X]\otimes(\,\cdot\,))),$$
$$\Phi_{\mathcal{E}}^!(\,\cdot\,) = \omega_M[\dim M]\otimes\mathbf{R}p_*(\mathcal{E}^\vee\otimes^{\mathbf{L}}(\,\cdot\,)),$$

where ω_X, ω_M are the canonical sheaves of X and M and $\mathcal{E}^\vee := \mathbf{R}^{\cdot}\underline{\mathcal{H}om}(\mathcal{E}, \mathcal{O}_{M\times X})$.

To study the problem of when two varieties have equivalent derived categories of coherent sheaves, and to describe their groups of auto-equivalence, it is desirable to have explicit formulae for all exact functors. There is a conjecture that they can all be represented by objects on the product, that is, are of the form (30). However, at present it is not known whether or not this assertion is true. Nevertheless, it turns out that a special case of this conjecture is valid. Namely, if a functor is fully faithful and has an adjoint functor, it can be represented by an object on the product. The present chapter is devoted to the proof of this fact. More exactly, the main theorem of this chapter is as follows.

Theorem 3.2.1. *Let F be an exact functor from the category $\mathbf{D}^b(M)$ to the category $\mathbf{D}^b(X)$, where M and X are smooth projective varieties. Suppose that F is fully faithful and has a right (or left) adjoint functor. Then there is an object $\mathcal{E} \in \mathbf{D}^b(M \times X)$ such that F is isomorphic to the functor $\Phi_{\mathcal{E}}$ defined by* (30), *and the object $\mathcal{E}$ is determined uniquely up to isomorphism.*

It follows at once that every equivalence is representable by an object on the product, because every equivalence has an adjoint, which coincides with a quasi-inverse functor.

Theorem 3.2.2. *Let M and X be two smooth projective varieties. Suppose that an exact functor $F\colon \mathbf{D}^b(M) \xrightarrow{\sim} \mathbf{D}^b(X)$ is an equivalence of triangulated categories. Then there exists an object $\mathcal{E} \in \mathbf{D}^b(M \times X)$, unique up to isomorphism, such that F is isomorphic to the functor $\Phi_{\mathcal{E}}$*

These results allow us to describe all equivalences between derived categories of coherent sheaves, and answer the question of when two distinct varieties have equivalent derived categories of coherent sheaves.

Before starting on the proof of these theorems, we make a remark. Let F be an exact functor from $\mathbf{D}^b(M)$ to $\mathbf{D}^b(X)$. We write F^* and $F^!$ respectively for the left and right adjoint functors of F, assuming that they exist. If a left adjoint F^* exists, the right adjoint $F^!$ also exists, and is defined by the formula

$$F^! = S_M \circ F^* \circ S_X^{-1},$$

where S_X and S_M are Serre functors of the categories $\mathbf{D}^b(X)$ and $\mathbf{D}^b(M)$. These functors exist and are equal to $(\,\cdot\,) \otimes \omega_X[\dim X]$ and $(\,\cdot\,) \otimes \omega_M[\dim M]$ respectively (see (6)).

Let F be an exact functor from a derived category $\mathbf{D}^b(\mathcal{A})$ to a derived category $\mathbf{D}^b(\mathcal{B})$. We say that F is *bounded* if there exist $z \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that the cohomology $H^i(F(A))$ vanishes for $i \notin [z, z+n]$ and for any object $A \in \mathcal{A}$.

Lemma 3.2.3. *Let M and X be projective varieties and M a smooth variety. If an exact functor $F\colon \mathbf{D}^b(M) \longrightarrow \mathbf{D}^b(X)$ has a left adjoint, then F is bounded.*

Proof. We denote by $G\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(M)$ the left adjoint of F and choose a very ample line bundle $\mathcal{L}$ on X. It defines an embedding $i\colon X \hookrightarrow \mathbb{P}^N$. For any $k < 0$, the sheaf $\mathcal{O}(k)$ on $\mathbb{P}^N$ has a right resolution in terms of the sheaves $\mathcal{O}(j)$ for $j = 0, 1, \dots, N$, of the form

$$\mathcal{O}(k) \xrightarrow{\sim} \bigl\{V_0 \otimes \mathcal{O} \longrightarrow V_1 \otimes \mathcal{O}(1) \longrightarrow \cdots \longrightarrow V_N \otimes \mathcal{O}(N) \longrightarrow 0\bigr\},$$

where all the V_j are vector spaces [2]. Restricting this resolution to X gives a resolution of the sheaf $\mathcal{L}^k$ in terms of the sheaves $\mathcal{L}^j$ for $j = 0, 1, \dots, N$. Since for any $j = 0, 1, \dots, N$ the non-zero cohomology of the objects $G(\mathcal{L}^j)$ belong to some interval, one can find an integer z' and a positive integer n' such that the cohomology $H^l(G(\mathcal{L}^k))$ vanishes for all $k \leqslant 0$ and for $l \notin [z', z' + n']$. This follows at once from the existence of a spectral sequence

$$E_1^{p,q} = V_p \otimes H^q(G(\mathcal{L}^p)) \Longrightarrow H^{p+q}(G(\mathcal{L}^i)).$$

Let $A \in \mathbf{D}^b(M)$ be some object. Since $\mathcal{L}$ is ample, it follows that, if for a chosen j we have $\operatorname{Hom}^j(\mathcal{L}^i, F(A)) = 0$ for any $i \ll 0$, then the cohomology $H^j(F(A))$ vanishes. By assumption, G is left adjoint to F. Hence,

$$\operatorname{Hom}^j(\mathcal{L}^i, F(A)) \cong \operatorname{Hom}^j(G(\mathcal{L}^i), A).$$

Now consider a sheaf $\mathcal{F}$ on M. Since for all $i < 0$ the cohomology of the objects $G(\mathcal{L}^i)$ is concentrated in the interval $[z', z'+n']$, it follows that $\operatorname{Hom}^j(G(\mathcal{L}^i), \mathcal{F}) = 0$ for any $i < 0$ and $j \notin [-z' - n', -z' + \dim M]$. (Here we use the fact that the homological dimension of the category $\operatorname{coh}(M)$ is equal to $\dim M$.) Hence, for the same values of j we have $H^j(F(\mathcal{F})) = 0$ for any sheaf $\mathcal{F}$. Therefore, the functor F is bounded.

Remark 3.2.4. After shifting F in the derived category if necessary, we assume from now on and throughout this chapter that for any sheaf $\mathcal{F}$ on M the cohomology $H^i(F(\mathcal{F}))$ is non-zero only for $i \in [-a, 0]$, where a is a fixed positive integer.

3.3. Construction of the object representing a fully faithful functor. In this section, starting from an exact fully faithful functor F, we construct a certain object $\mathcal{E} \in \mathbf{D}^b(M \times X)$; in the next section, we prove that the functors F and $\Phi_{\mathcal{E}}$ are isomorphic. The construction of $\mathcal{E}$ proceeds in a number of steps. We first consider a closed embedding $j\colon M \hookrightarrow \mathbb{P}^N$ and construct a certain object $\mathcal{E}' \in \mathbf{D}^b(\mathbb{P}^N \times X)$. We then prove that $\mathcal{E}'$ in fact comes from the subvariety $M \times X$, that is, there exists an object $\mathcal{E} \in \mathbf{D}^b(M \times X)$ such that $\mathcal{E}' = \mathbf{R}J_*\mathcal{E}$, where $J = (j \times \mathrm{id})$ is the closed embedding $M \times X$ in $\mathbb{P}^N \times X$.

We choose a very ample line bundle $\mathcal{L}$ on M such that $\mathrm{H}^i(\mathcal{L}^k) = 0$ for all $k > 0$ and all $i \neq 0$, and write j for the closed embedding of M in $\mathbb{P}^N$ defined by $\mathcal{L}$.

The product $\mathbb{P}^N \times \mathbb{P}^N$ has a so-called resolution of the diagonal (see [2]). This is a complex of sheaves of the form:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(-N) \boxtimes \Omega^N(N) \xrightarrow{d_{-N}} \mathcal{O}(-N+1) \boxtimes \Omega^{N-1}(N-1) \\ \longrightarrow \dots \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \xrightarrow{d_{-1}} \mathcal{O} \boxtimes \mathcal{O}. \end{aligned} \tag{31}$$

This complex is a resolution of the structure sheaf $\mathcal{O}_\Delta$, where Δ is the diagonal of the product $\mathbb{P}^N \times \mathbb{P}^N$.

Write F' for the functor from $\mathbf{D}^b(\mathbb{P}^N)$ to $\mathbf{D}^b(X)$ obtained as the composite $F \circ \mathbf{L}j^*$, and consider the diagram of projections

$$\begin{array}{ccc} \mathbb{P}^N \times X & \xrightarrow{\pi'} & X \\ {\scriptstyle q}\big\downarrow & & \\ \mathbb{P}^N & & \end{array}.$$

Write

$$d'_{-i} \in \mathrm{Hom}_{\mathbb{P}^N\times X}\Big(\mathcal{O}(-i)\boxtimes F'(\Omega^i(i)),\ \mathcal{O}(-i+1)\boxtimes F'(\Omega^{i-1}(i-1))\Big)$$

for the image of the morphism d_{-i} under the following composite map:

$$\begin{aligned}
&\mathrm{Hom}\big(\mathcal{O}(-i)\boxtimes\Omega^i(i),\ \mathcal{O}(-i+1)\boxtimes\Omega^{i-1}(i-1)\big)\\
&\qquad\xrightarrow{\sim}\mathrm{Hom}\big(\mathcal{O}\boxtimes\Omega^i(i),\ \mathcal{O}(1)\boxtimes\Omega^{i-1}(i-1)\big)\\
&\qquad\xrightarrow{\sim}\mathrm{Hom}\big(\Omega^i(i),\ \mathrm{H}^0(\mathcal{O}(1))\otimes\Omega^{i-1}(i-1)\big)\\
&\qquad\longrightarrow\mathrm{Hom}\big(F'(\Omega^i(i)),\ \mathrm{H}^0(\mathcal{O}(1))\otimes F'(\Omega^{i-1}(i-1))\big)\\
&\qquad\xrightarrow{\sim}\mathrm{Hom}\big(\mathcal{O}\boxtimes F'(\Omega^i(i)),\ \mathcal{O}(1)\boxtimes F'(\Omega^{i-1}(i-1))\big)\\
&\qquad\xrightarrow{\sim}\mathrm{Hom}\big(\mathcal{O}(-i)\boxtimes F'(\Omega^i(i)),\ \mathcal{O}(-i+1)\boxtimes F'(\Omega^{i-1}(i-1))\big).
\end{aligned}$$

One sees readily that the composite $d'_{-i+1}\circ d'_{-i}$ vanishes. Hence, we can consider the following bounded complex of objects of the derived category $\mathbf{D}^b(\mathbb{P}^N\times X)$:

$$\begin{aligned}
C^{\cdot} := \Big\{\mathcal{O}(-N)\boxtimes F'(\Omega^N(N)) &\xrightarrow{d'_{-N}}\cdots\\
&\longrightarrow\mathcal{O}(-1)\boxtimes F'(\Omega^1(1))\xrightarrow{d'_{-1}}\mathcal{O}\boxtimes F'(\mathcal{O})\Big\}.
\end{aligned}\tag{32}$$

For $l<0$ we have

$$\begin{aligned}
&\mathrm{Hom}^l\big(\mathcal{O}(-i)\boxtimes F'(\Omega^i(i)),\ \mathcal{O}(-k)\boxtimes F'(\Omega^k(k))\big)\\
&\qquad\cong\mathrm{Hom}^l\big(\mathcal{O}\boxtimes F'(\Omega^i(i)),\ \mathrm{H}^0(\mathcal{O}(i-k))\otimes F'(\Omega^k(k))\big)\\
&\qquad\cong\mathrm{Hom}^l\big(j^*(\Omega^i(i)),\ \mathrm{H}^0(\mathcal{O}(i-k))\otimes j^*(\Omega^k(k))\big)=0.
\end{aligned}$$

Thus, by Lemma 3.1.2, $C^{\cdot}$ has a convolution, and all convolutions are isomorphic. Write $\mathcal{E}'$ for a convolution of $C^{\cdot}$ and γ_0 for the morphism $\mathcal{O}\boxtimes F'(\mathcal{O})\xrightarrow{\gamma_0}\mathcal{E}'$. (In fact, we see below that all convolutions of $C^{\cdot}$ are canonically isomorphic.) Now let $\Phi_{\mathcal{E}'}$ be the functor from $\mathbf{D}^b(\mathbb{P}^N)$ to $\mathbf{D}^b(X)$ defined by (7).

Lemma 3.3.1. *For all $k\in\mathbb{Z}$ there are canonical isomorphisms*

$$f_k\colon F'(\mathcal{O}(k))\xrightarrow{\sim}\Phi_{\mathcal{E}'}(\mathcal{O}(k)),$$

and these isomorphisms are functorial; *that is, for any $\alpha\colon\mathcal{O}(k)\to\mathcal{O}(l)$ the diagram*

$$\begin{array}{ccc}
F'(\mathcal{O}(k)) & \xrightarrow{F'(\alpha)} & F'(\mathcal{O}(l))\\
{\scriptstyle f_k}\big\downarrow & & {\scriptstyle f_l}\big\downarrow\\
\Phi_{\mathcal{E}'}(\mathcal{O}(k)) & \xrightarrow{\Phi_{\mathcal{E}'}(\alpha)} & \Phi_{\mathcal{E}'}(\mathcal{O}(l))
\end{array}$$

is commutative.

Proof. Assume first that $k\geqslant 0$ and consider the resolution (31) of the diagonal $\Delta\subset\mathbb{P}^N\times\mathbb{P}^N$. Tensor it by $\mathcal{O}(k)\boxtimes\mathcal{O}$, then take its direct image under the projection to the second factor. As a result we obtain the following resolution of the sheaf $\mathcal{O}(k)$ on projective space $\mathbb{P}^N$:

$$\Big\{\mathrm{H}^0(\mathcal{O}(k-N))\otimes\Omega^N(N)\longrightarrow\cdots\longrightarrow\mathrm{H}^0(\mathcal{O}(k-1))\otimes\Omega^1(1)\longrightarrow\mathrm{H}^0(\mathcal{O}(k))\otimes\mathcal{O}\Big\}\xrightarrow{\delta_k}\mathcal{O}(k).$$

since F' is exact by assumption, it follows that $F'(\mathcal{O}(k))$ is a convolution of the complex

$$\mathrm{H}^0(\mathcal{O}(k-N))\otimes F'(\Omega^N(N))\longrightarrow\cdots\longrightarrow\mathrm{H}^0(\mathcal{O}(k-1))\otimes F'(\Omega^1(1))\longrightarrow\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O})$$

of objects of the category $\mathbf{D}^b(X)$. We denote this complex by $D^\cdot_k$.

Recall now that, by construction, $\mathcal{E}'$ is a convolution of the complex $C^\cdot$ (32). Consider the complex $C^\cdot_k := q^*\mathcal{O}(k)\otimes C^\cdot$ on $\mathbb{P}^N\times X$. Then $q^*\mathcal{O}(k)\otimes\mathcal{E}'$ is a convolution of $C^\cdot_k$. And there is a morphism $\gamma_k\colon \mathcal{O}(k)\boxtimes F'(\mathcal{O})\longrightarrow q^*\mathcal{O}(k)\otimes\mathcal{E}'$ canonically obtained from γ_0. The complex $\pi'_*(C^\cdot_k)$, the direct image of $(C^\cdot_k)$ under the projection to the second factor, is canonically isomorphic to $D^\cdot_k$. Thus, we see that the objects $F'(\mathcal{O}(k))$ and $\Phi_{\mathcal{E}'}(\mathcal{O}(k)) := \mathbf{R}\pi'_*(q^*\mathcal{O}(k)\otimes\mathcal{E}')$ are both convolutions of the same complex $D^\cdot_k$.

By assumption, the functor F is full and faithful. Hence, for locally free sheaves $\mathcal{G}$ and $\mathcal{H}$ on $\mathbb{P}^N$ we have the equality

$$\mathrm{Hom}^i(F'(\mathcal{G}),F'(\mathcal{H}))=\mathrm{Hom}^i(j^*(\mathcal{G}),j^*(\mathcal{H}))=0\quad\text{for } i<0.$$

This implies in particular that the complex $D^\cdot_k$ satisfies conditions (25) and (26) of Lemma 3.1.2. Hence, by the lemma, there exists a uniquely defined isomorphism $f_k\colon F'(\mathcal{O}(k))\xrightarrow{\sim}\Phi_{\mathcal{E}'}(\mathcal{O}(k))$ that makes the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_k)} & F'(\mathcal{O}(k)) \\
{\scriptstyle\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle f_k} \\
\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{\mathbf{R}\pi'_*(\gamma_k)} & \Phi_{\mathcal{E}'}(\mathcal{O}(k))
\end{array}.$$

We now prove that these isomorphisms are functorial. For any $\alpha\colon\mathcal{O}(k)\to\mathcal{O}(l)$ there are commutative squares of the form

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_k)} & F'(\mathcal{O}(k)) \\
{\scriptstyle\mathrm{H}^0(\alpha)\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle F'(\alpha)} \\
\mathrm{H}^0(\mathcal{O}(l))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_l)} & F'(\mathcal{O}(l))
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{\mathbf{R}\pi'_*(\gamma_k)} & \Phi_{\mathcal{E}'}(\mathcal{O}(k)) \\
{\scriptstyle\mathrm{H}^0(\alpha)\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_{\mathcal{E}'}(\alpha)} \\
\mathrm{H}^0(\mathcal{O}(l))\otimes F'(\mathcal{O}) & \xrightarrow{\mathbf{R}\pi'_*(\gamma_l)} & \Phi_{\mathcal{E}'}(\mathcal{O}(l))
\end{array}.$$

These three commutative squares imply the following equalities:

$$f_l \circ F'(\alpha) \circ F'(\delta_k) = f_l \circ F'(\delta_l) \circ (\mathrm{H}^0(\alpha) \otimes \mathrm{id}) = \mathbf{R}\pi'_*(\gamma_l) \circ (\mathrm{H}^0(\alpha) \otimes \mathrm{id}),$$
$$\Phi_{\mathcal{E}'}(\alpha) \circ f_k \circ F'(\delta_k) = \Phi_{\mathcal{E}'}(\alpha) \circ \mathbf{R}\pi'_*(\gamma_k) = \mathbf{R}\pi'_*(\gamma_l) \circ (\mathrm{H}^0(\alpha) \otimes \mathrm{id}).$$

The complexes $D^\cdot_k$ and $D^\cdot_l$ satisfy the conditions of Lemma 3.1.3, and hence, there is a unique morphism $h\colon F'(\mathcal{O}(k)) \to \Phi_{\mathcal{E}'}(\mathcal{O}(l))$ for which

$$h \circ F'(\delta_k) = \mathbf{R}\pi'_*(\gamma_l) \circ (\mathrm{H}^0(\alpha) \otimes \mathrm{id}).$$

Thus, the morphism h coincides simultaneously with $f_l \circ F'(\alpha)$ and with $\Phi_{\mathcal{E}'}(\alpha) \circ f_k$, which implies that these two are equal.

Now consider the case $k < 0$. Take the right resolution

$$\mathcal{O}(k) \xrightarrow{\sim} \bigl\{V_0^k \otimes \mathcal{O} \longrightarrow \cdots \longrightarrow V_N^k \otimes \mathcal{O}(N)\bigr\}$$

of the sheaf $\mathcal{O}(k)$ on $\mathbb{P}^N$. Applying Lemma 3.1.3 again, we see that the morphism of complexes

$$\begin{array}{ccccc}
V_0^k \otimes F'(\mathcal{O}) & \longrightarrow & \cdots & \longrightarrow & V_N^k \otimes F'(\mathcal{O}(N)) \\
{\scriptstyle \mathrm{id} \otimes f_0} \big\downarrow {\scriptstyle \wr} & & & & {\scriptstyle \mathrm{id} \otimes f_N} \big\downarrow {\scriptstyle \wr} \\
V_0^k \otimes \Phi_{\mathcal{E}'}(\mathcal{O}) & \longrightarrow & \cdots & \longrightarrow & V_N^k \otimes \Phi_{\mathcal{E}'}(\mathcal{O}(N))
\end{array}$$

gives a uniquely defined morphism $f_k\colon F'(\mathcal{O}(k)) \longrightarrow \Phi_{\mathcal{E}'}(\mathcal{O}(k))$. A direct check (which we omit) shows that these morphisms are functorial.

Remark 3.3.2. We note that the object $\mathcal{E}' \in \mathbf{D}^b(\mathbb{P}^N \times X)$ constructed from the functor F is uniquely determined.

We now prove the existence of an object in the category $\mathcal{E} \in \mathbf{D}^b(M \times X)$ such that $\mathbf{R}J_*\mathcal{E} \cong \mathcal{E}'$, where, as above, J is the embedding of $M \times X$ in $\mathbb{P}^N \times X$.

Let $\mathcal{L}$ be a very ample line bundle on M and $j\colon M \hookrightarrow \mathbb{P}^N$ the embedding into projective space it defines. We denote by A the graded algebra $\bigoplus_{i=0}^{\infty} \mathrm{H}^0(M, \mathcal{L}^i)$.

Set $B_0 = k$ and $B_1 = A_1$. For $m \geqslant 2$, we define B_m by the rule

$$B_m = \mathrm{Ker}\bigl(B_{m-1} \otimes A_1 \xrightarrow{u_{m-1}} B_{m-2} \otimes A_2\bigr), \tag{33}$$

where u_{m-1} is the natural map defined by induction.

Definition 3.3.3. We say that an algebra A is an *n-Koszul algebra* if the sequence of right A-modules

$$B_n \otimes_k A \longrightarrow B_{n-1} \otimes_k A \longrightarrow \cdots \longrightarrow B_1 \otimes_k A \longrightarrow A \longrightarrow k \longrightarrow 0$$

is exact. An algebra is called a *Koszul algebra* if it is an n-Koszul algebra for any n.

Suppose that A is an n-Koszul algebra. We set $R_0 = \mathcal{O}_M$ and for $m \geqslant 1$ we write R_m for the kernel of the canonical morphism

$$B_m \otimes \mathcal{O}_M \longrightarrow B_{m-1} \otimes \mathcal{L}$$

defined by the natural embedding $B_m \longrightarrow B_{m-1} \otimes A_1$. Using (33), we obtain a canonical morphism $R_m \longrightarrow A_1 \otimes R_{m-1}$ (in fact, one checks that there is an isomorphism $\operatorname{Hom}(R_m, R_{m-1}) \cong A_1^*$).

Moreover, if A is an n-Koszul algebra, then the following complex of sheaves is exact for $m \leqslant n$:

$$0 \longrightarrow R_m \longrightarrow B_m \otimes \mathcal{O}_M \longrightarrow B_{m-1} \otimes \mathcal{L} \longrightarrow \cdots \longrightarrow B_1 \otimes \mathcal{L}^{m-1} \longrightarrow \mathcal{L}^m \longrightarrow 0. \quad (34)$$

On the projective space $\mathbb{P}^N$ there is an exact complex of the form

$$0 \longrightarrow \Omega^m(m) \longrightarrow \bigwedge^m A_1 \otimes \mathcal{O} \longrightarrow \bigwedge^{m-1} A_1 \otimes \mathcal{O}(1) \longrightarrow \cdots \longrightarrow \mathcal{O}(m) \longrightarrow 0. \quad (35)$$

There is a canonical map $f_m \colon j^*\Omega^m(m) \longrightarrow R_m$. Indeed, since A is commutative, there are natural embeddings $\bigwedge^i A_1 \subset B_i$. Therefore, there exists a morphism from the complex (35) restricted to M to the complex (34), and hence a canonical map $f_m \colon j^*\Omega^m(m) \longrightarrow R_m$.

It is known that for any n there exists an l such that the Veronese algebra $A^l = \bigoplus_{i=0}^{\infty} \mathrm{H}^0(M, \mathcal{L}^{il})$ is n-Koszul; moreover, it was proved in [1] that the algebra A^l is in fact a Koszul algebra for $l \gg 0$.

However, in what follows, along with the n-Koszul property of the Veronese algebra, we need some additional properties. Namely, using the technique of [18] and replacing the sheaf $\mathcal{L}$ by a sufficiently high power $\mathcal{L}^j$, one can prove the following assertion.

Proposition 3.3.4. *For any integer n there is a very ample line bundle $\mathcal{L}$ such that*

1) *the algebra A is an n-Koszul algebra, that is, the sequence*

$$B_n \otimes_k A \longrightarrow B_{n-1} \otimes_k A \longrightarrow \cdots \longrightarrow B_1 \otimes_k A \longrightarrow A \longrightarrow k \longrightarrow 0$$

is exact;

2) *the complex of sheaves on M given by*

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_{k-1} \otimes R_1 \longrightarrow A_k \otimes R_0 \longrightarrow \mathcal{L}^k \longrightarrow 0$$

is exact for any $k \geqslant 0$ (if $k - i < 0$, then $A_{k-i} = 0$ by definition);

3) *the complex of sheaves on $M \times M$ of the form*

$$\mathcal{L}^{-n} \boxtimes R_n \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes R_1 \longrightarrow \mathcal{O}_M \boxtimes R_0 \longrightarrow \mathcal{O}_\Delta$$

is exact, that is, it gives an n-resolution of the diagonal on $M \times M$.

The proof of this proposition is given in § 3.5.

Write T_k for the kernel of the canonical morphism $A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1}$. In view of property 2) of Proposition 3.3.4 and the fact that $\operatorname{Ext}^{n+1}(\mathcal{L}^k, T_k) = 0$ for $n \gg 0$, we see that every convolution of the complex

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_k \otimes R_0$$

is canonically isomorphic to $T_k[n] \oplus \mathcal{L}^k$.

The canonical morphisms $R_k \longrightarrow A_1 \otimes R_{k-1}$ induce morphisms

$$\mathcal{L}^{-k} \boxtimes F(R_k) \longrightarrow \mathcal{L}^{-k+1} \boxtimes F(R_{k-1}).$$

This follows from the existence of isomorphisms

$$\begin{aligned}\operatorname{Hom}(\mathcal{L}^{-k} \boxtimes F(R_k), \mathcal{L}^{-k+1} \boxtimes F(R_{k-1})) &\cong \operatorname{Hom}(F(R_k), \mathrm{H}^0(\mathcal{L}) \otimes F(R_{k-1})) \\ &\cong \operatorname{Hom}(R_k, A_1 \otimes R_{k-1}).\end{aligned}$$

Moreover, we have the following complex of objects in the category $\mathbf{D}^b(M \times X)$:

$$\mathcal{L}^{-n} \boxtimes F(R_n) \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes F(R_1) \longrightarrow \mathcal{O}_M \boxtimes F(R_0). \tag{36}$$

By Lemma 3.1.2, the complex (36) has a convolution, and all its convolutions are isomorphic. We denote this convolution by $G \in \mathbf{D}^b(M \times X)$.

For any $k \geqslant 0$ the object $\mathbf{R}\pi_*(G \otimes p^*(\mathcal{L}^k))$ is a convolution of the complex

$$A_{k-n} \otimes F(R_n) \longrightarrow A_{k-n+1} \otimes F(R_{n-1}) \longrightarrow \cdots \longrightarrow A_k \otimes F(R_0). \tag{37}$$

On the other hand, the object $F(T_k[n] \oplus \mathcal{L}^k)$ is also a convolution of this complex, obviously satisfying the condition of Lemma 3.1.2. Hence, there is an isomorphism $\mathbf{R}\pi_*(G \otimes p^*(\mathcal{L}^k)) \cong F(T_k[n] \oplus \mathcal{L}^k)$.

It follows from Lemma 3.2.3 and Remark 3.2.4 that for all $k > 0$ the non-trivial cohomology sheaves $H^i(\mathbf{R}\pi_*(G \otimes p^*(\mathcal{L}^k))) = H^i(F(T_k)[n]) \oplus H^i(F(\mathcal{L}^k))$ are concentrated in the union $[-n-a, -n] \cup [-a, 0]$ (where a is the number defined in Remark 3.2.4). Since $\mathcal{L}$ is ample, it follows that the cohomology sheaves $H^i(G)$ are also concentrated in $[-n-a, -n] \cup [-a, 0]$. We can assume that $n > \dim M + \dim X + a$. Since the category of coherent sheaves on $M \times X$ has homological dimension $\dim M + \dim X$, we see in this case that $G \cong C \oplus \mathcal{E}$, where $\mathcal{E}, C$ are objects of $\mathbf{D}^b(M \times X)$ for which $H^i(\mathcal{E}) = 0$ for $i \notin [-a, 0]$ and $H^i(C) = 0$ for $i \notin [-n-a, -n]$. Hence, in particular, $\mathbf{R}\pi_*(\mathcal{E} \otimes p^*(\mathcal{L}^k)) \cong F(\mathcal{L}^k)$. Note that since the object G is uniquely determined as the convolution of the complex (36), the object $\mathcal{E}$ is also uniquely determined up to isomorphism.

We now show that there is an isomorphism $\mathbf{R}J_*\mathcal{E} \cong \mathcal{E}'$. For this, we consider the map of complexes over $\mathbf{D}^b(\mathbb{P}^N \times X)$,

$$\begin{tikzcd}
\mathcal{O}(-n) \boxtimes F'(\Omega^n(n)) \arrow[r] \arrow[d, "\mathrm{can} \boxtimes F(f_n)"] & \cdots \arrow[r] & \mathcal{O} \boxtimes F'(\mathcal{O}) \arrow[d, "\mathrm{can} \boxtimes F(f_0)"] \\
\mathbf{R}j_*\mathcal{L}^{-n} \boxtimes F(R_n) \arrow[r] & \cdots \arrow[r] & \mathbf{R}j_*\mathcal{O}_M \boxtimes F(R_0)
\end{tikzcd}.$$

Applying Lemma 3.1.3, we obtain the existence of a morphism $\varphi\colon K \longrightarrow \mathbf{R}J_*G$ between the convolutions.

If $N > n$, then the object K is not isomorphic to $\mathcal{E}'$, but there is a distinguished triangle

$$S \longrightarrow K \longrightarrow \mathcal{E}' \longrightarrow S[1].$$

As above, we can show that the cohomology sheaves $H^i(S)$ are non-zero only for $i \in [-n-a, -n]$. This implies that $\mathrm{Hom}(S, \mathbf{R}J_*\mathcal{E}) = 0$ and $\mathrm{Hom}(S[1], \mathbf{R}J_*\mathcal{E}) = 0$, because the cohomology $\mathbf{R}J_*\mathcal{E}$ is concentrated in the closed interval $[-a, 0]$. This implies the existence of a unique morphism $\psi\colon \mathcal{E}' \longrightarrow \mathbf{R}J_*\mathcal{E}$ such that the following diagram is commutative:

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \mathbf{R}J_*G \\ \big\downarrow & & \big\downarrow \\ \mathcal{E}' & \xrightarrow{\psi} & \mathbf{R}J_*\mathcal{E} \end{array} .$$

As we know,

$$\mathbf{R}\pi'_*(\mathcal{E}' \otimes q^*(\mathcal{O}(k))) \cong F(\mathcal{L}^k) \cong \mathbf{R}\pi_*(\mathcal{E} \otimes p^*(\mathcal{L}^k)).$$

Write ψ_k for the morphisms $\mathbf{R}\pi'_*(\mathcal{E}' \otimes q^*(\mathcal{O}(k))) \longrightarrow \mathbf{R}\pi_*(\mathcal{E} \otimes p^*(\mathcal{L}^k))$ induced by ψ. The ψ_k fit into a commutative diagram

$$\begin{array}{ccccc} S^kA_1 \otimes F(\mathcal{O}) & \xrightarrow{\text{can}} & F(\mathcal{L}^k) & \xrightarrow{\sim} & \mathbf{R}\pi'_*(\mathcal{E}' \otimes q^*(\mathcal{O}(k))) \\ {\scriptstyle\text{can}}\big\downarrow & & & & \big\downarrow{\scriptstyle\psi_k} \\ A_k \otimes F(\mathcal{O}) & \xrightarrow{\text{can}} & F(\mathcal{L}^k) & \xrightarrow{\sim} & \mathbf{R}\pi_*(\mathcal{E} \otimes p^*(\mathcal{L}^k)) \end{array} .$$

This implies that the morphisms ψ_k are isomorphisms for any $k \geqslant 0$. Hence, ψ is also an isomorphism. Thus, we have proved the following assertion.

Proposition 3.3.5. *There is an object $\mathcal{E} \in \mathbf{D}^b(M \times X)$ such that $\mathbf{R}J_*\mathcal{E} \cong \mathcal{E}'$, where $\mathcal{E}'$ is the object of $\mathbf{D}^b(\mathbb{P}^N \times X)$ constructed in* § 3.3; *and this $\mathcal{E}$ is unique up to isomorphism*

3.4. Proof of the main theorem. In the previous section, starting from a fully faithful functor F between the derived categories of coherent sheaves on varieties M and X, we constructed an object $\mathcal{E}$ on the product $M \times X$, and thus obtained a new functor $\Phi_{\mathcal{E}}$. The main objective of the present section is to show that these two functors F and $\Phi_{\mathcal{E}}$ are isomorphic. For this, we must construct a natural transformation between these functors which is an isomorphism. By construction, the transformation is already given on an ample sequence of line bundles on M. Our task is to extend this transformation to the entire derived category.

We start by proving some assertions on Abelian categories that we need below. Let $\mathcal{A}$ be a k-linear Abelian category (in what follows we always consider Abelian categories that are k-linear).

Definition 3.4.1. We say that a sequence of objects $\{P_i \mid i \in \mathbb{Z}_{\leqslant 0}\}$ (with negative indices) in an Abelian category $\mathcal{A}$ is *ample* if for every object $X \in \mathcal{A}$ there exists an integer N such that the following conditions hold for any index $i < N$:

a) the canonical morphism $\mathrm{Hom}(P_i, X) \otimes P_i \longrightarrow X$ is surjective,
b) $\mathrm{Ext}^j(P_i, X) = 0$ for all $j \neq 0$,
c) $\mathrm{Hom}(X, P_i) = 0$.

Example 3.4.2. For $\mathcal{L}$ an ample line bundle on a projective variety, the sequence $\{\mathcal{L}^i \mid i \in \mathbb{Z}_{\leqslant 0}\}$ is ample in the Abelian category of coherent sheaves.

Lemma 3.4.3. *Let $\{P_i\}$ be an ample sequence in an Abelian category $\mathcal{A}$. If an object X in the category $\mathbf{D}^b(\mathcal{A})$ satisfies the equality*

$$\operatorname{Hom}^{\cdot}(P_i, X) = 0 \quad \textit{for all } i \ll 0,$$

then X is the zero object.

Proof. It follows from the definition of ampleness that

$$\operatorname{Hom}(P_i, H^k(X)) \cong \operatorname{Hom}^k(P_i, X) = 0 \quad \text{for } i \ll 0.$$

However, the morphism $\operatorname{Hom}(P_i, H^k(X)) \otimes P_i \longrightarrow H^k(X)$ must be surjective for $i \ll 0$. Hence, $H^k(X) = 0$ for all k. This means that X is the zero object.

Lemma 3.4.4. *Let $\mathcal{A}$ be an Abelian category of finite homological dimension and $\{P_i\}$ an ample sequence in $\mathcal{A}$. If an object $X \in \mathbf{D}^b(\mathcal{A})$ is such that $\operatorname{Hom}^{\cdot}(X, P_i) = 0$ for any $i \ll 0$, then X is the zero object.*

Proof. Suppose that the object X is non-trivial. After shifting X in the derived category if necessary, we can assume that the rightmost non-zero cohomology of X is $H^0(X)$. Consider the canonical morphism $X \longrightarrow H^0(X)$. For some i_1, there exists a surjective map $P_{i_1}^{\oplus k_1} \longrightarrow H^0(X)$; write Y_1 for its kernel. By assumption, $\operatorname{Hom}^{\cdot}(X, P_{i_1}) = 0$, and hence also $\operatorname{Hom}^1(X, Y_1) \neq 0$. Next, take a surjective map $P_{i_2}^{\oplus k_2} \longrightarrow Y_1$, which exists for some $i_2 \ll 0$, and write Y_2 for its kernel. The condition $\operatorname{Hom}^{\cdot}(X, P_{i_2}) = 0$ again gives $\operatorname{Hom}^2(X, Y_2) \neq 0$. Continuing this procedure, we obtain a contradiction to the finite homological dimension of $\mathcal{A}$.

Lemma 3.4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories and suppose that $\mathcal{A}$ has finite homological dimension. Let $\{P_i\}$ be an ample sequence in $\mathcal{A}$. Suppose that F is an exact functor from $\mathbf{D}^b(\mathcal{A})$ to $\mathbf{D}^b(\mathcal{B})$ that has right and left adjoint functors $F^!$ and F^* respectively. If the maps*

$$\operatorname{Hom}^k(P_j, P_i) \xrightarrow{\sim} \operatorname{Hom}^k(F(P_j), F(P_i))$$

are isomorphisms for $j < i$ and for all k, then F is fully faithful.

Proof. Consider the canonical morphism $f_i \colon P_i \longrightarrow F^! F(P_i)$ and the distinguished triangle

$$P_i \xrightarrow{f_i} F^! F(P_i) \longrightarrow C_i \longrightarrow P_i[1].$$

By assumption, for $j < i$ we have isomorphisms

$$\operatorname{Hom}^k(P_j, P_i) \xrightarrow{\sim} \operatorname{Hom}^k(F(P_j), F(P_i)) \cong \operatorname{Hom}^k(P_j, F^! F(P_i)).$$

Hence, $\operatorname{Hom}^{\cdot}(P_j, C_i) = 0$ for $j < i$. By Lemma 3.4.3, $C_i = 0$. Thus, f_i is an isomorphism.

For an arbitrary object X we now consider the canonical map $g_X : F^*F(X) \longrightarrow X$ and the distinguished triangle

$$F^*F(X) \xrightarrow{g_X} X \longrightarrow C_X \longrightarrow F^*F(X)[1].$$

There is a sequence of isomorphisms

$$\mathrm{Hom}^k(X, P_i) \xrightarrow{\sim} \mathrm{Hom}^k(X, F^!F(P_i)) \cong \mathrm{Hom}^k(F^*F(X), P_i).$$

This implies that $\mathrm{Hom}^{\cdot}(C_X, P_i) = 0$ for all i. By Lemma 3.4.4 we get that $C_X = 0$. Thus, g_X is an isomorphism. Therefore, F is fully faithful.

We now state and prove the main proposition of this section, which we need in the proof of Theorem 3.2.1, the main result of this chapter. The proposition is also of independent interest.

Let $\mathcal{A}$ be an Abelian category with an ample sequence $\{P_i \mid i \in \mathbb{Z}_{\leqslant 0}\}$. Write j for the embedding of the full subcategory $\mathcal{C}$ with objects $\mathrm{Ob}\,\mathcal{C} := \{P_i \mid i \in \mathbb{Z}_{\leqslant 0}\}$ into $\mathbf{D}^b(\mathcal{A})$. In this situation, given a functor $F\colon \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$, one proves that, if there exists an isomorphism of $F_{|\mathcal{C}}$ with the identity functor on $\mathcal{C}$, then this transformation extends to an isomorphism on the entire category $\mathbf{D}^b(\mathcal{A})$.

Proposition 3.4.6. *Let $\mathcal{A}$ be an Abelian category and $\{P_i \mid i \in \mathbb{Z}_{\leqslant 0}\}$ an ample sequence in $\mathcal{A}$. Write j for the embedding of the full subcategory $\mathcal{C}$ with objects $\mathrm{Ob}\,\mathcal{C} := \{P_i \mid i \in \mathbb{Z}_{\leqslant 0}\}$ into $\mathbf{D}^b(\mathcal{A})$. Let $F\colon \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$ be some autoequivalence. Suppose that there exists an isomorphism of functors $f\colon j \xrightarrow{\sim} F_{|\mathcal{C}}$. Then f extends to an isomorphism $\mathrm{id} \xrightarrow{\sim} F$ on the entire category $\mathbf{D}^b(\mathcal{A})$.*

Proof. First, since F commutes with direct sums, the transformation f extends componentwise to direct sums of objects in the category $\mathcal{C}$. We note that an object $X \in \mathbf{D}^b(\mathcal{A})$ is isomorphic to an object of $\mathcal{A}$ if and only if $\mathrm{Hom}^j(P_i, X) = 0$ for $j \neq 0$ and all $i \ll 0$. It follows that in this case the object $F(X)$ is also isomorphic to an object of $\mathcal{A}$, because

$$\mathrm{Hom}^j(P_i, F(X)) \cong \mathrm{Hom}^j(F(P_i), F(X)) \cong \mathrm{Hom}^j(P_i, X) = 0$$

for $j \neq 0$ and for all $i \ll 0$.

Step 1. Let X be an object of the category $\mathcal{A}$. We fix a surjective morphism $v\colon P_i^{\oplus k} \longrightarrow X$. There exists an isomorphism $f_i\colon P_i^{\oplus k} \xrightarrow{\sim} F(P_i^{\oplus k})$ together with two distinguished triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & P_i^{\oplus k} & \xrightarrow{v} & X & \longrightarrow & Y[1] \\ & & \downarrow{\scriptstyle f_i} & & & & \\ F(Y) & \xrightarrow{F(u)} & F(P_i^{\oplus k}) & \xrightarrow{F(v)} & F(X) & \longrightarrow & F(Y)[1] \end{array}.$$

Let us prove that $F(v) \circ f_i \circ u = 0$. For this, consider a surjective morphism $w\colon P_j^{\oplus l} \longrightarrow Y$; it is enough to show that $F(v) \circ f_i \circ u \circ w = 0$. Let $f_j\colon P_j^{\oplus l} \xrightarrow{\sim} F(P_j^{\oplus l})$

be the canonical isomorphism. Using the commutation relations for f_i and f_j, we obtain the equalities

$$F(v) \circ f_i \circ u \circ w = F(v) \circ F(u \circ w) \circ f_j = F(v \circ u \circ w) \circ f_j = 0.$$

Since $\mathrm{Hom}(Y[1], F(X)) = 0$, there is a unique morphism $f_X \colon X \longrightarrow F(X)$ commuting with f_i by Lemma 3.1.1.

Now consider the mapping cone C_X of f_X. Using the isomorphisms

$$\mathrm{Hom}(P_i, X) \cong \mathrm{Hom}(F(P_i), F(X)) \cong \mathrm{Hom}(P_i, F(X)),$$

we see that $\mathrm{Hom}^j(P_i, C_X) = 0$ for all j and $i \ll 0$. Hence, $C_X = 0$ by Lemma 3.4.3, and f_X is an isomorphism.

Step 2. We now show that f_X does not depend on the choice of the covering $v \colon P_i^{\oplus k} \longrightarrow X$. Consider two such surjective morphisms $v_1 \colon P_{i_1}^{\oplus k_1} \longrightarrow X$ and $v_2 \colon P_{i_2}^{\oplus k_2} \longrightarrow X$. We can always fix up two surjective morphisms $w_1 \colon P_j^{\oplus l} \longrightarrow P_{i_1}^{\oplus k_1}$ and $w_2 \colon P_j^{\oplus l} \longrightarrow P_{i_2}^{\oplus k_2}$ such that the following diagram is commutative:

$$\begin{tikzcd} P_j^{\oplus l} \arrow[r, "w_2"] \arrow[d, "w_1"] & P_{i_2}^{\oplus k_2} \arrow[d, "v_2"] \\ P_{i_1}^{\oplus k_1} \arrow[r, "v_1"] & X \end{tikzcd}.$$

It is obviously enough to check that the transformations f_X constructed from v_1 and $v_1 \circ w_1$ coincide. For this, consider the commutative diagram

$$\begin{tikzcd} P_j^{\oplus l} \arrow[r, "w_1"] \arrow[d, "f_j"] & P_{i_1}^{\oplus k_1} \arrow[r, "v_1"] \arrow[d, "v_2"] & X \arrow[d, "f_X"] \\ F(P_j^{\oplus l}) \arrow[r, "F(w_1)"] & F(P_{i_1}^{\oplus k_1}) \arrow[r, "F(v_1)"] & F(X) \end{tikzcd}.$$

Here the isomorphism f_X is constructed from v_1. Both squares of the diagram commute. Since there only exists one morphism from X to $F(X)$ that commutes with f_j, it follows that the morphism f_X constructed from v_1 coincides with that constructed from $v_1 \circ w_1$.

Step 3. Now we have to check that the morphisms f_X define a natural transformation of functors on $\mathcal{A}$. That is, for any morphism $X \xrightarrow{\varphi} Y$, we must prove that

$$f_Y \circ \varphi = F(\varphi) \circ f_X.$$

Consider a surjective morphism $P_j^{\oplus l} \xrightarrow{v} Y$. We choose an index $i \ll 0$ and a surjective morphism $P_i^{\oplus k} \xrightarrow{u} X$ such that the composite $\varphi \circ u$ lifts to a morphism $\psi \colon P_i^{\oplus k} \longrightarrow P_j^{\oplus l}$. This is possible because for $i \ll 0$ the map $\mathrm{Hom}(P_i^{\oplus k}, P_j^{\oplus l}) \to \mathrm{Hom}(P_i^{\oplus k}, Y)$ is surjective. We obtain a commutative square

$$\begin{tikzcd} P_i^{\oplus k} \arrow[r, "u"] \arrow[d, "\psi"] & X \arrow[d, "\varphi"] \\ P_j^{\oplus l} \arrow[r, "v"] & Y \end{tikzcd}.$$

We write h_1 and h_2 for the composites $f_Y \circ \varphi$ and $F(\varphi) \circ f_X$ respectively. We have the equalities

$$h_1 \circ u = f_Y \circ \varphi \circ u = f_Y \circ v \circ \psi = F(v) \circ f_j \circ \psi = F(v) \circ F(\psi) \circ f_i$$

and

$$h_2 \circ u = F(\varphi) \circ f_X \circ u = F(\varphi) \circ F(u) \circ f_i = F(\varphi \circ u) \circ f_i = F(v \circ \psi) \circ f_i = F(v) \circ F(\psi) \circ f_i.$$

Thus, for $t = 1, 2$ the morphisms h_t make the following diagram commute:

$$\begin{tikzcd} Z \arrow[r] & P_i^{\oplus k} \arrow[r, "u"] \arrow[d, "F(\psi)\circ f_i"'] & X \arrow[r] \arrow[d, "h_t"] & Z[1] \\ F(W) \arrow[r] & F(P_j^{\oplus l}) \arrow[r, "F(v)"] & F(Y) \arrow[r] & F(W)[1] \end{tikzcd}.$$

Since $\mathrm{Hom}(Z[1], F(Y)) = 0$, it follows from Lemma 3.1.1 that $h_1 = h_2$. Thus, $f_Y \circ \varphi = F(\varphi) \circ f_X$.

Step 4. We define a transformation $f_{X[n]}\colon X[n] \longrightarrow F(X[n]) \cong F(X)[n]$ for any $X \in \mathcal{A}$ by the formula

$$f_{X[n]} = f_X[n].$$

One proves readily that the transformations defined in this way commute with any morphism $u \in \mathrm{Ext}^k(X, Y)$. Indeed, every element $u \in \mathrm{Ext}^k(X, Y)$ can be represented as a composite $u = u_0 u_1 \cdots u_k$ of certain elements $u_i \in \mathrm{Ext}^1(Z_i, Z_{i+1})$, where $Z_0 = X$ and $Z_k = Y$. Thus, it is enough to verify that $f_{X[n]}$ commutes with elements $u \in \mathrm{Ext}^1(X, Y)$. For this, consider the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f_Y"'] & Z \arrow[r] \arrow[d, "f_Z"] & X \arrow[r, "u"] & Y[1] \arrow[d, "f_Y[1]"] \\ F(Y) \arrow[r] & F(Z) \arrow[r] & F(X) \arrow[r, "F(u)"] & F(Y)[1] \end{tikzcd}.$$

By one of the axioms of triangulated category, there is a morphism $h\colon X \to F(X)$ such that (f_Y, f_Z, h) is a morphism of triangles. On the other hand, since $\mathrm{Hom}(Y[1], F(X)) = 0$, it follows from Lemma 3.1.1 that the morphism h is uniquely determined by the condition that it commutes with f_Z. However, f_X also commutes with f_Z. Hence, $h = f_X$, and thus

$$f_Y[1] \circ u = F(u) \circ f_X.$$

Step 5. We carry out the final part of the proof by induction on the length of the interval to which the non-trivial cohomology of the object belongs. For this, consider the full subcategory $j_n\colon \mathcal{D}_n \hookrightarrow \mathbf{D}^b(\mathcal{A})$ of $\mathbf{D}^b(\mathcal{A})$ consisting of objects with non-trivial cohomology in some interval of length n (the interval is not fixed). We now prove that there is a unique extension of the natural transform f to a natural functorial isomorphism $f_n\colon j_n \longrightarrow F_{|\mathcal{D}_n}$. We have already proved this above for $n=1$, as the basis of the induction.

Now to prove the inductive step, suppose that the assertion is already proved for some $n=a\geqslant 1$. Let X be an object of $\mathcal{D}_{a+1}$, and suppose for definiteness that the cohomology $H^p(X)$ is non-trivial for $p\in[-a,0]$. We take P_i in the ample sequence, where i is a sufficiently negative index such that

$$\begin{aligned} &\text{a)}\quad \operatorname{Hom}^j(P_i,H^p(X))=0 \text{ for all } p \text{ and } j\neq 0,\\ &\text{b)}\quad \text{there exists a surjective morphism } u\colon P_i^{\oplus k}\longrightarrow H^0(X),\\ &\text{c)}\quad \operatorname{Hom}(H^0(X),P_i)=0. \end{aligned} \tag{38}$$

It follows from a) and the standard spectral sequence that there is an isomorphism $\operatorname{Hom}(P_i,X)\xrightarrow{\sim}\operatorname{Hom}(P_i,H^0(X))$. Thus, there is a morphism $v\colon P_i^{\oplus k}\longrightarrow X$ whose composite with the canonical morphism $X\longrightarrow H^0(X)$ coincides with u. Consider the distinguished triangle

$$Y[-1]\longrightarrow P_i^{\oplus k}\xrightarrow{v} X\longrightarrow Y.$$

Since the object Y belongs to $\mathcal{D}_a$, it follows from the induction assumption that the isomorphism f_Y already exists and commutes with f_i. We have the diagram

$$\begin{tikzcd} P_i^{\oplus k} \arrow[r, "v"] \arrow[d, "f_i"] & X \arrow[r] \arrow[d, dotted, "f_X"] & Y \arrow[r] \arrow[d, "f_Y"] & P_i^{\oplus k}[1] \arrow[d, "f_i{[1]}"] \\ F(P_i^{\oplus k}) \arrow[r, "F(v)"] & F(X) \arrow[r] & F(Y) \arrow[r] & F(P_i^{\oplus k})[1] \end{tikzcd} \tag{39}$$

Next, the sequence of isomorphisms

$$\operatorname{Hom}(X,F(P_i^{\oplus k}))\cong\operatorname{Hom}(X,P_i^{\oplus k})\cong\operatorname{Hom}(H^0(X),P_i^{\oplus k})=0$$

allows us to apply Lemma 3.1.1 with $g=f_Y$, and it follows from this that there is a unique morphism $f_X\colon X\longrightarrow F(X)$ completing the diagram to a morphism of triangles. It is obvious that f_X is in fact an isomorphism, because f_i and f_Y are.

Step 6. We now have to prove that the isomorphism f_X does not depend on the choice of i and u. Suppose that we are given two surjective morphisms $u_1\colon P_{i_1}^{\oplus k_1}\longrightarrow H^0(X)$ and $u_2\colon P_{i_2}^{\oplus k_2}\longrightarrow H^0(X)$ satisfying a), b) and c). Then we can choose a sufficiently negative index j and surjective morphisms w_1 and w_2 that make the diagram

$$\begin{tikzcd} P_j^{\oplus l} \arrow[r, "w_2"] \arrow[d, "w_1"] & P_{i_2}^{\oplus k_2} \arrow[d, "u_2"] \\ P_{i_1}^{\oplus k_1} \arrow[r, "u_1"] & H^0(X) \end{tikzcd}$$

commute. Write $v_1\colon P_{i_1}^{\oplus k_1}\longrightarrow X$ and $v_2\colon P_{i_2}^{\oplus k_2}\longrightarrow X$ for the morphisms corresponding to u_1 and u_2. Since $\operatorname{Hom}(P_j,X)\xrightarrow{\sim}\operatorname{Hom}(P_j,H^0(X))$, we see that $v_2w_2=v_1w_1$.

There is a morphism $\varphi\colon Y_j \longrightarrow Y_{i_1}$ such that the triple $(w_1, \mathrm{id}, \varphi)$ is a morphism of triangles

$$\begin{tikzcd} P_j^{\oplus l} \arrow[r, "v_1 \circ w_1"] \arrow[d, "w_1"'] & X \arrow[r, "y"] \arrow[d, "\mathrm{id}"] & Y_j \arrow[r] \arrow[d, "\varphi"] & P_j^{\oplus l}[1] \arrow[d, "w_1[1]"] \\ P_{i_1}^{\oplus k_1} \arrow[r, "v_1"] & X \arrow[r, "y_1"] & Y_{i_1} \arrow[r] & P_{i_1}^{\oplus k_1}[1] \end{tikzcd} \quad ,$$

that is, $\varphi y = y_1$.

Since Y_j and Y_{i_1} only have non-trivial cohomology in the interval $[-a, -1]$, by induction we have the following commutative square:

$$\begin{tikzcd} Y_j \arrow[r, "\varphi"] \arrow[d, "f_{Y_j}"'] & Y_{i_1} \arrow[d, "f_{Y_{i_1}}"] \\ F(Y_j) \arrow[r, "F(\varphi)"] & F(Y_{i_1}) \end{tikzcd} \quad .$$

Write f_X^j, $f_X^{i_1}$ and $f_X^{i_2}$ for the morphisms constructed by the above rule; these can be completed to a commutative diagram (39) for $v = v_1 w_1$, $v = v_1$ and $v = v_2$ respectively. We have already proved in Lemma 3.1.1 above that the morphism $f_X^{i_1}$ is uniquely determined by the condition

$$F(y_1) f_X^{i_1} = f_{Y_{i_1}} y_1.$$

On the other hand, we have the relations

$$F(y_1) f_X^j = F(\varphi y) f_X^j = F(\varphi) F(y) f_X^j = F(\varphi) F_{Y_j} y = f_{Y_{i_1}} \varphi y = f_{Y_{i_1}} y_1,$$

which imply at once that $f_X^j = f_X^{i_1}$. In the same way we get $f_X^j = f_X^{i_2}$. Hence, the morphism f_X does not depend on the choices of the index i and the morphism $u\colon P_i^{\oplus k} \longrightarrow H^0(X)$, and is thus absolutely uniquely defined.

Step 7. We have thus obtained an extension of f_a to $\mathcal{D}_{a+1}$. It remains to show that this extension is again a natural transformation from j_{a+1} to $F_{|\mathcal{D}_{a+1}}$; that is, that for any morphism $\varphi\colon X \longrightarrow Y$ with X and Y in $\mathcal{D}_{a+1}$, we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"] \\ F(X) \arrow[r, "F(\varphi)"] & F(Y) \end{tikzcd} \quad . \tag{40}$$

We will reduce this problem to the case in which both objects X and Y belong to $\mathcal{D}_a$. There are two cases.

Case 1. We consider the case when the highest non-trivial cohomology of the object X (which we can assume to be $H^0(X)$ without loss of generality) has index strictly greater than that for Y. As above, we take a surjective morphism $u\colon P_i^{\oplus k} \longrightarrow H^0(X)$ satisfying a), b) and c) and construct a lift of u to a morphism $v\colon P_i^{\oplus k} \longrightarrow X$. We have a distinguished triangle

$$P_i^{\oplus k} \xrightarrow{v_1} X \xrightarrow{\alpha} Z \longrightarrow P_i^{\oplus k}[1].$$

If i is sufficiently negative, then $\mathrm{Hom}(P_i^{\oplus k}, Y) = 0$. Applying $\mathrm{Hom}(-, Y)$ to this triangle, we see that there exists a morphism $\psi\colon Z \longrightarrow Y$ for which $\varphi = \psi\alpha$. It is known that the isomorphism f_X constructed above satisfies the relation

$$F(\alpha) f_X = f_Z \alpha.$$

If we assume that

$$F(\psi) f_Z = f_Y \psi,$$

then we obtain

$$F(\varphi) f_X = F(\psi) F(\alpha) f_X = F(\psi) f_Z \alpha = f_Y \psi\alpha = f_Y \varphi.$$

This means that, to check that the square (40) is commutative, we can replace X by Z. But the upper bound for the non-trivial cohomology of Z is one less than for X. Moreover, one can see that, if X belongs to $\mathcal{D}_k$ with $k > 1$, then Z belongs to $\mathcal{D}_{k-1}$, and, if X belongs to $\mathcal{D}_1$, then Z also belongs to $\mathcal{D}_1$, but the index for its non-trivial cohomology is one less than for X.

Case 2. We now consider the other case: the highest non-trivial cohomology of Y (which we can again assume to be $H^0(Y)$) has index greater than or equal to that of X. Take a surjective morphism $u\colon P_i^{\oplus k} \longrightarrow H^0(Y)$ satisfying conditions a), b) and c) and construct a morphism $v\colon P_i^{\oplus k} \longrightarrow Y$, which is uniquely determined by u. Consider the distinguished triangle

$$P_i^{\oplus k} \xrightarrow{\ v\ } Y \xrightarrow{\ \beta\ } W \longrightarrow P_i^{\oplus k}. \tag{41}$$

Write ψ for the composite $\beta \circ \varphi$.

If we now assume that

$$F(\psi) f_X = f_W \psi,$$

then, since $F(\beta) f_Y = f_W \beta$, we obtain

$$F(\beta)(f_Y \varphi - F(\varphi) f_X) = f_W \beta\varphi - f(\beta\varphi) f_X = f_W \psi - F(\psi) f_X = 0. \tag{42}$$

We again choose i to be sufficiently negative, so that the vanishing condition $\mathrm{Hom}(X, P_i^{\oplus k}) = 0$ is satisfied. Since $F(P_i^{\oplus k})$ is isomorphic to $P_i^{\oplus k}$, we have the equality $\mathrm{Hom}(X, F(P_i^{\oplus k})) = 0$. Now applying $\mathrm{Hom}(X, F(-))$ to the triangle (41), we see that $F(\beta)$ defines an embedding of $\mathrm{Hom}(X, F(Y))$ into $\mathrm{Hom}(X, F(W))$. It now follows at once from (42) that $f_Y \varphi = F(\varphi) f_X$.

Thus, to check that the square (40) is commutative, we can replace Y by an object W that has upper bound of the non-trivial cohomology one less than Y. If Y belongs to $\mathcal{D}_k$ with $k > 1$, then W belongs to $\mathcal{D}_{k-1}$. If Y belongs to $\mathcal{D}_1$, then W also belongs to $\mathcal{D}_1$, but has non-trivial cohomology of index one less than Y.

Suppose now that X and Y belong to the category $\mathcal{D}_{a+1}$ with $a > 1$. Depending on which of the cases 1) or 2) is applicable, we can replace either X or Y by an object that already belongs to $\mathcal{D}_a$. Repeating this procedure if necessary, we can reduce the upper bound of the cohomology of this object until the other case becomes applicable. Then we will be able to reduce the length of the non-trivial cohomology of the other object, and arrive at the situation in which both objects already belong to $\mathcal{D}_a$. This is our induction step.

In conclusion we note that during our construction, the isomorphisms f_X were uniquely determined at each point. Hence, the natural transformation from id to F that we have constructed is unique. This completes the proof of the proposition.

Proof of Theorem 3.2.1. 1) *Existence*. Starting from the functor F, we can use Proposition 3.3.5 and Lemma 3.3.1 to construct an object $\mathcal{E} \in \mathbf{D}^b(M \times X)$ for which there exists an isomorphism of functors $\overline{f}\colon F_{|\mathcal{C}} \xrightarrow{\sim} \Phi_{\mathcal{E}\,|\mathcal{C}}$ on the full subcategory $\mathcal{C} \subset \mathbf{D}^b(M)$ with $\operatorname{Ob}\mathcal{C} = \{\mathcal{L}^i \mid i \in \mathbb{Z}\}$, where $\mathcal{L}$ is a very ample bundle on M for which $\mathrm{H}^i(M, \mathcal{L}^k) = 0$ for $k > 0$ and $i \neq 0$.

By Lemma 3.4.5, $\Phi_{\mathcal{E}}$ is fully faithful. Moreover, since there are isomorphisms

$$F^!(\overline{f})\colon F^! \circ F_{|\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} F^! \circ \Phi_{\mathcal{E}\,|\mathcal{C}},$$
$$\Phi^*_{\mathcal{E}}(\overline{f})\colon \Phi^*_{\mathcal{E}} \circ F_{|\mathcal{C}} \xrightarrow{\sim} \Phi^*_{\mathcal{E}} \circ \Phi_{\mathcal{E}\,|\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}},$$

it follows again from Lemma 3.4.5 that the functors $F^! \circ \Phi_{\mathcal{E}}$ and $\Phi^*_{\mathcal{E}} \circ F$ are also fully faithful. Since they are adjoint to one another, it follows that they are in fact equivalences.

Consider again the isomorphism $F^!(\overline{f})\colon F^! \circ F_{|\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} F^! \circ \Phi_{\mathcal{E}\,|\mathcal{C}}$ on the subcategory $\mathcal{C}$. By Proposition 3.4.6, it extends to an isomorphism on the entire category $\mathbf{D}^b(M)$, that is, $\mathrm{id} \xrightarrow{\sim} F^! \circ \Phi_{\mathcal{E}}$.

Since $F^!$ is right adjoint to F, we obtain a morphism of functors $f\colon F \longrightarrow \Phi_{\mathcal{E}}$ for which $f_{|\mathcal{C}} = \overline{f}$. It remains to show that f is an isomorphism. Indeed, take the mapping cone C_Z of the canonical morphism $f_Z\colon F(Z) \longrightarrow \Phi_{\mathcal{E}}(Z)$. Since $F^!(f_Z)$ is an isomorphism, we see that $F^!(Z) = 0$. Hence, $\operatorname{Hom}(F(Y), C_Z) = 0$ for any object Y. Moreover, since $F(\mathcal{L}^k) \cong \Phi_{\mathcal{E}}(\mathcal{L}^k)$ for all k, we obtain a sequence of isomorphisms

$$\operatorname{Hom}^i(\mathcal{L}^k, \Phi^!_{\mathcal{E}}(C_Z)) = \operatorname{Hom}^i(\Phi_{\mathcal{E}}(\mathcal{L}^k), C_Z) = \operatorname{Hom}^i(F(\mathcal{L}^k), C_Z) = 0$$

for all k and i.

Applying Lemma 3.6 gives at once the equality $\Phi^!_{\mathcal{E}}(C_Z) = 0$. It follows that $\operatorname{Hom}(\Phi_{\mathcal{E}}(Z), C_Z) = 0$. Therefore, the triangle for the morphism f_Z must be split, that is, $F(Z) = C_Z[-1] \oplus \Phi_{\mathcal{E}}(Z)$. However, we have already proved above that $\operatorname{Hom}(F(Y), C_Z) = 0$ for any Y, and hence also for $Z[1]$. However, this can only happen if $C_Z = 0$, and f_Z is an isomorphism.

2) *Uniqueness*. The uniqueness of the object representing F in fact follows from our construction, because each time we construct some object it is unique. However, let us go through this once more. Suppose that there exist two objects $\mathcal{E}_1$ and $\mathcal{E}_2$ in $D^b(M \times X)$ for which $\Phi_{\mathcal{E}_1} \cong F \cong \Phi_{\mathcal{E}_2}$. Consider the functor $F' = \mathbf{L}j^* \circ F$, where, as above, $j\colon M \longrightarrow \mathbb{P}^N$ is an embedding by a suitable very ample line bundle. The objects $\mathbf{R}J_*\mathcal{E}_i$ for $i = 1, 2$ must both be convolutions of the complex (32)

$$C^\cdot := \bigl\{\mathcal{O}(-N) \boxtimes F'(\Omega^N(N)) \xrightarrow{d'_{-N}} \cdots \longrightarrow \mathcal{O}(-1) \boxtimes F'(\Omega^1(1)) \xrightarrow{d'_{-1}} \mathcal{O} \boxtimes F'(\mathcal{O})\bigr\}.$$

However, as we proved above, all convolutions of this complex are isomorphic by Lemma 3.1.2. Thus, $\mathbf{R}J_*\mathcal{E}_1 \cong \mathbf{R}J_*\mathcal{E}_2$. Applying Proposition 3.3.5, we now see that the objects $\mathcal{E}_1$ and $\mathcal{E}_2$ are themselves also isomorphic.

3.5. Appendix: the n-Koszul property of a homogeneous coordinate algebra. The facts collected in this appendix are not original, and are well known in one form or another. However, in the absence of a good reference, we are obliged to present our own proof of the assertion used in the main text in the form we need it. Here we mainly use the technique of [18].

Let X be a smooth projective variety and $\mathcal{L}$ a very ample line bundle on X satisfying the additional condition $\mathrm{H}^i(X,\mathcal{L}^k)=0$ for all $k>0$ and $i\neq 0$. We write A for the homogeneous coordinate algebra of X with respect to $\mathcal{L}$, that is, $A=\bigoplus_{k=0}^{\infty}\mathrm{H}^0(X,\mathcal{L}^k)$.

Consider the variety X^n for some $n\in\mathbb{N}$. In what follows we write $\pi_i^{(n)}$ for the projection of X^n to the ith factor and $\pi_{ij}^{(n)}$ for its projection to the product of the ith and jth factors. Define a subvariety $\Delta^{(n)}_{(i_1,\dots,i_k)(i_{k+1},\dots,i_m)}\subset X^n$ as follows:

$$\Delta^{(n)}_{(i_1,\dots,i_k)(i_{k+1},\dots,i_m)} := \bigl\{(x_1,\dots,x_n)\bigm| x_{i_1}=\dots=x_{i_k};\ x_{i_{k+1}}=\dots=x_m\bigr\}.$$

For brevity, we write $S_i^{(n)}$ instead of $\Delta^{(n)}_{(n,n-1,\dots,i)}$; obviously, $S_i^{(n)}\cong X^i$.

Now set

$$T_i^{(n)} := \bigcup_{k=1}^{i-1}\Delta^{(n)}_{(n,n-1,\dots,i)(k,k-1)}, \qquad \Sigma^{(n)} := \bigcup_{k=1}^{n}\Delta^{(n)}_{(k,k-1)}.$$

(By definition, $T_1^{(n)}$ and $T_2^{(n)}$ are the empty subset.) It is clear that $T_i^{(n)}\subset S_i^{(n)}$.

We write $J_{\Sigma^{(n)}}$ for the ideal sheaf of the subscheme $\Sigma^{(n)}\subset X^n$ and $\mathcal{I}_i^{(n)}$ for the sheaf on X^n which is the kernel of the natural map $\mathcal{O}_{S_i^{(n)}}\longrightarrow\mathcal{O}_{T_i^{(n)}}\longrightarrow 0$.

Let us temporarily fix m and $k\leqslant m$. Let s be the embedding of the subvariety $S_k^{(m)}\cong X^{k-1}\times X$ into X^n, which by the definition of $S_k^{(m)}$ is the identity on the first $k-1$ factors and the diagonal on the final kth factor. We write p for the projection of $S_k^{(m)}$ to X^{k-1}, which is the product of the first $k-1$ factors.

Lemma 3.5.1. *The sheaf $\mathcal{I}_1^{(m)}$ is isomorphic to $\mathcal{O}_{\Delta^{(n)}_{(n,\dots,1)}}$; and $\mathcal{I}_k^{(m)}$ for $k>1$ is isomorphic to $s_*p^*(J_{\Sigma^{(k-1)}})$. In particular, for $k>1$ there are isomorphisms*

a) $\mathrm{H}^j(X^m,\mathcal{I}_k^{(m)}\otimes(\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}^i))$
$$= \mathrm{H}^j(X^{k-1},J_{\Sigma^{(k-1)}}\otimes(\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}))\otimes A_{m-k+i} \quad \textit{for all } i>0;$$
b) $\mathbf{R}^j\pi_{1*}^{(m)}(\mathcal{I}_k^{(m)}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}^i))$
$$\cong \mathbf{R}^j\pi_{1*}^{(k-1)}(J_{\Sigma^{(k-1)}}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}))\otimes A_{m-k+i} \quad \textit{for all } i>0;$$
c) $\mathbf{R}^j\pi_{1m*}^{(m)}(\mathcal{I}_k^{(m)}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}^i))$
$$\cong \mathbf{R}^j\pi_{1*}^{(k-1)}(J_{\Sigma^{(k-1)}}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}))\boxtimes\mathcal{L}^{m-k+i} \quad \textit{for all } i.$$

Proof. The assertion that $\mathcal{I}_k^{(m)}$ is isomorphic to $s_*p^*(J_{\Sigma^{(k-1)}})$ for $k>1$ follows at once from the definition of $\mathcal{I}_k^{(m)}$ and of the subschemes $T_k^{(m)}$ and $S_k^{(m)}$. The rest follows at once from this.

By induction on n, one sees readily that the complex

$$P_n^{\cdot}\colon 0\longrightarrow J_{\Sigma^{(n)}}\longrightarrow\mathcal{I}_n^{(n)}\longrightarrow\mathcal{I}_{n-1}^{(n)}\longrightarrow\dots\longrightarrow\mathcal{I}_2^{(n)}\longrightarrow\mathcal{I}_1^{(n)}\longrightarrow 0$$

on X^n is exact. For example, for $n=2$ this complex is the short exact sequence on $X\times X$,

$$P_2^\cdot\colon 0\longrightarrow J_\Delta\longrightarrow \mathcal{O}_{X\times X}\longrightarrow \mathcal{O}_\Delta\longrightarrow 0.$$

Lemma 3.5.2. *Let X be a smooth projective variety with ample line bundle $\mathcal{M}$. Then for any positive integer k there exists an i such that the bundle $\mathcal{L}=\mathcal{M}^i$ has the following properties for all $1<m\leqslant k$:*

$$\left.\begin{aligned} &\text{a)}\quad \mathrm{H}^j(X^m, J_{\Sigma(m)}\otimes(\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L}))=0 && \text{for}\quad j\neq 0;\\ &\text{b)}\quad \mathbf{R}^j\pi^{(m)}_{1*}(J_{\Sigma(m)}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L}))=0 && \text{for}\quad j\neq 0;\\ &\text{c)}\quad \mathbf{R}^j\pi^{(m)}_{1m*}(J_{\Sigma(m)}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L}\boxtimes\mathcal{O}))=0 && \text{for}\quad j\neq 0.\end{aligned}\right\}\tag{43}$$

Proof. For any m, the line bundles $\mathcal{M}\boxtimes\cdots\boxtimes\mathcal{M}$, $\mathcal{O}\boxtimes\mathcal{M}\boxtimes\cdots\boxtimes\mathcal{M}$, and $\mathcal{O}\boxtimes\mathcal{M}\boxtimes\cdots\boxtimes\mathcal{M}\boxtimes\mathcal{O}$ on X^m are ample, $\pi_1^{(m)}$-ample, and $\pi_{1m}^{(m)}$-ample, respectively. Therefore, for each of them there is an integer such that properties a), b), and c) hold for all powers of these bundles larger than this integer. Take the maximum of these numbers over all $m\leqslant k$, and denote it by i. Then properties a), b), and c) also hold for $\mathcal{L}=\mathcal{M}^i$.

We introduce the following notation:

$$\begin{aligned} B_n &:= \mathrm{H}^0(X^n, J_{\Sigma(n)}\otimes(\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L}))\\ \text{and}\quad R_{n-1} &:= \mathbf{R}^0\pi^{(n)}_{1*}(J_{\Sigma(n)}\otimes(\mathcal{O}\boxtimes\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L})).\end{aligned}$$

Proposition 3.5.3. *Let $\mathcal{L}$ be a very ample bundle on a smooth projective variety X satisfying condition* (43) *for all m with $1<m\leqslant n+\dim X+2$. Then*

1) *A is an n-Koszul algebra, that is, the sequence*

$$B_n\otimes_k A\longrightarrow B_{n-1}\otimes_k A\longrightarrow\cdots\longrightarrow B_1\otimes_k A\longrightarrow A\longrightarrow k\longrightarrow 0$$

is exact;

2) *the complex of sheaves on X of the form*

$$A_{k-n}\otimes R_n\longrightarrow A_{k-n+1}\otimes R_{n-1}\longrightarrow \cdots\longrightarrow A_{k-1}\otimes R_1\longrightarrow A_k\otimes R_0\longrightarrow \mathcal{L}^k\longrightarrow 0$$

is exact for any $k\geqslant 0$ (if $k-i<0$, then $A_{k-i}=0$ by definition);

3) *the complex of sheaves*

$$\mathcal{L}^{-n}\boxtimes R_n\longrightarrow\cdots\longrightarrow\mathcal{L}^{-1}\boxtimes R_1\longrightarrow\mathcal{O}_M\boxtimes R_0\longrightarrow\mathcal{O}_\Delta$$

on $X\times X$ is exact; *that is, it gives an n-resolution of the diagonal of $X\times X$.*

Proof. 1) First, combining Lemmas 3.5.1 and 3.5.2, for any $1 < m \leqslant n+\dim X+2$ we see that

$$\begin{aligned} &1) \quad \mathrm{H}^0(X^m, \mathcal{J}_k^{(m)} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L}^i)) \\ &\qquad = \mathrm{H}^0(X^{k-1}, J_{\Sigma^{(k-1)}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})) \otimes A_{m-k+i} = B_{k-1} \otimes A_{m-k+i}; \\ &2) \quad \mathrm{H}^j(X^m, \mathcal{J}_k^{(m)} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L}^i)) = 0 \quad \text{for} \quad j \neq 0. \end{aligned} \tag{44}$$

Consider the complexes $P_m^{\boldsymbol{\cdot}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})$ for $m \leqslant n+\dim X+1$. Applying H^0 to them and using condition (44), we obtain the exact sequences

$$0 \longrightarrow B_m \longrightarrow B_{m-1} \otimes_k A_1 \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m-1} \longrightarrow A_m \longrightarrow 0$$

for any $m \leqslant n+\dim X+1$.

We now set $m = n+\dim X+1$ and write $W_m^{\boldsymbol{\cdot}}$ for the complex

$$\mathcal{J}_m^{(m)} \longrightarrow \mathcal{J}_{m-1}^{(m)} \longrightarrow \cdots \longrightarrow \mathcal{J}_2^{(m)} \longrightarrow \mathcal{J}_1^{(m)} \longrightarrow 0,$$

which is a right resolution of $J_{\Sigma^{(m)}}$. We take the complex

$$W_m^{\boldsymbol{\cdot}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^i)$$

and apply the functor H^0 to it. We obtain the sequence

$$B_{m-1} \otimes_k A_i \longrightarrow B_{m-2} \otimes_k A_{i+1} \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m+i-2} \longrightarrow A_{m+i-1} \longrightarrow 0.$$

It follows from (44), 2) that the cohomology of this complex equals

$$\mathrm{H}^j(X^m, J_{\Sigma^{(m)}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^i)).$$

And (43), b) gives us that

$$\begin{aligned} &\mathrm{H}^j(X^m, J_{\Sigma^{(m)}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^i)) \\ &\qquad\qquad = \mathrm{H}^j(X, \mathbf{R}^0\pi_{m*}^{(m)}(J_{\Sigma^{(m)}} \otimes (\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{O})) \otimes \mathcal{L}^i). \end{aligned}$$

Hence, this cohomology is trivial for $j > \dim X$, and thus there is an exact sequence of the form

$$B_n \otimes_k A_{m-n+i-1} \longrightarrow B_{n-1} \otimes_k A_{m-n+i} \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m+i-2} \longrightarrow A_{m+i-1} \longrightarrow 0$$

for $i \geqslant 1$. However, the exactness for $i \leqslant 1$ has been proved above. Thus, A is an n-Koszul algebra.

2) The proof is similar to that of 1). We have isomorphisms

$$\begin{aligned} &1) \quad \mathbf{R}^0\pi_{1*}^{(m)}(\mathcal{J}_k^{(m)} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})) \\ &\qquad \cong \mathbf{R}^0\pi_{1*}^{(k-1)}(J_{\Sigma^{(k-1)}} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})) \otimes A_{m-k+1}; \\ &2) \quad \mathbf{R}^j\pi_{1*}^{(m)}(\mathcal{J}_k^{(m)} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})) = 0 \quad \text{for all } j \neq 0. \end{aligned} \tag{45}$$

Applying the functor $\mathbf{R}^0\pi_{1*}^{(m)}$ to the complexes $P_m^{\cdot} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})$ for $m \leqslant n + \dim X + 2$, we obtain an exact complex of sheaves on X,

$$0 \longrightarrow R_{m-1} \longrightarrow A_1 \otimes R_{m-2} \longrightarrow \cdots \longrightarrow A_{m-2} \otimes R_1 \longrightarrow A_{m-1} \otimes R_0 \longrightarrow \mathcal{L}^{m-1} \longrightarrow 0$$

for $m \leqslant n + \dim X + 2$.

We consider the case $m = n + \dim X + 2$. Applying the functor $\mathbf{R}^0\pi_{1*}^{(m)}$ to

$$W_m^{\cdot} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^i),$$

gives the complex

$$A_i \otimes R_{m-2} \longrightarrow \cdots \longrightarrow A_{m+i-3} \otimes R_1 \longrightarrow A_{m+i-2} \otimes R_0 \longrightarrow \mathcal{L}^{m+i-2} \longrightarrow 0.$$

By property (45), its cohomology is

$$\begin{aligned} &\mathbf{R}^j\pi_{1*}^{(m)}(J_{\Sigma^{(m)}} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^i)) \\ &\qquad \cong \mathbf{R}^j p_{1*}(\mathbf{R}^0\pi_{1m*}^{(m)}(J_{\Sigma^{(m)}} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{O})) \otimes (\mathcal{O} \boxtimes \mathcal{L}^i)), \end{aligned}$$

which is trivial for $j > \dim X$. Thus, the sequence of sheaves

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_{k-1} \otimes R_1 \longrightarrow A_k \otimes R_0 \longrightarrow \mathcal{L}^k \longrightarrow 0$$

on X is exact for all $k \geqslant 0$.

3) Consider the complex $W_{n+2}^{\cdot} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{L}^{-n})$ on X^{n+2}. Applying $\mathbf{R}^0\pi_{1(n+2)*}^{(n+2)}$ to it, we obtain the following complex on $X \times X$:

$$\mathcal{L}^{-n} \boxtimes R_n \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes R_1 \longrightarrow \mathcal{O}_M \boxtimes R_0 \longrightarrow \mathcal{O}_\Delta. \tag{46}$$

Recalling condition c) of Lemma 3.5.1 and condition (43), b), we see that its cohomology sheaves are isomorphic to

$$\mathbf{R}^j\pi_{1(n+2)*}^{(n+2)}(J_{\Sigma^{(n+2)}} \otimes (\mathcal{O} \boxtimes \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L} \boxtimes \mathcal{O})) \otimes (\mathcal{O} \boxtimes \mathcal{L}^{-n}),$$

which vanish for $j > 0$ by (43), c). That is, the complex (46) is exact.

CHAPTER 4

Derived categories of coherent sheaves on K3 surfaces

4.1. K3 surfaces and the Mukai lattice. This chapter is entirely taken up with derived categories of coherent sheaves on K3 surfaces over the field of complex numbers. The main question we are interested in, and answer in this chapter, is as follows: when do two distinct K3 surfaces have equivalent categories of coherent sheaves? As before, we view derived categories as triangulated categories, and equivalences are understood as equivalences between triangulated categories.

We recall that for smooth projective varieties with ample canonical or anticanonical class there is a procedure (see Theorem 2.1.3) for recovering the variety from its derived category of coherent sheaves. However, for varieties of other types this is a non-trivial question, and is especially interesting for varieties with trivial canonical class.

We start with the main facts we need concerning K3 surfaces. Recall that a K3 surface is a smooth compact algebraic surface S with $K_S = 0$ and $\mathrm{H}^1(S,\mathbb{Z}) = 0$. These surfaces are actually simply connected. One can show that the second cohomology $\mathrm{H}^2(S,\mathbb{Z})$ is torsion-free and is an even lattice of rank 22 with respect to the intersection form. Moreover, it follows from the Noether formula that $p_g(S) = 1$ and $h^{1,1}(S) = 20$.

One of the main invariants of a K3 surface is its Néron–Severi group $\mathrm{NS}(S) \subset \mathrm{H}^2(S,\mathbb{Z})$, which coincides in this case with the Picard group $\mathrm{Pic}(S)$. The rank of $\mathrm{NS}(S)$ is $\leqslant h^{1,1} = 20$. We write T_S for the lattice of transcendental cycles which, by definition, is the orthogonal complement to the Néron–Severi lattice $\mathrm{NS}(S)$ in the second cohomology $\mathrm{H}^2(S,\mathbb{Z})$.

We denote by td_S the Todd class of the surface S; this class is an element of the form $1 + 2w$ in $\mathrm{H}^*(S,\mathbb{Q})$, where $1 \in \mathrm{H}^0(S,\mathbb{Z})$ is the identity of the cohomology ring $\mathrm{H}^*(S,\mathbb{Z})$ and $w \in \mathrm{H}^4(S,\mathbb{Z})$ is the fundamental cocycle of S. We consider the positive square root $\sqrt{\mathrm{td}_S} = 1 + w$; for any K3 surface it belongs to the integral cohomology ring $\mathrm{H}^*(S,\mathbb{Z})$.

One introduces the Chern character for any coherent sheaf on S, and extends it by additivity to the entire derived category of coherent sheaves. If F is an object of $\mathbf{D}^b(S)$, its Chern character

$$\mathrm{ch}(F) = r(F) + c_1(F) + \frac{1}{2}(c_1^2 - 2c_2)$$

belongs to the integral cohomology $\mathrm{H}^*(S,\mathbb{Z})$. For any object F we define the element

$$v(F) = \mathrm{ch}(F)\cdot\sqrt{\mathrm{td}_S} \in \mathrm{H}^*(S,\mathbb{Z})$$

and call it the *vector associated with* F (or the *Mukai vector of* F).

We define a symmetric bilinear form on the cohomology lattice $\mathrm{H}^*(S,\mathbb{Z})$ by the rule

$$(u,u') = r\cdot s' + s\cdot r' - \alpha\cdot\alpha' \in \mathrm{H}^4(S,\mathbb{Z}) \cong \mathbb{Z}$$

for any pair $u = (r,\alpha,s)$, $u' = (r',\alpha',s') \in \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$. The cohomology lattice $\mathrm{H}^*(S,\mathbb{Z})$ together with this bilinear form $(\,\cdot\,,\,\cdot\,)$ is called the *Mukai lattice* and denoted by $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. Note that on H^2 the bilinear form $(\,\cdot\,,\,\cdot\,)$ differs from the usual intersection form by the minus sign. Thus, the Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is isomorphic to the lattice $\mathrm{U}\perp -\mathrm{H}^2(S,\mathbb{Z})$, where U is the hyperbolic lattice $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\perp$ means orthogonal direct sum.

For any two objects F and G, the pairing $(v(F),v(G))$ is by definition the component in H^4 of the element $\mathrm{ch}(F)^\vee\cdot\mathrm{ch}(G)\cdot\mathrm{td}_S$. Hence, by the Grothendieck Riemann–Roch theorem we have the equality

$$(v(F),v(G)) = \chi(F,G) := \sum_i (-1)^i \dim\mathrm{Ext}^i(F,G).$$

The lattices $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ and T_S admit natural Hodge structures. Here by a Hodge structure we mean that the spaces $\widetilde{\mathrm{H}}(S,\mathbb{C})$ and $T_S\otimes\mathbb{C}$ have a fixed one-dimensional subspace $\mathrm{H}^{2,0}(S)$.

Definition 4.1.1. Let S_1 and S_2 be two K3 surfaces. We say that the Mukai lattices of S_1 and S_2 (or their lattices of transcendental cycles) are *Hodge isometric* if there is an isometry between the lattices that takes the one-dimensional subspace $\mathrm{H}^{2,0}(S_1)$ to $\mathrm{H}^{2,0}(S_2)$.

Let $\mathcal{E}$ in $\mathbf{D}^b(S_1 \times S_2)$ be an arbitrary object of the derived category of the product. Consider the algebraic cycle

$$Z_{\mathcal{E}} := p^* \sqrt{\mathrm{td}_{S_1}} \cdot \mathrm{ch}(\mathcal{E}) \cdot \pi^* \sqrt{\mathrm{td}_{S_2}} \tag{47}$$

on the product $S_1 \times S_2$, where p and π are the projections in the diagram

$$\begin{array}{ccc} S_1 \times S_2 & \xrightarrow{\pi} & S_2 \\ {\scriptstyle p}\big\downarrow & & \\ S_1 & & \end{array} \quad .$$

In the case of K3 surfaces the cycle $Z_{\mathcal{E}}$, which is *a priori* rational, is in fact integral:

Lemma 4.1.2 [31]. *For any object $\mathcal{E} \in \mathbf{D}^b(S_1 \times S_2)$, both the Chern character $\mathrm{ch}(\mathcal{E})$ and the cycle $Z_{\mathcal{E}}$ are integral, that is, they belong to $\mathrm{H}^*(S_1 \times S_2, \mathbb{Z})$.*

Thus, the cycle $Z_{\mathcal{E}}$ defines a map from the integral cohomology lattice of S_1 to that of S_2,

$$\begin{array}{cccc} f_{Z_{\mathcal{E}}}\colon & \mathrm{H}^*(S_1, \mathbb{Z}) & \longrightarrow & \mathrm{H}^*(S_2, \mathbb{Z}) \\ & \cup & & \cup \\ & \alpha & \longmapsto & \pi_*(Z_{\mathcal{E}} \cdot p^*(\alpha)) \end{array} \quad . \tag{48}$$

The following proposition is analogous to Theorem 4.9 in [31].

Proposition 4.1.3. *For an object $\mathcal{E}$, if the functor $\Phi_{\mathcal{E}} \colon \mathbf{D}^b(S_1) \longrightarrow \mathbf{D}^b(S_2)$ is fully faithful, then*

1) *$f_{Z_{\mathcal{E}}}$ is an isometry between the lattices $\widetilde{\mathrm{H}}(S_1, \mathbb{Z})$ and $\widetilde{\mathrm{H}}(S_2, \mathbb{Z})$,*
2) *the inverse map of f coincides with the homomorphism*

$$\begin{array}{cccc} f'\colon & \mathrm{H}^*(S_2, \mathbb{Z}) & \longrightarrow & \mathrm{H}^*(S_1, \mathbb{Z}) \\ & \cup & & \cup \\ & \beta & \longmapsto & p_*(Z_{\mathcal{E}}^{\vee} \cdot \pi^*(\beta)) \end{array}$$

defined by the cycle

$$Z_{\mathcal{E}}^{\vee} = p^* \sqrt{\mathrm{td}_{S_1}} \cdot \mathrm{ch}(\mathcal{E}^{\vee}) \cdot \pi^* \sqrt{\mathrm{td}_{S_2}},$$

where $\mathcal{E}^{\vee} := \mathbf{R}^{\cdot}\underline{\mathcal{H}om}(\mathcal{E}, \mathcal{O}_{S_1 \times S_2})$.

Proof. The left and right adjoint functors to $\Phi_{\mathcal{E}}$ are isomorphic; they are given by the formula

$$\Phi_{\mathcal{E}}^* = \Phi_{\mathcal{E}}^! = \mathbf{R}p_*(\mathcal{E}^{\vee} \otimes^{\mathbf{L}} \pi^*(\,\cdot\,))[2].$$

Since $\Phi_{\mathcal{E}}$ is fully faithful, it follows that the composite $\Phi_{\mathcal{E}}^* \circ \Phi_{\mathcal{E}}$ is isomorphic to the identity functor $\mathrm{id}_{\mathbf{D}^b(S_1)}$. The identity functor $\mathrm{id}_{\mathbf{D}^b(S_1)}$ is defined by the structure sheaf $\mathcal{O}_{\Delta}$ of the diagonal $\Delta \subset S_1 \times S_1$.

By the projection formula and the Grothendieck Riemann–Roch theorem, one finds that the composite $f'\circ f$ is represented by the cycle $p_1^*\sqrt{\mathrm{td}_{S_1}}\cdot\mathrm{ch}(\mathcal{O}_\Delta)\cdot p_2^*\sqrt{\mathrm{td}_{S_1}}$, where p_1 and p_2 are the projections of $S_1\times S_1$ to its factors. Using the Grothendieck Riemann–Roch theorem again, we see that this cycle is equal to Δ. Hence, the composite $f'\circ f$ is the identity map, and thus f is an isomorphism from $\mathrm{H}^*(S_1,\mathbb{Z})$ to $\mathrm{H}^*(S_2,\mathbb{Z})$, because both groups are free Abelian groups of the same rank.

Write $\nu_S\colon S\longrightarrow\operatorname{Spec}\mathbb{C}$ for the structure morphism of S. Then we can express the pairing (α,α') on $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ as $\nu_*(\alpha^\vee\cdot\alpha')$. It follows from the projection formula that

$$\begin{aligned}(\alpha,f(\beta))&=\nu_{S_2,*}(\alpha^\vee\cdot\pi_*(\pi^*\sqrt{\mathrm{td}_{S_2}}\cdot\mathrm{ch}(\mathcal{E})\cdot p^*\sqrt{\mathrm{td}_{S_1}}\cdot p^*(\beta)))\\&=\nu_{S_2,*}\pi_*(\pi^*(\alpha^\vee)\cdot p^*(\beta)\cdot\mathrm{ch}(\mathcal{E})\cdot\sqrt{\mathrm{td}_{S_1\times S_2}})\\&=\nu_{S_1\times S_2,*}(\pi^*(\alpha^\vee)\cdot p^*(\beta)\cdot\mathrm{ch}(\mathcal{E})\cdot\sqrt{\mathrm{td}_{S_1\times S_2}})\end{aligned}$$

for arbitrary $\alpha\in\mathrm{H}^*(S_2,\mathbb{Z})$ and $\beta\in\mathrm{H}^*(S_1,\mathbb{Z})$. In the same way we see that

$$(\beta,f'(\alpha))=\nu_{S_1\times S_2,*}(p^*(\beta^\vee)\cdot\pi^*(\alpha)\cdot\mathrm{ch}(\mathcal{E})^\vee\cdot\sqrt{\mathrm{td}_{S_1\times S_2}}).$$

Hence, $(\alpha,f(\beta))=(f'(\alpha),\beta)$. Since $f'\circ f$ is the identity map, it follows that

$$(f(\alpha),f(\alpha'))=(f'f(\alpha),\alpha')=(\alpha,\alpha').$$

Thus, f is an isometry.

4.2. The criterion for equivalence of derived categories of coherent sheaves. In this section we give a criterion for the derived categories of coherent sheaves on two K3 surfaces to be equivalent as triangulated categories. The form of this criterion is very reminiscent of the Torelli theorem for K3 surfaces, which says that two K3 surfaces S_1 and S_2 are isomorphic if and only if their lattices of second cohomology are Hodge isometric, that is, there is an isometry

$$\mathrm{H}^2(S_1,\mathbb{Z})\xrightarrow{\sim}\mathrm{H}^2(S_2,\mathbb{Z})$$

whose extension to complex cohomology takes $\mathrm{H}^{2,0}(S_1)$ to $\mathrm{H}^{2,0}(S_2)$ (see [39], [27]).

The main result of this chapter is as follows.

Theorem 4.2.1. *Let S_1 and S_2 be two smooth projective $K3$ surfaces over the field of complex numbers $\mathbb{C}$. Then the derived categories of coherent sheaves $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ are equivalent as triangulated categories if and only if there exists a Hodge isometry $f\colon\widetilde{\mathrm{H}}(S_1,\mathbb{Z})\xrightarrow{\sim}\widetilde{\mathrm{H}}(S_2,\mathbb{Z})$ between the Mukai lattices of S_1 and S_2.*

There is another version of this theorem (Theorem 4.2.4) which may also be of interest.

We break up the proof of Theorem 4.2.1 into two propositions. The proof of the first proposition depends essentially on the main Theorem 3.2.1 of the preceding chapter, since it uses the fact that every equivalence of derived categories can be represented by an object on the product.

Proposition 4.2.2. *Let S_1 and S_2 be two $K3$ surfaces whose derived categories of coherent sheaves are equivalent. Then there is a Hodge isometry between the lattices of transcendental cycles T_{S_1} and T_{S_2}.*

Proof. By Theorem 3.2.2, there is an object $\mathcal{E}$ on the product $S_1 \times S_2$ that defines the equivalence. It follows from Proposition 4.1.3 that $f_{Z_{\mathcal{E}}}$ defines a Hodge isometry between the Mukai lattices $\widetilde{\mathrm{H}}(S_1, \mathbb{Z})$ and $\widetilde{\mathrm{H}}(S_2, \mathbb{Z})$. Since the cycle Z is algebraic, we obtain two isometries

$$f_{\mathrm{alg}}\colon\ -\mathrm{NS}(S_1)\perp \mathrm{U} \xrightarrow{\ \sim\ } -\mathrm{NS}(S_2)\perp \mathrm{U} \quad \text{and} \quad f_{\tau}\colon T_{S_1} \xrightarrow{\ \sim\ } T_{S_2},$$

where $\mathrm{NS}(S_1)$ and $\mathrm{NS}(S_2)$ are the Néron–Severi lattices and T_{S_1} and T_{S_2} the lattices of transcendental cycles. It is obvious that f_τ is a Hodge isometry.

The proof of the converse uses in an essential way the results of [31], which studied moduli spaces of bundles on K3 surfaces, and it also uses Theorem 2.1.5, which gave a criterion for a functor to be fully faithful (see [7]).

Proposition 4.2.3. *Let S_1 and S_2 be two projective $K3$ surfaces. Suppose that there exists a Hodge isometry*

$$f\colon \widetilde{\mathrm{H}}(S_2, \mathbb{Z}) \xrightarrow{\ \sim\ } \widetilde{\mathrm{H}}(S_1, \mathbb{Z}).$$

Then the bounded derived categories of coherent sheaves $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ are equivalent.

Proof. We set $v = f(0,0,1) = (r,l,s)$ and $u = f(1,0,0) = (p,k,q)$. Without loss of generality we can assume that $r > 1$. Indeed, a Mukai lattice has two types of Hodge isometries. The first type is multiplication by the Chern character $\exp(m)$ of a line bundle:

$$\varphi_m(r,l,s) = \Bigl(r, l + rm, s + (m,l) + \frac{r}{2}m^2\Bigr).$$

The second type is the transposition of r and s. Using these two types of permutations, one can replace f in such a way that r becomes greater than 1.

The vector $v \in \mathrm{U} \perp -\mathrm{NS}(S_1)$ is obviously isotropic, that is, $(v,v) = 0$. In his brilliant paper [31] Mukai proved that, in this case, there is a polarization A on the K3 surface S_1 such that the moduli space $\mathcal{M}_A(v)$ of vector bundles whose Mukai vector coincides with v and that are stable with respect to A is a smooth projective K3 surface. Moreover, since there is a vector $u \in \mathrm{U} \perp -\mathrm{NS}(S_1)$ such that $(v,u) = 1$, it follows that $\mathcal{M}_A(v)$ is a fine moduli space. Hence, there exists a universal bundle $\mathcal{E}$ on the product $S_1 \times \mathcal{M}_A(v)$.

The universal bundle $\mathcal{E}$ defines a functor $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(\mathcal{M}_A(v)) \longrightarrow \mathbf{D}^b(S_1)$. One sees readily that this functor satisfies the conditions of Theorem 2.1.5. Indeed, we have $\Phi_{\mathcal{E}}(\mathcal{O}_t) = \mathcal{E}_t$, where $\mathcal{E}_t$ is a stable bundle on S_1 for which $v(\mathcal{E}_t) = v$. All the sheaves $\mathcal{E}_t$ are simple, and we of course have $\mathrm{Ext}^i(\mathcal{E}_t, \mathcal{E}_t) = 0$ for $i \notin \{0,2\}$. This gives condition 2) of Theorem 2.1.5.

Since the $\mathcal{E}_t$ are stable, it follows that $\mathrm{Hom}(\mathcal{E}_{t_1}, \mathcal{E}_{t_2}) = 0$. By Serre duality, also $\mathrm{Ext}^2(\mathcal{E}_{t_1}, \mathcal{E}_{t_2}) = 0$. Since the vector v is isotropic, we also have $\mathrm{Ext}^1(\mathcal{E}_{t_1}, \mathcal{E}_{t_2}) = 0$.

Thus, the sheaves $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$ are orthogonal for any two distinct points t_1 and t_2. Theorem 2.1.5 gives us that the functor $\Phi_{\mathcal{E}}$ is fully faithful.

In fact, $\Phi_{\mathcal{E}}$ is not just fully faithful, but an equivalence of categories. This can be shown by the following argument, which is based on the proof of Theorem 3.3 of [9]. Write $\mathcal{D}$ for the image of $\mathbf{D}^b(\mathcal{M}_A(v))$ in $\mathbf{D}^b(S_1)$. Since it is an admissible subcategory (see Definition 2.2.2), it admits right and left orthogonals; since the canonical class of a K3 surface is trivial, it follows that these orthogonals coincide. Thus, the semi-orthogonal decomposition of the form $\langle \mathcal{D}^{\perp}, \mathcal{D}\rangle$ is completely orthogonal. Consider a very ample line bundle $\mathcal{L}$ on $\mathcal{M}_A(v)$. All the powers $\mathcal{L}^i$ are indecomposable objects, and therefore belong to one or other of the subcategories $\mathcal{D}$ or $\mathcal{D}^{\perp}$, and they all belong to the same one, because no pair of these objects is completely orthogonal. However, the powers $\{\mathcal{L}^i\}$ form an ample sequence (see Definition 3.4.1). By Lemma 3.4.3, the orthogonal to a subcategory generated by an ample sequence is 0. Thus, since $\mathcal{D}$ is non-trivial, it follows that $\mathcal{D}^{\perp} = 0$. Hence, $\Phi_{\mathcal{E}}$ is an equivalence.

Next, the cycle $Z_{\mathcal{E}}$ defined by (47) induces a Hodge isometry

$$g\colon \widetilde{\mathrm{H}}(\mathcal{M}_A(v), \mathbb{Z}) \to \widetilde{\mathrm{H}}(S_1, \mathbb{Z})$$

for which $g(0,0,1) = v = (r,l,s)$. Hence, $f^{-1}\circ g$ is also a Hodge isometry, and takes $(0,0,1)$ to $(0,0,1)$. Thus, $f^{-1}\cdot g$ induces a Hodge isometry between the second cohomology lattices of S_2 and $\mathcal{M}_A(v)$. Therefore, by the Torelli theorem ([39], [27]), S_2 and $\mathcal{M}_A(v)$ are isomorphic.

This proposition together with Proposition 4.1.3 prove Theorem 4.2.1. There is another version of Theorem 4.2.1, which gives a criterion for equivalence of derived categories in terms of the lattices of transcendental cycles.

Theorem 4.2.4. *Let S_1 and S_2 be two smooth projective K3 surfaces over $\mathbb{C}$. Then the derived categories of coherent sheaves $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ are equivalent as triangulated categories if and only if there exists a Hodge isometry $f_\tau\colon \mathrm{T}_{S_1} \xrightarrow{\sim} \mathrm{T}_{S_2}$ between the lattices of transcendental cycles of S_1 and S_2.*

This assertion is a corollary of Theorem 4.2.1 and the following proposition.

Proposition 4.2.5 [33]. *Let $\varphi_1, \varphi_2\colon \mathrm{T} \longrightarrow \mathrm{H}$ be two primitive embeddings of the lattice T into an even unimodular lattice H. Suppose that the orthogonal complement $\mathrm{N} := \varphi_1(T)^{\perp}$ in H contains the hyperbolic lattice $\mathrm{U} = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ as a sublattice. Then φ_1 and φ_2 are equivalent, that is, there is an isometry γ of H such that $\varphi_1 = \gamma\varphi_2$.*

Indeed, suppose that there is a Hodge isometry

$$f_\tau\colon \mathrm{T}_{S_2} \xrightarrow{\sim} \mathrm{T}_{S_1}\,.$$

As we know, the orthogonal complement to the lattice of transcendental cycles T_S in the Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is isomorphic to the lattice $-\mathrm{NS}(S) \perp \mathrm{U}$. Thus, by the previous proposition (Proposition 4.2.5), there is an isometry

$$f\colon \widetilde{\mathrm{H}}(S_2, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S_1, \mathbb{Z})$$

such that $f_{|\,\mathrm{T}_{S_2}} = f_\tau$. Thus, the isometry f is also a Hodge isometry. Therefore, by Theorem 4.2.1, the derived categories of coherent sheaves on S_1 and S_2 are equivalent.

CHAPTER 5

Abelian varieties

5.1. Equivalences between categories of coherent sheaves on Abelian varieties. In this chapter we study derived categories of coherent sheaves on Abelian varieties and their groups of auto-equivalences. Let A be an Abelian variety of dimension n over a field k. We write $m\colon A\times A\to A$ for the composition morphism, which is assumed to be defined over k, and e for the k-point which is the identity of the group structure. For any k-point $a\in A$ there is a translation automorphism $m(\,\cdot\,,a)\colon A\to A$, which we denote by T_a.

We write $\widehat{A}$ for the dual Abelian variety, which is the moduli space of line bundles on A belonging to $\operatorname{Pic}^0(A)$. Moreover, $\widehat{A}$ is a fine moduli space. Therefore, there exists a universal line bundle $\mathcal{P}$ on the product $A\times\widehat{A}$, called the *Poincaré bundle*. It is uniquely determined by the condition that for any k-point $\alpha\in\widehat{A}$ the restriction of $\mathcal{P}$ to $A\times\{\alpha\}$ is isomorphic to the line bundle in $\operatorname{Pic}^0(A)$ corresponding to α, and, in addition, the restriction $\mathcal{P}_{|\{e\}\times\widehat{A}}$ should be trivial.

Definition 5.1. In what follows we denote the line bundle on A corresponding to a k-point $\alpha\in\widehat{A}$ by $\mathcal{P}_\alpha$. Moreover, given a number of Abelian varieties $A_1,\dots,A_m$ and a k-point $(\alpha_1,\dots,\alpha_m)\in\widehat{A}_1\times\dots\times\widehat{A}_m$, we denote by $\mathcal{P}_{(\alpha_1,\dots,\alpha_k)}$ the line bundle $\mathcal{P}_{\alpha_1}\boxtimes\dots\boxtimes\mathcal{P}_{\alpha_k}$ on the product $A_1\times\dots\times A_k$.

For any homomorphism $f\colon A\to B$ of Abelian varieties one defines a dual homomorphism $\widehat{f}\colon\widehat{B}\to\widehat{A}$. Pointwise, it acts by taking a point $\beta\in\widehat{B}$ to $\alpha\in\widehat{A}$ if and only if the line bundle $f^*\mathcal{P}_\beta$ on A coincides with the bundle $\mathcal{P}_\alpha$.

The double dual or bidual Abelian variety $\widehat{\widehat{A}}$ is naturally identified with A using the Poincaré bundles on $A\times\widehat{A}$ and on $\widehat{A}\times\widehat{\widehat{A}}$. In other words, there is a unique isomorphism $\kappa_A\colon A\xrightarrow{\sim}\widehat{\widehat{A}}$ such that the pull-back of the Poincaré bundle $\mathcal{P}_{\widehat{A}}$ under the isomorphism $1\times\kappa_A\colon\widehat{A}\times A\xrightarrow{\sim}\widehat{A}\times\widehat{\widehat{A}}$ coincides with the Poincaré bundle $\mathcal{P}_A$, that is, $(1\times\kappa_A)^*P_{\widehat{A}}\cong\mathcal{P}_A$. Thus, $\widehat{\ }$ is an involution on the category of Abelian varieties; that is, it is a contravariant functor whose square is isomorphic to the identity functor: $\kappa\colon\mathrm{id}\xrightarrow{\sim}\widehat{\widehat{\ }}$.

The Poincaré bundle $\mathcal{P}$ gives an example of an exact equivalence between the derived categories of coherent sheaves on two varieties A and $\widehat{A}$ that are not in general isomorphic. Consider the projections

$$A\xleftarrow{p}A\times\widehat{A}\xrightarrow{q}\widehat{A}$$

and the functor $\Phi_{\mathcal{P}}\colon\mathbf{D}^b(A)\longrightarrow\mathbf{D}^b(\widehat{A})$ defined by (7):

$$\Phi_{\mathcal{P}}(\,\cdot\,)=\mathbf{R}q_*(\mathcal{P}\otimes p^*(\,\cdot\,)).$$

The following proposition was proved in [29].

Proposition 5.1.2 [29]. *Let $\mathcal{P}$ be the Poincaré bundle on $A \times \widehat{A}$. Then the functor $\Phi_{\mathcal{P}} \colon \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(\widehat{A})$ is an exact equivalence, and there exists an isomorphism of functors*

$$\Psi_{\mathcal{P}} \circ \Phi_{\mathcal{P}} \cong (-1_A)^*[n],$$

where (-1_A) is the group inverse of A.

Remark 5.1.3. In [29] this assertion was proved for Abelian varieties over an algebraically closed field. However, it also holds over an arbitrary field because the dual variety and the Poincaré bundle are always defined over the same field (see, for example, [32]). And the assertion concerning the equivalence of categories will follow from Lemma 5.1.9.

Consider a k-point $(a, \alpha) \in A \times \widehat{A}$. To any such point one can assign a functor from $\mathbf{D}^b(A)$ to itself by the rule

$$\Phi_{(a,\alpha)}(\,\cdot\,) := T_{a*}(\,\cdot\,) \otimes \mathcal{P}_\alpha = T^*_{-a}(\,\cdot\,) \otimes \mathcal{P}_\alpha. \tag{49}$$

The functor $\Phi_{(a,\alpha)}$ is represented by the sheaf

$$S_{(a,\alpha)} = \mathcal{O}_{\Gamma_a} \otimes p_2^*(\mathcal{P}_\alpha) \tag{50}$$

on the product $A \times A$, where Γ_a stands for the graph of the translation automorphism T_a. The functor $\Phi_{(a,\alpha)}$ is obviously an auto-equivalence.

The set of all functors $\Phi_{(a,\alpha)}$ parametrized by $A \times \widehat{A}$ can be collected into a single functor from $\mathbf{D}^b(A \times \widehat{A})$ to $\mathbf{D}^b(A \times A)$ that takes the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of a point to $S_{(a,\alpha)}$. (We note that this condition does not define the functor uniquely, but only uniquely up to multiplication by a line bundle lifted from $A \times \widehat{A}$.)

We define the required functor $\Phi_{S_A} \colon \mathbf{D}^b(A \times \widehat{A}) \longrightarrow \mathbf{D}^b(A \times A)$ as the composite of two other functors.

Consider the object $\mathrm{P}_A = p_{14}^*\mathcal{O}_\Delta \otimes p_{23}^*\mathcal{P} \in \mathbf{D}^b((A \times \widehat{A}) \times (A \times A))$, and write $\mu_A \colon A \times A \longrightarrow A \times A$ for the morphism taking (a_1, a_2) to $(a_1, m(a_1, a_2))$. We obtain two functors,

$$\Phi_{\mathrm{P}_A} \colon \mathbf{D}^b(A \times \widehat{A}) \longrightarrow \mathbf{D}^b(A \times A), \qquad \mathbf{R}\mu_{A*} \colon \mathbf{D}^b(A \times A) \longrightarrow \mathbf{D}^b(A \times A).$$

Definition 5.1.4. The functor Φ_{S_A} is the composite $\mathbf{R}\mu_{A*} \circ \Phi_{\mathrm{P}_A}$.

Proposition 5.1.5. *The functor Φ_{S_A} is an equivalence of categories. For any k-point $(a, \alpha) \in A \times \widehat{A}$ it takes*

a) *the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of (a, α) to the sheaf $S_{(a,\alpha)}$ defined by* (50),

b) *the line bundle $\mathcal{P}_{(\alpha,a)}$ on $A \times \widehat{A}$ to the object $\mathcal{O}_{\{-a\} \times A} \otimes p_2^*\mathcal{P}_\alpha[n]$.*

Proof. By definition, Φ_{S_A} is the composite of the functors $\mathbf{R}\mu_{A*}$ and Φ_{P_A}, which are equivalences; this is obvious for the first functor and for the second it follows from Propositions 2.1.7 and 5.1.2.

The functor Φ_{P_A} takes the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of a point to $\mathcal{O}_{A \times \{a\}} \otimes p_1^*\mathcal{P}_\alpha$. Moreover, $\mathbf{R}\mu_{A*}$ takes $\mathcal{O}_{A \times \{a\}} \otimes p_1^*\mathcal{P}_\alpha$ to $\mathcal{O}_{\Gamma_a} \otimes p_1^*(\mathcal{P}_\alpha)$.

In the same way, applying Proposition 5.1.2, we see that Φ_{P_A} takes the line bundle $\mathcal{P}_{(\alpha,a)}$ to the object $\mathcal{O}_{\{-a\}\times A}\otimes p_2^*\mathcal{P}_\alpha[n]$, and $\mathbf{R}\mu_{A*}$ takes $\mathcal{O}_{\{-a\}\times A}\otimes p_2^*\mathcal{P}_\alpha[n]$ to itself.

Suppose that A and B are two Abelian varieties whose derived categories of coherent sheaves are equivalent. Let us fix some equivalence. By Theorem 3.2.2, it can be represented by an object on the product. Thus, there is an object $\mathcal{E}\in\mathbf{D}^b(A\times B)$ and an equivalence $\Phi_{\mathcal{E}}\colon\mathbf{D}^b(A)\xrightarrow{\sim}\mathbf{D}^b(B)$.

Consider the functor

$$Ad_{\mathcal{E}}\colon\mathbf{D}^b(A\times A)\xrightarrow{\sim}\mathbf{D}^b(B\times B),$$

defined by (11), which is an equivalence. And consider the composite of functors $\Phi_{S_B}^{-1}\circ Ad_{\mathcal{E}}\circ\Phi_{S_A}$.

Definition 5.1.6. We denote by $\mathcal{J}(\mathcal{E})$ the object representing the functor

$$\Phi_{S_B}^{-1}\circ Ad_{\mathcal{E}}\circ\Phi_{S_A}.$$

Thus, we have the commutative diagram

$$\begin{tikzcd}
\mathbf{D}^b(A\times\widehat{A}) \arrow[r, "\Phi_{S_A}"] \arrow[d, "\Phi_{\mathcal{J}(\mathcal{E})}"'] & \mathbf{D}^b(A\times A) \arrow[d, "Ad_{\mathcal{E}}"] \\
\mathbf{D}^b(B\times\widehat{B}) \arrow[r, "\Phi_{S_B}"] & \mathbf{D}^b(B\times B)
\end{tikzcd}\quad . \tag{51}$$

The following theorem allows us to compute the object $\mathcal{J}(\mathcal{E})$; it is the main tool for describing Abelian varieties having equivalent derived categories of coherent sheaves.

Theorem 5.1.7. *There exists a homomorphism of Abelian varieties $f_{\mathcal{E}}\colon A\times\widehat{A}\longrightarrow B\times\widehat{B}$ which is an isomorphism, and a line bundle $\mathcal{L}_{\mathcal{E}}$ on $A\times\widehat{A}$ such that the object $\mathcal{J}(\mathcal{E})$ is isomorphic to $i_*(\mathcal{L}_{\mathcal{E}})$, where i is the embedding of $A\times\widehat{A}$ in $(A\times\widehat{A})\times(B\times\widehat{B})$ as the graph of the isomorphism $f_{\mathcal{E}}$.*

Before proceeding to the proof of the theorem, we state two lemmas that allow us to assume that the field k is algebraically closed. We write $\overline{k}$ for the algebraic closure of k, set $\overline{X}:=X\times_{\mathrm{Spec}(k)}\mathrm{Spec}(\overline{k})$, and write $\overline{\mathcal{F}}$ for the inverse image of $\mathcal{F}$ under the morphism $\overline{X}\longrightarrow X$.

Lemma 5.1.8 [37]. *Let $\mathcal{F}$ be a coherent sheaf on a smooth variety X. Suppose that there exist a closed subvariety $j\colon Z\hookrightarrow\overline{X}$ and an invertible sheaf $\mathcal{L}$ on Z such that $\overline{\mathcal{F}}\cong j_*\mathcal{L}$. Then there exist a closed subvariety $i\colon Y\hookrightarrow X$ and an invertible sheaf $\mathcal{M}$ on Y such that $\mathcal{F}\cong i_*\mathcal{M}$ and $j=\overline{\imath}$.*

The next lemma tells us that the property that a functor is fully faithful (or an equivalence) is stable under field extensions.

Lemma 5.1.9 [37]. *Let X and Y be smooth projective varieties over a field k and $\mathcal{E}$ an object of the derived category $\mathbf{D}^b(X \times Y)$. Consider a field extension F/k and the varieties*

$$X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F), \qquad Y' = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F).$$

Let $\mathcal{E}'$ be the lift of $\mathcal{E}$ to the category $\mathbf{D}^b(X' \times Y')$. Then the functor $\Phi_{\mathcal{E}} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ is fully faithful (or an equivalence) if and only if $\Phi_{\mathcal{E}'} : \mathbf{D}^b(X') \longrightarrow \mathbf{D}^b(Y')$ is fully faithful (respectively, an equivalence).

Proof of Theorem 5.1.7. Using Lemmas 5.1.8 and 5.1.9, we can pass to the algebraic closure of the field k.

Step 1. Write $e \in A \times \widehat{A}$ and $e' \in B \times \widehat{B}$ for the closed points which are the identity elements of the group structures. We consider the skyscraper sheaf $\mathcal{O}_e$ and evaluate its image under the functor $\Phi_{\mathcal{J}(\mathcal{E})}$. By definition,

$$\Phi_{\mathcal{J}(\mathcal{E})} = \Phi_{S_B}^{-1} \circ Ad_{\mathcal{E}} \circ \Phi_{S_A}.$$

By Proposition 5.1.5, the functor Φ_{S_A} takes $\mathcal{O}_e$ to the structure sheaf $\mathcal{O}_{\Delta(A)}$ of the diagonal in $A \times A$. Since the structure sheaf of the diagonal represents the identity functor, it follows from (12) that $Ad_{\mathcal{E}}(\mathcal{O}_{\Delta(A)})$ is the structure sheaf $\mathcal{O}_{\Delta(B)}$ of the diagonal in $B \times B$. In turn, this sheaf goes to the structure sheaf $\mathcal{O}_{e'}$ under the action of the functor $\Phi_{S_B}^{-1}$, by Proposition 5.1.5 again.

Step 2. Thus, we see that

$$\mathcal{J}(\mathcal{E}) \otimes^{\mathbf{L}} \mathcal{O}_{\{e\} \times (B \times \widehat{B})} \cong \mathcal{O}_{\{e\} \times \{e'\}}.$$

It follows from this that there is an affine neighbourhood $U = \mathrm{Spec}(R)$ of e in the Zariski topology such that the object $\mathcal{J}' := \mathcal{J}(\mathcal{E})_{|U \times (B \times \widehat{B})}$ is a coherent sheaf whose support intersects the fibre $\{e\} \times (B \times \widehat{B})$ at the point $\{e\} \times \{e'\}$. We recall that the support of any coherent sheaf is a closed subset.

Consider now some affine neighbourhood $V = \mathrm{Spec}(S)$ of the point e' in $B \times \widehat{B}$. The intersection of the support of $\mathcal{J}'$ with the complement $B \times \widehat{B} \setminus V$ is a closed subset whose projection to $A \times \widehat{A}$ is a closed subset not containing the point e.

Thus, reducing U if necessary, we can assume that it is still affine and the support of $\mathcal{J}'$ is contained in $U \times V$. This means that there is a coherent sheaf $\mathcal{F}$ on $U \times V$ such that $j_*(\mathcal{F}) = \mathcal{J}'$, where j is the embedding of $U \times V$ into $U \times (B \times \widehat{B})$. We denote by M the finitely generated $R \otimes S$-module corresponding to the sheaf $\mathcal{F}$, that is, $\mathcal{F} = \widetilde{M}$. Moreover, we note that M is a finitely generated R-module, because the direct image under projection of a coherent sheaf $\mathcal{J}' = j_*\mathcal{F}$ is a coherent sheaf.

Let m be the maximal ideal of R corresponding to the point e. As we know,

$$M \otimes_R R/m \cong R/m.$$

Hence, there exists a homomorphism of R-modules $\varphi \colon R \longrightarrow M$ which becomes an isomorphism after tensoring with R/m. Thus, the supports of the coherent sheaves $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ do not contain the point e. Therefore, replacing U

by a smaller affine neighbourhood of e disjoint from the supports of the sheaves $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, we see that φ is an isomorphism. Hence, there is a subscheme $X(U) \subset U \times (B \times \widehat{B})$ such that the projection $X(U) \longrightarrow U$ is an isomorphism and

$$\mathcal{J}' = \mathcal{J}(\mathcal{E})_{|U\times(B\times\widehat{B})} \cong \mathcal{O}_{X(U)}.$$

Step 3. We have thus proved that for any closed point $(a,\alpha) \in U$,

$$\Phi_{\mathcal{J}(\mathcal{E})}(\mathcal{O}_{(a,\alpha)}) \cong \mathcal{O}_{(b,\beta)}$$

for some closed point $(b,\beta) \in B \times \widehat{B}$. If we now consider an arbitrary closed point $(a,\alpha) \in A \times \widehat{A}$, we can always express it as a sum $(a,a') = (a_1,\alpha_1) + (a_2,\alpha_2)$, where the points (a_1,α_1) and (a_2,α_2) belong to U. Write (b_1,β_1) and (b_2,β_2) for the images of these points under the functor $\Phi_{\mathcal{J}(\mathcal{E})}$. As we know, the functor Φ_{S_A} takes the structure sheaf $\mathcal{O}_{(a,\alpha)}$ to the sheaf $S_{(a,\alpha)}$. We denote by $\mathcal{G}$ the object $Ad_{\mathcal{E}}(S_{(a,\alpha)})$. We compute it using (12). We have an isomorphism

$$\Phi_{\mathcal{G}} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)} \circ \Phi_{\mathcal{E}}^{-1}.$$

However, the functor $\Phi_{(a,\alpha)}$, equal by definition (49) to $T_a^*(\,\cdot\,)\otimes\mathcal{P}_\alpha$, can be expressed as the composite $\Phi_{(a_1,\alpha_1)}\Phi_{(a_2,\alpha_2)}$. We thus obtain a chain of isomorphisms

$$\begin{aligned}\Phi_{\mathcal{G}} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)} \circ \Phi_{\mathcal{E}}^{-1} &\cong \Phi_{\mathcal{E}} \circ \Phi_{(a_1,\alpha_1)} \circ \Phi_{\mathcal{E}}^{-1} \\ &\cong \Phi_{\mathcal{E}} \circ \Phi_{(a_2,\alpha_2)} \circ \Phi_{\mathcal{E}}^{-1} \cong \Phi_{(b_1,\beta_1)} \circ \Phi_{(b_2,\beta_2)} \cong \Phi_{(b,\beta)},\end{aligned}$$

where $(b,\beta) = (b_1,\beta_1) + (b_2,\beta_2)$. Therefore, the object $\mathcal{G}$ is isomorphic to $S_{(b,\beta)}$. We finally obtain

$$\Phi_{\mathcal{J}(\mathcal{E})}(\mathcal{O}_{(a,\alpha)}) \cong \mathcal{O}_{(b,\beta)} \quad \text{for any closed point } (a,\alpha) \in A \times \widehat{A}.$$

Now repeating the procedure of Step 2, for any closed point (a,a') we can find a neighbourhood W and a subscheme $X(W) \subset W \times (B \times \widehat{B})$ such that the projection $X(W) \longrightarrow W$ is an isomorphism, and $\mathcal{J}_{|W\times(B\times\widehat{B})} \cong \mathcal{O}_{X(W)}$. Gluing all these neighbourhoods together, we find a subvariety $i\colon X \hookrightarrow (A\times\widehat{A})\times(B\times\widehat{B})$ such that the projection $X \longrightarrow A \times \widehat{A}$ is an isomorphism, and the sheaf $\mathcal{J}(\mathcal{E})$ is isomorphic to $i_*\mathcal{L}$, where $\mathcal{L}$ is a line bundle on X. The subvariety X defines a homomorphism from $A \times \widehat{A}$ to $B \times \widehat{B}$ which induces an equivalence of derived categories. Hence, this homomorphism is an isomorphism.

In particular, it follows at once from the theorem that, if two Abelian varieties A and B have equivalent derived categories of coherent sheaves, then the varieties $A \times \widehat{A}$ and $B \times \widehat{B}$ are isomorphic. We show below that this isomorphism must satisfy a certain additional condition (see Proposition 5.1.15).

Corollary 5.1.10. *The isomorphism $f_{\mathcal{E}}$ takes a k-point $(a,\alpha) \in A \times \widehat{A}$ to a point $(b,\beta) \in B \times \widehat{B}$ if and only if the equivalences*

$$\Phi_{(a,\alpha)} \colon \mathbf{D}^b(A) \xrightarrow{\sim} \mathbf{D}^b(A), \qquad \Phi_{(b,\beta)} \colon \mathbf{D}^b(B) \xrightarrow{\sim} \mathbf{D}^b(B),$$

defined by the formula (49) *are related as follows*:

$$\Phi_{(b,\beta)} \circ \Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)},$$

or, *in terms of the objects*,

$$T_{b*}\mathcal{E} \otimes \mathcal{P}_\beta \cong T_{-a*}\mathcal{E} \otimes \mathcal{P}_\alpha = T_a^*\mathcal{E} \otimes \mathcal{P}_\alpha.$$

Proof. By Theorem 5.1.7, $\Phi_{\mathcal{J}(\mathcal{E})}$ takes the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of (a,α) to the structure sheaf $\mathcal{O}_{(b,\beta)}$ of $(b,\beta) = f_{\mathcal{E}}(a,\alpha)$. It follows from Proposition 5.1.5 that Φ_{S_A} takes $\mathcal{O}_{(a,\alpha)}$ to $S_{(a,\alpha)}$. In turn, the sheaf $S_{(a,\alpha)}$ represents the functor

$$\Phi_{(a,\alpha)} = T_{a*}(\,\cdot\,) \otimes \mathcal{P}_\alpha.$$

Now using diagram (51), we see that $f_{\mathcal{E}}$ takes (a,α) to (b,β) if and only if $S_{(b,\beta)} \cong Ad_{\mathcal{E}}(S_{(a,\alpha)})$. Applying formula (12), we see that $\Phi_{(b,\beta)} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)} \circ \Phi_{\mathcal{E}}^{-1}$.

In what follows we need an explicit formula for the object $\mathcal{J}(\mathcal{E})$ in the special case when $A = B$ and the equivalence $\Phi_{\mathcal{E}}$ is equal to $\Phi_{(a,\alpha)}$ defined by the formula (49).

Proposition 5.1.11. *Let $A = B$. Consider the object $S_{(a,\alpha)}$ on $A \times A$ representing the equivalence $\Phi_{(a,\alpha)}$ given by* (49). *Then $\mathcal{J}(S_{(a,\alpha)})$ is equal to $\Delta_*\mathcal{P}_{(\alpha,-a)}$, where Δ is the diagonal embedding of $A \times \widehat{A}$ into $(A \times \widehat{A}) \times (A \times \widehat{A})$ and $\mathcal{P}_{(\alpha,a)}$ is the line bundle on $A \times \widehat{A}$ defined in* 5.1.1.

Proof. It follows from Proposition 5.1.5 that Φ_{S_A} takes $\mathcal{O}_{(a',\alpha')}$ to the sheaf $S_{(a',\alpha')}$ on $A \times A$ (50). Moreover, $Ad_{S_{(a,\alpha)}}$ takes $S_{(a',\alpha')}$ to itself because, by formula (12), the object $Ad_{S_{(a,\alpha)}}(S_{(a',\alpha')})$ represents the functor

$$\Phi_{(a,\alpha)} \circ \Phi_{(a',\alpha')} \circ \Phi_{(a,\alpha)}^{-1},$$

which is in turn isomorphic to $\Phi_{(a',\alpha')}$ because all such functors commute with one another. Thus, we see that the functor defined by $\mathcal{J}(S_{(a,\alpha)})$ takes the structure sheaf of every point to itself, and thus the sheaf $\mathcal{J}(S_{(a,\alpha)})$ is some line bundle L concentrated on the diagonal.

Now to find the line bundle L, we ask where the functor sends the bundle $\mathcal{P}_{(\alpha',a')}$. Applying Proposition 5.1.5 again, we see that the functor Φ_{S_A} takes $\mathcal{P}_{(\alpha',a')}$ to the object $\mathcal{O}_{\{-a'\}\times A} \otimes p_2^*(\mathcal{P}_{\alpha'})[n]$. Next, one sees readily that this goes to the object $\mathcal{O}_{\{-a'+a\}\times A} \otimes p_2^*(\mathcal{P}_{\alpha'+\alpha})[n]$ under the functor $Ad_{S_{(a,\alpha)}}$. Hence, under the action of the functor given by the sheaf $\mathcal{J}(S_{(a,\alpha)})$, the bundle $\mathcal{P}_{(\alpha',a')}$ goes to the bundle $\mathcal{P}_{(\alpha'+\alpha,a'-a)}$. That is, L is isomorphic to $\mathcal{P}_{(\alpha,-a)}$.

For Abelian varieties A and B, write $\mathrm{Eq}(A, B)$ for the set of all exact equivalences from $\mathbf{D}^b(A)$ to $\mathbf{D}^b(B)$ up to isomorphism. We introduce two groupoids $\mathfrak{A}$ and $\mathfrak{D}$ (that is, categories in which all morphisms are invertible). In both, the objects are Abelian varieties. The morphisms in $\mathfrak{A}$ are isomorphisms between Abelian varieties regarded as algebraic groups. The morphisms in $\mathfrak{D}$ are exact equivalences between the derived categories of coherent sheaves on Abelian varieties; that is,

$$\mathrm{Mor}_{\mathfrak{A}}(A, B) := \mathrm{Iso}(A, B) \quad \text{and} \quad \mathrm{Mor}_{\mathfrak{D}}(A, B) := \mathrm{Eq}(A, B).$$

Theorem 5.1.7 provides a map from the set $\mathrm{Eq}(A, B)$ to the set $\mathrm{Iso}(A\times\widehat{A}, B\times\widehat{B})$, taking an equivalence $\Phi_{\mathcal{E}}$ to the isomorphism $f_{\mathcal{E}}$. We consider the map F from $\mathfrak{D}$ to $\mathfrak{A}$ that assigns to an Abelian variety A the variety $A\times\widehat{A}$ and acts on the morphisms as described above.

Proposition 5.1.12. *The map $F\colon \mathfrak{D} \longrightarrow \mathfrak{A}$ is a functor.*

Proof. To prove the assertion, we need only show that F respects composition of morphisms. Consider three Abelian varieties A, B, and C. Let $\mathcal{E}$ and $\mathcal{F}$ be objects of $\mathbf{D}^b(A \times B)$ and $\mathbf{D}^b(B \times C)$ respectively, such that the functors

$$\Phi_{\mathcal{E}}\colon \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(B) \quad \text{and} \quad \Phi_{\mathcal{F}}\colon \mathbf{D}^b(B) \longrightarrow \mathbf{D}^b(C)$$

are equivalences. We denote by $\mathcal{G}$ the object of $\mathbf{D}^b(A \times C)$ that represents the composite of these functors.

The relation (10) gives an isomorphism $Ad_{\mathcal{G}} \cong Ad_{\mathcal{F}} \circ Ad_{\mathcal{E}}$. Hence, we see that

$$\Phi_{\mathcal{J}(\mathcal{F})} \circ \Phi_{\mathcal{J}(\mathcal{E})} \cong (\Phi_{S_A}^{-1} \circ Ad_{\mathcal{F}} \circ \Phi_{S_A}) \circ (\Phi_{S_A}^{-1} \circ Ad_{\mathcal{E}} \circ \Phi_{S_A}) \cong \Phi_{S_A}^{-1} \circ Ad_{\mathcal{G}} \circ \Phi_{S_A} \cong \Phi_{\mathcal{J}(\mathcal{G})}.$$

By Theorem 5.1.7, all the objects $\mathcal{J}(\mathcal{E})$, $\mathcal{J}(\mathcal{F})$, and $\mathcal{J}(\mathcal{G})$ are line bundles concentrated on the graphs of the isomorphisms $f_{\mathcal{E}}$, $f_{\mathcal{F}}$, and $f_{\mathcal{G}}$, respectively. Thus, we obtain the relation $f_{\mathcal{G}} = f_{\mathcal{F}} \cdot f_{\mathcal{E}}$.

Corollary 5.1.13. *Let A be an Abelian variety and $\Phi_{\mathcal{E}}$ an auto-equivalence of the derived category $\mathbf{D}^b(A)$. Then the correspondence $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ defines a group homomorphism*

$$\gamma_A\colon \mathrm{Auteq}\,\mathbf{D}^b(A) \longrightarrow \mathrm{Aut}(A \times \widehat{A}).$$

Thus, there is a functor $F\colon \mathfrak{D} \longrightarrow \mathfrak{A}$. Our next objective is to describe this functor. For this, we must determine which elements of $\mathrm{Iso}(A \times \widehat{A}, B \times \widehat{B})$ can be realized as $f_{\mathcal{E}}$ for some $\mathcal{E}$, and also answer the question of when $f_{\mathcal{E}_1} = f_{\mathcal{E}_2}$ holds for two equivalences $\mathcal{E}_1$ and $\mathcal{E}_2$.

Consider an arbitrary morphism $f\colon A\times\widehat{A} \longrightarrow B\times\widehat{B}$. It is convenient to represent it as a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where α is a morphism from A to B, β from $\widehat{A}$ to B, γ from A to $\widehat{B}$, and δ from $\widehat{A}$ to $\widehat{B}$. Each morphism f defines two other morphisms $\widehat{f}$ and $\widetilde{f}$ from $B \times \widehat{B}$ to $A \times \widehat{A}$ having the following matrix forms:

$$\widehat{f} = \begin{pmatrix} \widehat{\delta} & \widehat{\beta} \\ \widehat{\gamma} & \widehat{\alpha} \end{pmatrix} \quad \text{and} \quad \widetilde{f} = \begin{pmatrix} \widehat{\delta} & -\widehat{\beta} \\ -\widehat{\gamma} & \widehat{\alpha} \end{pmatrix}.$$

We define the set $U(A\times\widehat{A}, B\times\widehat{B})$ to be the subset of $\mathrm{Iso}(A\times\widehat{A}, B\times\widehat{B})$ consisting of f such that $\widetilde{f}$ coincides with the inverse of f, that is,

$$U(A\times\widehat{A}, B\times\widehat{B}) := \bigl\{f\in \mathrm{Iso}(A\times\widehat{A}, B\times\widehat{B}) \bigm| \widetilde{f}=f^{-1}\bigr\}.$$

If $B=A$, then we denote this set by $U(A\times\widehat{A})$. We note that $U(A\times\widehat{A})$ is a subgroup of $\mathrm{Aut}(A\times\widehat{A})$.

Definition 5.1.14. We say that an isomorphism $f\colon A\times\widehat{A}\xrightarrow{\sim} B\times\widehat{B}$ is *isometric* if it belongs to $U(A\times\widehat{A}, B\times\widehat{B})$.

Proposition 5.1.15. *For any equivalence $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(A)\xrightarrow{\sim}\mathbf{D}^b(B)$ the isomorphism $f_{\mathcal{E}}$ is isometric.*

Proof. Passing to the algebraic closure if necessary, we can assume that k is algebraically closed. To verify the equality $\widetilde{f}_{\mathcal{E}}=f_{\mathcal{E}}^{-1}$, it is enough to establish that these morphisms coincide at closed points. Suppose that $f_{\mathcal{E}}$ takes $(a,\alpha)\in A\times\widehat{A}$ to $(b,\beta)\in B\times\widehat{B}$. We must show that $\widetilde{f}_{\mathcal{E}}(b,\beta)=(a,\alpha)$, or, equivalently, that $\widehat{f}_{\mathcal{E}}(-b,\beta)=(-a,\alpha)$.

The isomorphism $f_{\mathcal{E}}$ is given by the Abelian subvariety $X\hookrightarrow A\times\widehat{A}\times B\times\widehat{B}$. Hence, we must show that $\mathcal{P}_{(0,0,\beta,-b)}\otimes\mathcal{O}_X\cong\mathcal{P}_{(\alpha,-a,0,0)}\otimes\mathcal{O}_X$, or, equivalently, that

$$\mathcal{J}':=\mathcal{P}_{(-\alpha,a,\beta,-b)}\otimes\mathcal{J}(\mathcal{E})$$

is isomorphic to the sheaf $\mathcal{J}(\mathcal{E})$.

By Proposition 5.1.11, the functor given by $\mathcal{J}'$ is the composite of the functors represented by the objects $\mathcal{J}(S_{(-a,-\alpha)})$, $\mathcal{J}(\mathcal{E})$, and $\mathcal{J}(S_{(b,\beta)})$. Thus, $\mathcal{J}'$ coincides with $\mathcal{J}(\mathcal{E}')$, where $\mathcal{E}'$ is the object of $\mathbf{D}^b(A\times B)$ representing the functor

$$\Phi_{(b,\beta)}\circ\Phi_{\mathcal{E}}\circ\Phi_{(-a,-\alpha)}.$$

By Corollary 5.1.10, this composite is isomorphic to the functor $\Phi_{\mathcal{E}}$. This means that the object $\mathcal{E}'$ is isomorphic to $\mathcal{E}$, and hence $\mathcal{J}'=\mathcal{J}(\mathcal{E}')\cong\mathcal{J}(\mathcal{E})$.

As a corollary of Theorem 5.1.7 and Proposition 5.1.15 we get the following result.

Theorem 5.1.16. *Let A and B be two Abelian varieties over a field k. If the derived categories of coherent sheaves $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ are equivalent as triangulated categories, then there is an isometric isomorphism between $A\times\widehat{A}$ and $B\times\widehat{B}$.*

The converse holds for Abelian varieties over an algebraically closed field of characteristic 0, as proved in [38]. We give another proof of this fact in § 5.3.

Corollary 5.1.17. *For any Abelian variety A there are only finitely many non-isomorphic Abelian varieties whose derived categories of coherent sheaves are equivalent to $\mathbf{D}^b(A)$ (as triangulated categories).*

Proof. It was proved in [26] that, for any Abelian variety Z, there are only finitely many Abelian varieties up to isomorphism admitting an embedding in Z as Abelian subvarieties. Applying this assertion to $Z=A\times\widehat{A}$ and using Theorem 5.1.16, we obtain the desired result.

5.2. Objects representing equivalences, and groups of auto-equivalences. It follows from Propositions 5.1.12 and 5.1.15 that there exists a homomorphism from the group $\operatorname{Auteq}\mathbf{D}^b(A)$ of exact auto-equivalences to the group $U(A\times\widehat{A})$ of isometric automorphisms. In this section we describe the kernel of this homomorphism. As we know from Proposition 5.1.11, all the equivalences $\Phi_{(a,\alpha)}[n]$ belong to the kernel. We show that the kernel consists exactly of these. To prove this, we need an assertion which is of independent interest: we prove that for an Abelian variety, if a functor of the form $\Phi_{\mathcal{E}}$ is an equivalence, then the object $\mathcal{E}$ on the product is actually a sheaf, up to a shift in the derived category. We note that this is false, for example, for K3 surfaces.

Lemma 5.2.1. *Let $\mathcal{E}$ be an object on $A\times B$ defining an equivalence $\Phi_{\mathcal{E}}\colon\mathbf{D}^b(A)\longrightarrow\mathbf{D}^b(B)$. Consider the projection $q\colon(A\times\widehat{A})\times(B\times\widehat{B})\longrightarrow A\times B$ and write K for the direct image $\mathbf{R}q_*\mathcal{J}(\mathcal{E})$, where $\mathcal{J}(\mathcal{E})$ is the object defined in* 5.1.6. *Then K is isomorphic to the object $\mathcal{E}\otimes(\mathcal{E}^\vee{}_{|(0,0)})$, where $\mathcal{E}^\vee{}_{|(0,0)}$ stands for the complex of vector spaces which is the inverse image of the object $\mathbf{R}^{\cdot}\underline{\mathcal{H}om}(\mathcal{E},\mathcal{O}_{A\times B})$ under the embedding of the point $(0,0)$ into the Abelian variety $A\times B$.*

Proof. Consider the Abelian variety

$$Z=(A\times\widehat{A})\times(A\times A)\times(B\times B)\times(B\times\widehat{B})$$

and the object

$$H=p_{1234}^*S_A\otimes p_{35}^*\mathcal{E}^\vee[n]\otimes p_{46}^*\mathcal{E}\otimes p_{5678}^*S_B^\vee[2n].$$

It follows from Proposition 2.1.2 on composition of functors and from diagram (51) that $\mathcal{J}(\mathcal{E})\cong p_{1278*}H$, and hence the object K equals $p_{17*}H$. To evaluate the latter object, we first consider the projection of Z to

$$V=A\times(A\times A)\times(B\times B)\times B,$$

and denote it by v. Now, to evaluate v_*H, we recall that the functor Φ_{S_A} is the composite of Φ_{P_A} and $\mathbf{R}\mu_{A*}$, where

$$\mathrm{P}_A=p_{14}^*\mathcal{O}_\Delta\otimes p_{23}^*\mathcal{P}\in\mathbf{D}^b((A\times\widehat{A})\times(A\times A)).$$

One sees readily that $p_{134*}\mathrm{P}_A\cong\mathcal{O}_{T_A}[-n]$, where $T\subset A\times A\times A$ is the subvariety isomorphic to A and consisting of the points $(a,0,a)$. Next, taking into account the equality $\mu_A(a_1,a_2)=(a_1,m(a_1,a_2))$, we can see that $p_{134*}S_A$ is also isomorphic to $\mathcal{O}_{T_A}[-n]$. We verify the equality $p_{134*}S_B^\vee[2n]=\mathcal{O}_{T_B}$ in the same way.

Thus, we have

$$v_*H\cong p_{123}^*\mathcal{O}_{T_A}\otimes p_{24}^*\mathcal{E}^\vee\otimes p_{35}^*\mathcal{E}\otimes p_{456}^*\mathcal{O}_{T_B}$$

on V. Consider the embedding

$$j\colon A\times A\times B\times B\longrightarrow V\quad\text{given by}\quad(a_1,a_2,b_1,b_2)\mapsto(a_1,0,a_2,0,b_1,b_2).$$

The object v_*H is isomorphic to $j_*\mathcal{M}$, where

$$\mathcal{M}=(\mathcal{E}^\vee{}_{|(0,0)})\otimes p_{12}^*\mathcal{O}_{\Delta_A}\otimes p_{23}^*\mathcal{E}\otimes p_{34}^*\mathcal{O}_{\Delta_B}.$$

Finally, we see that $K\cong p_{14*}\mathcal{M}\cong(\mathcal{E}^\vee{}_{|(0,0)})\otimes\mathcal{E}$.

Proposition 5.2.2. *Let A and B be Abelian varieties and $\mathcal{E}$ an object of $\mathbf{D}^b(A\times B)$ such that the functor $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(A) \xrightarrow{\sim} \mathbf{D}^b(B)$ is an exact equivalence. Then $\mathcal{E}$ has only one non-trivial cohomology sheaf, that is, it is isomorphic to an object $\mathcal{F}[n]$, where $\mathcal{F}$ is a sheaf on $A \times B$.*

Proof. Consider the projection

$$q\colon (A\times \widehat{A})\times (B\times \widehat{B}) \longrightarrow A\times B$$

and write q' for its restriction to the Abelian subvariety X which is the support of the sheaf $\mathcal{J}(\mathcal{E})$ and the graph of the isomorphism $f_{\mathcal{E}}$. By Theorem 5.1.7, $\mathcal{J}(\mathcal{E})$ equals $i_*(L)$, where L is a line bundle on X. We write K for the object $\mathbf{R}^{\cdot}q_*\mathcal{J}(\mathcal{E}) = \mathbf{R}^{\cdot}q'_*L$. The morphism q' is a homomorphism of Abelian varieties; set $d = \dim \operatorname{Ker}(q')$. Then $\dim \operatorname{Im}(q') = 2n-d$, so that the cohomology sheaves $H^j(K)$ are trivial for $j\notin [0,d]$.

On the other hand, by Lemma 5.2.1, K is isomorphic to $\mathcal{E}\otimes(\mathcal{E}^{\vee}{}_{|(0,0)})$.

After shifting $\mathcal{E}$ in the derived category if necessary, we can assume that the rightmost non-zero cohomology sheaf of $\mathcal{E}$ is $H^0(\mathcal{E})$. Let $H^{-i}(\mathcal{E})$ for $i\geqslant 0$ be the leftmost non-zero cohomology sheaf of $\mathcal{E}$, and $H^k(\mathcal{E}^{\vee})$ the highest non-zero cohomology sheaf of $\mathcal{E}^{\vee}$. Replacing $\mathcal{E}$ by $T^*_{(a,b)}\mathcal{E}$ if necessary, we can assume that the point $(0,0)$ belongs to the support of $H^k(\mathcal{E}^{\vee})$. Since the support of $\mathcal{E}$ coincides with the support of K, it follows that the supports of all cohomology sheaves $\mathcal{E}$ belong to $\operatorname{Im}(q')$. In particular, we have the inequality $\operatorname{codim}\operatorname{Supp} H^{-i}(\mathcal{E}) \geqslant d$. Hence, the cohomology sheaf of the object $(H^{-i}(\mathcal{E}))^{\vee}[-i]$ of degree less than $i+d$ is trivial.

The canonical morphism $H^{-i}(\mathcal{E})[i] \longrightarrow \mathcal{E}$ induces a non-trivial morphism

$$\mathcal{E}^{\vee} \longrightarrow (H^{-i}(\mathcal{E}))^{\vee}[-i].$$

Since the indices of the non-trivial cohomology sheaves of the second object belong to the ray $[i+d,\infty)$, we see that $k\geqslant i+d$, where, as above, $H^k(\mathcal{E}^{\vee})$ is the highest non-zero cohomology sheaf of $\mathcal{E}^{\vee}$. Thus, the object

$$K = \mathcal{E}^{\vee}{}_{|(0,0)}\otimes \mathcal{E} \tag{52}$$

has non-trivial cohomology sheaf with the same index $k\geqslant i+d$. On the other hand, we already know that all the cohomology sheaves $H^j(K)$ are trivial for $j\notin[0,d]$. This is only possible if $i=0$. Thus, the object $\mathcal{E}$ has only one non-trivial cohomology sheaf, with index 0, and hence it is isomorphic to a sheaf.

We now consider the case $B\cong A$. Let $\mathcal{E}$ be a sheaf on $A\times A$ such that $\Phi_{\mathcal{E}}$ is an auto-equivalence. We want to describe all the sheaves $\mathcal{E}$ for which $f_{\mathcal{E}}$ is the identity map, that is, its graph X is the diagonal in $(A\times\widehat{A})\times(A\times\widehat{A})$. Thus, the object

$$K = \mathcal{E}^{\vee}{}_{|(0,0)}\otimes\mathcal{E} = \mathbf{R}^{\cdot}q_*\mathcal{J}(\mathcal{E})$$

is of the form $\Delta_*(\mathcal{M})$, where $\mathcal{M}$ is an object on A and $\Delta\colon A\longrightarrow A\times A$ is the diagonal embedding.

We assume first that $(0,0)$ belongs to the support of $\mathcal{E}$. Hence, $\mathcal{E}^{\vee}{}_{|(0,0)}$ is a non-trivial complex of vector spaces. Then the condition $K = \Delta_*(\mathcal{M})$ implies the existence of a sheaf E on A such that $\mathcal{E} \cong \Delta_*(E)$. Hence, $\Phi_{\mathcal{E}}(\,\cdot\,) \cong E \otimes (\,\cdot\,)$. Since $\Phi_{\mathcal{E}}$ is an auto-equivalence, E is a line bundle. One sees readily that the condition $f_{\mathcal{E}} = \mathrm{id}$ can only hold if $E \in \mathrm{Pic}^0(A)$.

If $(0,0)$ does not belong to $\mathrm{Supp}\,\mathcal{E}$, we replace $\mathcal{E}$ by the sheaf $\mathcal{E}' := T_{(a_1,a_2)*}\mathcal{E}$ in such a way that its support contains $(0,0)$. It follows from Proposition 5.1.11 that $f_{\mathcal{E}'} = f_{\mathcal{E}}$. As shown above, there is an isomorphism $\mathcal{E}' \cong \Delta_*(E')$, where $E' \in \mathrm{Pic}^0(A)$. Hence, $\mathcal{E} \cong T_{(a_1-a_2,0)*}\Delta_*(E')$. We thus obtain the corollary.

Proposition 5.2.4. *Let A be an Abelian variety. The kernel of the homomorphism*

$$\gamma_A\colon\ \mathrm{Auteq}\,\mathbf{D}^b(A) \longrightarrow U(A \times \widehat{A})$$

consists of the auto-equivalences of the form $\Phi_{(a,\alpha)}[i] = T_{a*}(\,\cdot\,) \otimes \mathcal{P}_\alpha[i]$, and hence is isomorphic to the group $\mathbb{Z} \oplus (A \times \widehat{A})_k$, where $(A \times \widehat{A})_k$ is the group of k-points of the Abelian variety $A \times \widehat{A}$.

Corollary 5.2.4. *Let A and B be two Abelian varieties and $\mathcal{E}_1$ and $\mathcal{E}_2$ objects on the product $A \times B$ that define equivalences between their derived categories of coherent sheaves. In this case if $f_{\mathcal{E}_1} = f_{\mathcal{E}_2}$, then*

$$\mathcal{E}_2 \cong T_{a*}\mathcal{E}_1 \otimes \mathcal{P}_\alpha[i]$$

for some k-point $(a,\alpha) \in A \times \widehat{A}$.

5.3. Semi-homogeneous vector bundles. In the previous sections we showed that an equivalence $\Phi_{\mathcal{E}}$ from $\mathbf{D}^b(A)$ to $\mathbf{D}^b(B)$ induces an isometric isomorphism of varieties $A \times \widehat{A}$ and $B \times \widehat{B}$. In this section we assume that the field k is algebraically closed and $\mathrm{char}(k) = 0$. Under these assumptions, using the technique of [30] and the results of [7], we will show that every isometric isomorphism $f\colon A \times \widehat{A} \longrightarrow B \times \widehat{B}$ can be realized in this way. The fact that the existence of an isometric isomorphism between the varieties $A \times \widehat{A}$ and $B \times \widehat{B}$ implies the equivalence of the derived categories $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ was proved in [38]. Thus, we will give another proof of this fact.

We first recall that every line bundle $\mathcal{L}$ on an Abelian variety D gives a map $\varphi_{\mathcal{L}}$ from D to $\widehat{D}$ that sends a point d to the point corresponding to the bundle $T_d^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(D)$. This correspondence defines an embedding of $\mathrm{NS}(D)$ into $\mathrm{Hom}(D, \widehat{D})$. Moreover, it is known that the map $\varphi\colon D \longrightarrow \widehat{D}$ belongs to the image of $\mathrm{NS}(D)$ if and only if $\widehat{\varphi} = \varphi$.

Semi-homogeneous bundles on an Abelian variety allow us to generalize the above phenomenon as follows. To every element of $\mathrm{NS}(D) \otimes \mathbb{Q}$ one assigns a correspondence on $D \times \widehat{D}$, and every such correspondence is obtained from a semi-homogeneous bundle (see Proposition 5.3.6 and Lemma 5.1.10 below).

We first recall the definitions of homogeneous and semi-homogeneous bundles on Abelian varieties and some facts concerning them.

Definition 5.3.1. A vector bundle $\mathcal{E}$ on an Abelian variety D is *homogeneous* if $T_d^*(\mathcal{E}) \cong \mathcal{E}$ for every point $d \in D$.

Definition 5.3.2. We say that a vector bundle $\mathcal{F}$ on an Abelian variety D is *unipotent* if there is a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_D$ for all $i = 1, \dots, n$.

The following proposition characterizes homogeneous vector bundles.

Proposition 5.3.3 ([28], [30]). *Let $\mathcal{E}$ be a vector bundle on an Abelian variety D. Then the following conditions are equivalent*:

(i) *$\mathcal{E}$ is homogeneous*,
(ii) *there exist line bundles $\mathcal{P}_i \in \mathrm{Pic}^0(D)$ and unipotent bundles $\mathcal{F}_i$ such that $\mathcal{E} \cong \bigoplus_i(\mathcal{F}_i \otimes \mathcal{P}_i)$.*

Definition 5.3.4. A vector bundle $\mathcal{E}$ on an Abelian variety D is said to be *semi-homogeneous* if for every point $d \in D$ there exists a line bundle $\mathcal{L}$ on D such that $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{L}$. (We note that, in this case, the bundle $\mathcal{L}$ belongs to $\mathrm{Pic}^0(D)$.)

We recall that a vector bundle on a variety is *simple* if its endomorphism algebra coincides with the field k.

The following assertion was proved in [30].

Proposition 5.3.5 ([30], Theorem 5.8). *Let $\mathcal{E}$ be a simple vector bundle on an Abelian variety D. Then the following conditions are equivalent*:

(1) $\dim \mathrm{H}^j(D, \underline{\mathcal{E}nd}(\mathcal{E})) = \binom{n}{j}$ *for any j, $j = 0, \dots, n$,*
(2) *$\mathcal{E}$ is a semi-homogeneous bundle*,
(3) *$\underline{\mathcal{E}nd}(\mathcal{E})$ is a homogeneous bundle*,
(4) *there exists an isogeny $\pi\colon Y \longrightarrow D$ and a line bundle $\mathcal{L}$ on Y such that $\mathcal{E} \cong \pi_*(\mathcal{L})$.*

Let $\mathcal{E}$ be a vector bundle on an Abelian variety D. We write $\mu(\mathcal{E})$ for the equivalence class $\frac{\det(\mathcal{E})}{r(\mathcal{E})}$ in $\mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$. To every element $\mu = \frac{[\mathcal{L}]}{l} \in \mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$, and hence to each bundle $\mathcal{E}$, we can assign a correspondence $\Phi_\mu \subset D \times \widehat{D}$ given by $\Phi_\mu = \mathrm{Im}\big[D \xrightarrow{(l,\varphi_{\mathcal{L}})} D \times \widehat{D}\big]$, where $\varphi_{\mathcal{L}}$ is the well-known map from D to $\widehat{D}$ that takes d to the point corresponding to the bundle $T_d^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(D)$. If the bundle is a line bundle $\mathcal{L}$, we obtain the graph of the map $\varphi_{\mathcal{L}} : D \longrightarrow \widehat{D}$. We write q_1 and q_2 for the projections of Φ_μ to D and $\widehat{D}$ respectively.

The paper [30] contains a complete description of all simple semi-homogeneous bundles.

Proposition 5.3.6 ([30], 7.10). *Let $\mu = \frac{[\mathcal{L}]}{l}$, where $[\mathcal{L}]$ is the equivalence class of a bundle $\mathcal{L}$ in $\mathrm{NS}(D)$ and l a positive integer. Then*

(1) *there is a simple semi-homogeneous vector bundle E with slope $\mu(\mathcal{E}) = \mu$;*
(2) *every simple semi-homogeneous vector bundle $\mathcal{E}'$ with slope $\mu(\mathcal{E}') = \mu$ is of the form $\mathcal{E} \otimes \mathcal{M}$ for some line bundle $\mathcal{M} \in \mathrm{Pic}^0(D)$;*
(3) *we have the equalities $r(\mathcal{E})^2 = \deg(q_1)$ and $\chi(\mathcal{E})^2 = \deg(q_2)$.*

The following assertion enables one to characterize all semi-homogeneous vector bundles in terms of simple bundles.

Proposition 5.3.7 ([30], 6.15, 6.16). *Every semi-homogeneous vector bundle $\mathcal{F}$ with slope μ has a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are simple semi-homogeneous vector bundles with the same slope μ. Every simple semi-homogeneous bundle is stable.

The next two lemmas concerning semi-homogeneous bundles are direct corollaries of the above assertions, and will be useful in what follows.

Lemma 5.3.8. *Two simple semi-homogeneous bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ with the same slope μ are either isomorphic or orthogonal to each other*; *that is, either $\mathcal{E}_1 = \mathcal{E}_2$ or*

$$\mathrm{Ext}^i(\mathcal{E}_1, \mathcal{E}_2) = 0 \quad \textit{and} \quad \mathrm{Ext}^i(\mathcal{E}_2, \mathcal{E}_1) = 0 \quad \textit{for all } i.$$

Proof. It follows from Proposition 5.3.6 that $\mathcal{E}_2 \cong \mathcal{E}_1 \otimes \mathcal{M}$, and hence $\underline{\mathcal{H}om}(\mathcal{E}_1, \mathcal{E}_2)$ is a homogeneous bundle. By Proposition 5.3.3, every homogeneous bundle can be represented as the sum of unipotent bundles twisted by line bundles in $\mathrm{Pic}^0(D)$. Therefore, either all the cohomology $\underline{\mathcal{H}om}(\mathcal{E}_1, \mathcal{E}_2)$ vanishes, and hence the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are orthogonal, or $\underline{\mathcal{H}om}(\mathcal{E}_1, \mathcal{E}_2)$ admits a non-zero section. In the last case we obtain a non-zero homomorphism from $\mathcal{E}_1$ to $\mathcal{E}_2$. However, these two bundles are stable and have the same slope. Thus, every non-zero homomorphism is actually an isomorphism.

Lemma 5.3.9. *Let $\mathcal{E}$ be a simple semi-homogeneous vector bundle on an Abelian variety D. Then $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{P}_\delta$ if and only if $(d, \delta) \in \Phi_\mu$.*

Proof. We first show that for any point $(d, \delta) \in \Phi_\mu$ there is an isomorphism $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{P}_\delta$. Indeed, set $l = r(\mathcal{E})$ and $\mathcal{L} = \det(\mathcal{E})$. By definition of Φ_μ, we know that we can express $(d, \delta) = (lx, \varphi_{\mathcal{L}}(x))$ for some point $x \in D$. Since $\mathcal{E}$ is semi-homogeneous, there is a line bundle $\mathcal{M} \in \mathrm{Pic}^0(D)$ such that

$$T_x^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{M}. \tag{53}$$

Equating determinants, we obtain the equality $T_x^*(\mathcal{L}) \cong \mathcal{L} \otimes \mathcal{M}^{\otimes l}$. By definition of the map $\varphi_{\mathcal{L}}$, this means that $\mathcal{P}_{\varphi_{\mathcal{L}}(x)} = \mathcal{M}^{\otimes l}$. On the other hand, iterating the equality (53) l times, we obtain

$$T_{lx}^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{M}^{\otimes l} = \mathcal{E} \otimes \mathcal{P}_{\varphi_{\mathcal{L}}(x)}.$$

Hence, $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{P}_\delta$ because $(d, \delta) = (lx, \varphi_{\mathcal{L}}(x))$.

Now for the converse. We introduce the subgroup $\Sigma^0(\mathcal{E}) \subset \widehat{D}$ given by the condition

$$\Sigma^0(\mathcal{E}) := \{\delta \in \widehat{D} \mid \mathcal{E} \otimes \mathcal{P}_\delta \cong \mathcal{E}\}. \tag{54}$$

Since $\mathcal{E}$ is semi-homogeneous, $\underline{\mathcal{E}nd}(\mathcal{E})$ is homogeneous by Proposition 5.3.5. Thus, $\underline{\mathcal{E}nd}(\mathcal{E})$ can be represented as a sum $\bigoplus_i(\mathcal{F}_i \otimes \mathcal{P}_i)$, where all the $\mathcal{F}_i$ are unipotent. Hence, $\mathrm{H}^0(\underline{\mathcal{E}nd}(\mathcal{E}) \otimes \mathcal{P}) \neq 0$ for at most r^2 line bundles $\mathcal{P} \in \mathrm{Pic}^0(D)$. That is, the order of the group $\Sigma^0(\mathcal{E})$ does not exceed r^2. On the other hand, it is known that $q_2(\mathrm{Ker}(q_1)) \subset \Sigma^0(\mathcal{E})$. Hence, we obtain the equalities $\mathrm{ord}\,\Sigma^0(\mathcal{E}) = r^2$ and $q_2(\mathrm{Ker}(q_1)) = \Sigma^0(\mathcal{E})$.

We now assume that $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{P}_\delta$ for some point $(d, \delta) \in D \times \widehat{D}$. Consider a point $\delta' \in \widehat{D}$ such that $(d, \delta') \in \Phi_\mu$. As already proved above, there is then an isomorphism $T_d^*(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{P}_{\delta'}$. Hence, $\mathcal{E} \otimes \mathcal{P}_{(\delta-\delta')} \cong \mathcal{E}$, and thus $\delta - \delta' \in \Sigma^0(\mathcal{E})$. However, since $\Sigma^0(\mathcal{E}) = q_2(\mathrm{Ker}(q_1))$, it follows that the point $(0, \delta - \delta')$ belongs to Φ_μ. Thus, the point (d, δ) also belongs to Φ_μ.

Let f be an isometric isomorphism. We now present a construction which shows how to construct from f an object $\mathcal{E}$ on the product such that $\mathcal{E}$ defines an equivalence of derived categories and for which $f_{\mathcal{E}}$ coincides with f.

Construction 5.3.10. We fix an isometric isomorphism $f\colon A \times \widehat{A} \longrightarrow B \times \widehat{B}$ and write Γ for its graph. As above, we represent the isomorphism f in the matrix form

$$f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Suppose that $y\colon \widehat{A} \longrightarrow B$ is an isogeny. In this case one assigns to the map f an element $g \in \mathrm{Hom}(A \times B, \widehat{A} \times \widehat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ of the form:

$$g = \begin{pmatrix} y^{-1}x & -y^{-1} \\ -\widehat{y}^{-1} & wy^{-1} \end{pmatrix}.$$

The element g defines a certain correspondence on $(A \times B) \times (\widehat{A} \times \widehat{B})$. One sees readily that the condition that f is isometric implies the equality $\widehat{g} = g$. This means that g in fact belongs to the image of $\mathrm{NS}(A \times B) \otimes_{\mathbb{Z}} \mathbb{Q}$ under its canonical embedding into $\mathrm{Hom}(A \times B, \widehat{A} \times \widehat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (see, for example, [32]). Hence, there exists $\mu = \frac{[\mathcal{L}]}{l} \in \mathrm{NS}(A \times B)$ such that Φ_μ coincides with the graph of the correspondence g. Proposition 5.3.6 tells us that for any μ one constructs a simple semi-homogeneous bundle $\mathcal{E}$ on $A \times B$ with slope $\mu(\mathcal{E}) = \mu$.

We show presently that the functor $\Phi_{\mathcal{E}}$ from $\mathbf{D}^b(A)$ to $\mathbf{D}^b(B)$ is an equivalence and $f_{\mathcal{E}} = f$. However, first let us compare the graphs Γ and Φ_μ. If a point (a, α, b, β) belongs to Γ, then

$$\begin{aligned} b &= x(a) + y(\alpha), \\ \beta &= z(a) + w(\alpha), \end{aligned} \quad \text{and hence} \quad \begin{aligned} \alpha &= -y^{-1}x(a) + y^{-1}(b), \\ \beta &= (z - wy^{-1}x)(a) + wy^{-1}(b). \end{aligned}$$

Since f is isometric, we have the equality $(z - wy^{-1}x) = -\widehat{y}^{-1}$. Thus, a point (a, α, b, β) belongs to the graph Γ if and only if $(a, -\alpha, b, \beta)$ belongs to Φ_μ. Therefore,

$$\Phi_\mu = (1_A, -1_{\widehat{A}}, 1_B, 1_{\widehat{B}})\Gamma.$$

In particular, since f is an isomorphism, it follows that the projections of Φ_μ to $A \times \widehat{A}$ and $B \times \widehat{B}$ are isomorphisms.

Proposition 5.3.11. *Let $\mathcal{E}$ be the semi-homogeneous bundle on $A \times B$ constructed from an isometric isomorphism f as just described. Then the functor $\Phi_{\mathcal{E}} : \mathbf{D}^b(A) \to \mathbf{D}^b(B)$ is an equivalence.*

Proof. We write $\mathcal{E}_a$ for the restriction of $\mathcal{E}$ to the fibre $\{a\} \times B$. By Theorem 2.1.5, to prove that the functor $\Phi_{\mathcal{E}}$ is fully faithful, it is enough to show that all the bundles $\mathcal{E}_a$ are simple and mutually orthogonal for distinct points.

First note that by Proposition 5.3.6, the rank of $\mathcal{E}$ is equal to the square root of the degree of the map $\Phi_\mu \longrightarrow A \times B$, that is, $\sqrt{\deg(\beta)}$.

Since $\mathcal{E}$ is semi-homogeneous, we see at once that all the bundles $\mathcal{E}_a$ are also semi-homogeneous. Moreover, the slope $\mu(\mathcal{E}_a)$ of the restriction is equal to $\delta\beta^{-1} \in \mathrm{NS}(B) \otimes \mathbb{Q} \subset \mathrm{Hom}(B, \widehat{B}) \otimes \mathbb{Q}$. For brevity we denote $\delta\beta^{-1}$ by ν, considering it as an element of $\mathrm{NS}(B) \otimes \mathbb{Q}$. By Proposition 5.3.6, there is a simple semi-homogeneous bundle $\mathcal{F}$ on B with the given slope $\mu(\mathcal{F}) = \nu$. Obviously, in this case the map Φ_ν is $\mathrm{Im}\big[\widehat{A} \xrightarrow{(\beta,\delta)} B \times \widehat{B}\big]$. Since f is an isomorphism, it follows that $\widehat{A} \xrightarrow{(\beta,\delta)} B \times \widehat{B}$ is an embedding. Hence, applying Proposition 5.3.6 again, we obtain the equality $r(\mathcal{F}) = \sqrt{\deg(\beta)} = r(\mathcal{E}_a)$. Thus, the two bundles $\mathcal{F}$ and $\mathcal{E}_a$ are semi-homogeneous and have the same slope and the same rank. Moreover, the bundle $\mathcal{F}$ is simple. It follows from Propositions 5.3.7 and 5.3.6, (2) that $\mathcal{E}_a$ is also a simple bundle.

Next, it follows from Lemma 5.3.8 that for two points $a_1, a_2 \in A$, the bundles $\mathcal{E}_{a_1}$ and $\mathcal{E}_{a_2}$ are either orthogonal or isomorphic. Suppose that they are isomorphic. Since $\mathcal{E}$ is semi-homogeneous, it follows that

$$T^*_{(a_2-a_1,0)}\mathcal{E} \cong \mathcal{E} \otimes \mathcal{P}_{(\alpha,\beta)} \tag{55}$$

for some point $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. In particular, we obtain

$$\mathcal{E}_{a_2} \otimes \mathcal{P}_\beta \cong \mathcal{E}_{a_1} \cong \mathcal{E}_{a_2}.$$

Hence, $\mathcal{P}_\beta \in \Sigma^0(\mathcal{E}_a)$ (see (54)).

By Lemma 5.3.9 and Proposition 5.3.6, the orders of the groups $\Sigma^0(\mathcal{E})$ and $\Sigma^0(\mathcal{E}_a)$ are equal to r^2. We claim that the natural map $\sigma \colon \Sigma^0(\mathcal{E}) \longrightarrow \Sigma^0(\mathcal{E}_a)$ is an isomorphism. Indeed, otherwise there would exist a point $\alpha' \in \widehat{A}$ such that $\mathcal{E} \otimes \mathcal{P}_{\alpha'} \cong \mathcal{E}$. Then $(0, \alpha', 0, 0) \in \Phi_\mu$ by Lemma 5.3.9. This contradicts the fact that the projection $\Phi_\mu \longrightarrow B \times \widehat{B}$ is an isomorphism.

Now if σ is an isomorphism, there is a point $\alpha' \in \widehat{A}$ such that $\mathcal{E} \otimes \mathcal{P}_{(\alpha',\beta)} \cong \mathcal{E}$. It follows from (55) that

$$T^*_{(a_2-a_1,0)}\mathcal{E} \cong \mathcal{E} \otimes \mathcal{P}_{(\alpha-\alpha',0)}.$$

By Lemma 5.3.9 this means that the point $(a_2 - a_1, \alpha - \alpha', 0, 0)$ belongs to Φ_μ. Since the projection $\Phi_\mu \longrightarrow B \times \widehat{B}$ is an isomorphism, we again obtain the equality $a_2 - a_1 = 0$. Thus, for two distinct points a_1 and a_2 the corresponding bundles $\mathcal{E}_{a_1}$ and $\mathcal{E}_{a_2}$ are orthogonal. Hence, the functor $\Phi_{\mathcal{E}} : \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(B)$ is fully faithful. For the same reason, the adjoint functor $\Psi_{\mathcal{E}^\vee}$ is also fully faithful. Hence, $\Phi_{\mathcal{E}}$ is an equivalence.

Proposition 5.3.12. *Let $\mathcal{E}$ be the semi-homogeneous bundle constructed from an isometric isomorphism $f\colon A\times\widehat{A}\longrightarrow B\times\widehat{B}$ as described above. Then $f_{\mathcal{E}}=f$.*

Proof. We write X for the graph of the morphism $f_{\mathcal{E}}$. It follows from Corollary 5.1.10 that the point (a,α,b,β) belongs to X if and only if

$$T_{b*}\mathcal{E}\otimes\mathcal{P}_\beta\cong T_a^*\mathcal{E}\otimes\mathcal{P}_\alpha,$$

which is equivalent to the equality

$$T^*_{(a,b)}\mathcal{E}\cong\mathcal{E}\otimes\mathcal{P}_{(-\alpha,\beta)}.$$

Hence, by Lemma 5.3.9 we see that $X=(1_A,-1_{\widehat{A}},1_B,1_{\widehat{B}})\Phi_\mu$, where $\mu=\mu(\mathcal{E})$ is the slope of $\mathcal{E}$. On the other hand, by Construction 5.3.10, the graph Γ of f is also equal to $(1_A,-1_{\widehat{A}},1_B,1_{\widehat{B}})\Phi_\mu$. Thus, the isomorphisms $f_{\mathcal{E}}$ and f coincide.

When constructing a bundle $\mathcal{E}$ from an isomorphism f, we assumed that the map $y\colon\widehat{A}\longrightarrow B$ is an isogeny. If this is not the case, then we represent f as the composite of two maps $f_1\in U(A\times\widehat{A},B\times\widehat{B})$ and $f_2\in U(A\times\widehat{A})$ for which y_1 and y_2 are isogenies. One sees readily that this is always possible. Now for any map f_i we find the corresponding object $\mathcal{E}_i$, consider the composite of the functors $\Phi_{\mathcal{E}_i}$, and take the object representing it. The assertions proved in this section and in the previous ones can be collected in the form of the following theorems.

Theorem 5.3.13. *Let A and B be two Abelian varieties over an algebraically closed field of characteristic* 0. *Then the bounded derived categories of coherent sheaves $\mathbf{D}^b(A)$ and $\mathbf{D}^b(B)$ are equivalent as triangulated categories if and only if there is an isometric isomorphism $f\colon A\times\widehat{A}\xrightarrow{\sim}B\times\widehat{B}$.*

Theorem 5.3.14. *Let A be an Abelian variety over an algebraically closed field of characteristic* 0. *Then the group of exact auto-equivalences of the derived category* $\operatorname{Auteq}\mathbf{D}^b(A)$ *fits in the following exact sequence of groups*:

$$0\longrightarrow\mathbb{Z}\oplus(A\times\widehat{A})_k\longrightarrow\operatorname{Auteq}\mathbf{D}^b(A)\longrightarrow U(A\times\widehat{A})\longrightarrow 1.$$

Thus, the group $\operatorname{Auteq}\mathbf{D}^b(A)$ has a normal subgroup $(A\times\widehat{A})_k$ which consists of the functors of the form $T_{a*}(\,\cdot\,)\otimes\mathcal{P}_\alpha$, where $(a,\alpha)\in A\times\widehat{A}$. The quotient by this subgroup is a central extension of $U(A\times\widehat{A})$ by $\mathbb{Z}$.

This central extension is described by a 2-cocycle, a formula for which can be found in [37].

Example 5.3.15. Consider an Abelian variety A for which the endomorphism ring $\operatorname{End}(A)$ is isomorphic to $\mathbb{Z}$. Then the Néron–Severi group $\operatorname{NS}(A)$ is isomorphic to $\mathbb{Z}$. Write $\mathcal{L}$ and $\mathcal{M}$ for generators of $\operatorname{NS}(A)$ and $\operatorname{NS}(\widehat{A})$ respectively. The composite $\varphi_{\mathcal{M}}\circ\varphi_{\mathcal{L}}$ equals $N\cdot\mathrm{id}_A$ for some $N>0$. In this case the group $U(A\times\widehat{A})$ coincides with the congruence subgroup $\Gamma_0(N)\subset\mathrm{SL}(2,\mathbb{Z})$. Next, let B be another Abelian variety such that $B\times\widehat{B}$ is isomorphic to $A\times\widehat{A}$. One sees readily that every isomorphism of this kind is isometric. The Abelian variety B can be represented as the image of some morphism $A\xrightarrow{(k\cdot\mathrm{id},m\varphi_{\mathcal{L}})}A\times\widehat{A}$. We can assume that $\gcd(k,m)=1$.

We write ψ for the morphism from A to B defined in this way. The kernel of ψ is $\operatorname{Ker}(m\varphi_{\mathcal{L}}) \cap A_k$. Since $\gcd(k, m) = 1$, we have in fact $\operatorname{Ker}(\psi) = \operatorname{Ker}(\varphi_{\mathcal{L}}) \cap A_k$. On the other hand, we have an inclusion $\operatorname{Ker}(\varphi) \subset A_N$. Thus, without loss of generality we can assume that k is a divisor of N. Every divisor k of N induces an Abelian variety of the form $B := A/(\operatorname{Ker}(\varphi_{\mathcal{L}}) \cap A_k)$. Obviously, two distinct divisors of N give non-isomorphic Abelian varieties. Moreover, one sees readily that an embedding of B in $A \times \widehat{A}$ splits if and only if $\gcd(k, N/k) = 1$. Hence, the number of Abelian varieties B such that $\mathbf{D}^b(B) \simeq \mathbf{D}^b(A)$ equals 2^s, where s is the number of prime divisors of N.

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