

DERIVED CATEGORIES OF COHERENT SHEAVES ON ABELIAN VARIETIES AND EQUIVALENCES BETWEEN THEM

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1.X.2001

ABSTRACT. We study derived categories of coherent sheaves on abelian varieties. We give a criterion for the equivalence of the derived categories on two abelian varieties. We describe the autoequivalence group for the derived category of coherent sheaves of an abelian variety.

INTRODUCTION

For every algebraic variety X we have the abelian category $\text{coh}(X)$ of coherent sheaves on X . Morphisms between varieties induce the inverse image functors between the abelian categories of coherent sheaves. If the morphism is proper, then the direct image functor is also defined. These functors are not exact. They have left and right derived functors respectively. To take all the derived functors into account, one must pass from the abelian categories to their derived categories. For example, to every smooth projective varieties there corresponds the so-called bounded derived category of coherent sheaves $D^b(X)$, and to every morphism between such varieties there correspond the derived direct and inverse image functors between the derived categories of coherent sheaves.

The question is how many information is lost in the passage from varieties to the derived categories of coherent sheaves. This passage actually preserves almost all information. For example, in some cases one can restore a variety from its derived category (see [2] or Theorem 1.2 of this paper). Nevertheless, some classes of varieties contain examples of two different varieties with equivalent derived categories of coherent sheaves.

In this paper we study the case of abelian varieties.

Let A be an abelian variety, and let \hat{A} be the dual abelian variety. As shown in [9], the derived categories of coherent sheaves $D^b(A)$ and $D^b(\hat{A})$ are equivalent and their equivalence, which is called the *Fourier–Mukai transform*, can be defined in terms of the Poincaré line bundle P_A on the product $A \times \hat{A}$ by the following rule:

$$F \mapsto \mathbf{R}p_{2*}(P_A \otimes p_1^*(F)).$$

1991 *Mathematics Subject Classification.* 18E30, 14K05.

This work is done under partial financial support of the Russian Foundation for Basic Research (grant no. 99-01-01144), grant in support of leading scientific schools (no. 00-15-96085), and American Foundation of Civil Research and Development for countries of the former Soviet Union (grant no. RM1-2089).

This construction of Mukai was generalized in [13] as follows.

Consider abelian varieties A , B and an isomorphism f between the abelian varieties $A \times \hat{A}$ and $B \times \hat{B}$. We write f as a matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

where x is a homomorphism from A to B , y is a homomorphism from \hat{A} to B and so on. The isomorphism f is called *isometric* if its inverse map is given by

$$f^{-1} = \begin{pmatrix} \hat{w} & -\hat{y} \\ -\hat{z} & \hat{x} \end{pmatrix}.$$

It is proved in [13] that if A , B are abelian varieties over an algebraically closed field and there is an isometric isomorphism between $A \times \hat{A}$ and $B \times \hat{B}$, then the derived categories of coherent sheaves $D^b(A)$ and $D^b(B)$ are equivalent.

In this paper we prove the equivalence of these conditions over an algebraically closed field of characteristic 0. In other words, we prove that the derived categories $D^b(A)$ and $D^b(B)$ are equivalent if and only if there is an isometric isomorphism of $A \times \hat{A}$ onto $B \times \hat{B}$. Part “only if” of this assertion actually holds over an arbitrary field (Theorem 2.19). As a corollary of this theorem, we obtain that for every given abelian variety A there are only finitely many non-isomorphic abelian varieties whose derived categories are isomorphic to $D^b(A)$ (Corollary 2.20). The proof essentially uses the main theorem of [12], which states that every exact equivalence between the derived categories of coherent sheaves on smooth projective varieties can be represented by an object on the product.

Representing equivalences by objects on the product, we construct a map from the set of all exact equivalences between $D^b(A)$ and $D^b(B)$ to the set of isometric isomorphisms of $A \times \hat{A}$ onto $B \times \hat{B}$. Then we show that this map is functorial (Proposition 2.15). In particular, we get a homomorphism from the group of exact autoequivalences of $D^b(A)$ to the group $U(A \times \hat{A})$ of isometric automorphisms of $A \times \hat{A}$.

The kernel of this homomorphism is described in §3. It is isomorphic to the direct sum of the free abelian group \mathbb{Z} and the group of k -points of the variety $A \times \hat{A}$ (Proposition 3.3). This fact is technically based on Proposition 3.2, which states that the object on the product of abelian varieties which determines an equivalence is actually a sheaf up to a shift in the derived category. We note that this result holds for abelian varieties only (for example, it is not true for K3-surfaces) and is a key to the description of the autoequivalence group in the case of abelian varieties.

In §4 we assume that the ground field is algebraically closed and $\text{char}(k) = 0$ and give another proof of the results of [13]. This proof is based on the results of [10] that describe semihomogeneous bundles on abelian varieties. In particular, we get an exact sequence of groups

$$0 \rightarrow \mathbb{Z} \oplus (A \times \hat{A})_k \rightarrow \text{Auteq } D^b(A) \rightarrow U(A \times \hat{A}) \rightarrow 1.$$

In conclusion we describe the central extension of $U(A \times \hat{A})$ by the group \mathbb{Z} and give a formula for the 2-cocycle that determines this extension.

The author is grateful to the Max Planck Institute of Mathematics for the hospitality and stimulating atmosphere. It is also a pleasure for the author to express his gratitude to the Foundation for Promotion of National Science

§ 1. PRELIMINARY INFORMATION

Let X be an algebraic variety over a field k with structure sheaf \mathcal{O}_X . For every variety we have the abelian category $\text{coh}(X)$ of coherent sheaves on it.

We denote by $D^b(X)$ the bounded derived category of the abelian category $\text{coh}(X)$. It is obtained from the category of bounded complexes of coherent sheaves by inversion of all quasi-isomorphisms, that is, those maps between complexes that induce isomorphisms in cohomology (see, for example, [3]).

Every derived category has the structure of a triangulated category. This means that the additive category \mathcal{D} is endowed with

- a) an additive shift functor $[1]: \mathcal{D} \rightarrow \mathcal{D}$, which is an autoequivalence,
- b) a class of distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, which must satisfy certain axioms (see [15]).

An additive functor $F: \mathcal{D} \rightarrow \mathcal{D}'$ between triangulated categories is called *exact* if it commutes with the shift functors and transforms every distinguished triangle of \mathcal{D} to a distinguished triangle of \mathcal{D}' .

In what follows we assume that all varieties are smooth and projective. Any morphism $f: X \rightarrow Y$ between smooth projective varieties induces two exact functors: the direct image functor $\mathbf{R}f_*: D^b(X) \rightarrow D^b(Y)$ and the inverse image functor $\mathbf{L}f^*: D^b(Y) \rightarrow D^b(X)$. Moreover, each object $\mathcal{F} \in D^b(X)$ determines the exact functor of tensor product

$$\mathbf{L} \otimes \mathcal{F}: D^b(X) \rightarrow D^b(X).$$

Using these functors, we can introduce a large class of exact functors between the categories $D^b(X)$ and $D^b(Y)$.

Let X, Y be smooth projective varieties over the field k . We consider the Cartesian product $X \times Y$ and denote by p, q the projections of $X \times Y$ onto X, Y respectively:

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y.$$

Each object $\mathcal{E} \in D^b(X \times Y)$ determines an exact functor $\Phi_{\mathcal{E}}$ from the derived category $D^b(X)$ to the derived category $D^b(Y)$ by the formula

$$\Phi_{\mathcal{E}}(\cdot) := \mathbf{R}q_*(\mathcal{E} \otimes^{\mathbf{L}} p^*(\cdot)). \quad (1.1)$$

Moreover, the same object $\mathcal{E} \in D^b(X \times Y)$ determines another functor $\Psi_{\mathcal{E}}$ from the derived category $D^b(Y)$ to the derived category $D^b(X)$ by the formula similar to (1.1):

$$\Psi_{\mathcal{E}}(\cdot) := \mathbf{R}p_*(\mathcal{E} \otimes^{\mathbf{L}} q^*(\cdot)).$$

It is easily verified that $\Phi_{\mathcal{E}}$ has left and right adjoint functors $\Phi_{\mathcal{E}}^*$ and $\Phi_{\mathcal{E}}^!$ respectively. They are given by

$$\Phi_{\mathcal{E}}^* \cong \Psi_{\mathcal{E}^\vee \otimes q^* \omega_Y[\dim Y]}, \quad \Phi_{\mathcal{E}}^! \cong \Psi_{\mathcal{E}^\vee \otimes p^* \omega_X[\dim X]}. \quad (1.2)$$

Here ω_X and ω_Y are the canonical sheaves on X and Y , and \mathcal{E}^\vee is a convenient notation for $\mathbf{R}Hom(\mathcal{E}, \mathcal{O}_{X \times Y})$.

Let now X, Y, Z be three smooth projective varieties, and let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be objects of the derived categories $D^b(X \times Y), D^b(Y \times Z), D^b(X \times Z)$ respectively. We consider the following diagram of projections:

The objects $\mathcal{E}, \mathcal{F}, \mathcal{G}$ determine the functors

$$\begin{aligned}\Phi_{\mathcal{E}} &: D^b(X) \rightarrow D^b(Y), \\ \Phi_{\mathcal{F}} &: D^b(Y) \rightarrow D^b(Z), \\ \Phi_{\mathcal{G}} &: D^b(X) \rightarrow D^b(Z)\end{aligned}$$

by formula (1.1), that is,

$$\begin{aligned}\Phi_{\mathcal{E}} &:= \mathbf{R}\pi_{12*}^2(\mathcal{E} \otimes^{\mathbf{L}} \pi_{12}^1(\cdot)), \\ \Phi_{\mathcal{F}} &:= \mathbf{R}\pi_{23*}^3(\mathcal{F} \otimes^{\mathbf{L}} \pi_{23}^2(\cdot)), \\ \Phi_{\mathcal{G}} &:= \mathbf{R}\pi_{13*}^3(\mathcal{G} \otimes^{\mathbf{L}} \pi_{13}^1(\cdot)).\end{aligned}$$

We consider the object $p_{12}^*\mathcal{E} \otimes p_{23}^*\mathcal{F} \in D^b(X \times Y \times Z)$. In what follows we always denote it by $\mathcal{E} \boxtimes_Y \mathcal{F}$. The following assertion from [9] yields the composition law for those exact functors between derived categories that are represented by objects on the product.

Proposition 1.1. *The composite functor $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}}$ is isomorphic to the functor $\Phi_{\mathcal{G}}$ represented by the object*

$$\mathcal{G} = \mathbf{R}p_{13*}(\mathcal{E} \boxtimes_Y \mathcal{F}). \quad (1.3)$$

Thus, to every smooth projective variety there corresponds the derived category of coherent sheaves on it, and to every object $\mathcal{E} \in D^b(X \times Y)$ on the product of such varieties there corresponds the exact functor $\Phi_{\mathcal{E}}$ from the triangulated category $D^b(X)$ to the triangulated category $D^b(Y)$ with the composition law described above.

The following two questions are fundamental in understanding this correspondence.

1) When are derived categories of coherent sheaves on two different smooth projective varieties equivalent as triangulated categories?

2) What is the group of exact autoequivalences of the derived category of coherent sheaves on a given variety X ?

Some results in this direction are already known. For example, there is a complete answer to these questions in the case when either canonical or anticanonical sheaf of the variety is ample.

Theorem 1.2 [2]. *Let X be a smooth projective variety whose canonical (or anticanonical) sheaf is ample. Suppose that the category $D^b(X)$ is equivalent (as a triangulated category) to the derived category $D^b(X')$ for some smooth algebraic variety X' . Then X' is isomorphic to X .*

Theorem 1.3 [2]. *Let X be a smooth projective variety whose canonical (or anticanonical) sheaf is ample. Then the group of isomorphism classes of exact autoequivalences of the category $D^b(X)$ is generated by automorphisms of the variety, twists by line bundles and shifts in the derived category.*

The group of exact autoequivalences may also be described. For any variety X , the group $\text{Auteq } D^b(X)$ of exact autoequivalences always contains a subgroup $G(X)$ which is a semidirect product of its normal subgroup $G_1 = \text{Pic}(X) \oplus \mathbb{Z}$ and the subgroup $G_2 = \text{Aut } X$ with its natural action on G_1 . Under the inclusion $G(X) \subset \text{Auteq } D^b(X)$, the generator of \mathbb{Z} is mapped to the shift functor $[1]$, each line bundle $\mathcal{L} \in \text{Pic}(X)$ is mapped to the functor $\otimes \mathcal{L}$, and each automorphism $f: X \rightarrow X$ induces the autoequivalence $\mathbf{R}f_*$.

Under the hypotheses of Theorem 1.3, we can additionally assert that the group $\text{Auteq } D^b(X)$ of exact autoequivalences coincides with $G(X)$, that is,

$$\text{Auteq } D^b(X) \cong \text{Aut } X \ltimes (\text{Pic}(X) \oplus \mathbb{Z}).$$

To study the cases of equivalence of the derived categories of coherent sheaves on two varieties and to describe their autoequivalence groups, it is desirable to have explicit formulae for all exact functors. It is conjectured that they are all represented by objects on the product, that is, are given by (1.1). This conjecture is proved in the particular case of equivalences.

Theorem 1.4 [12]. *Let X, Y be smooth projective varieties. Suppose that $F: D^b(X) \xrightarrow{\sim} D^b(Y)$ is an exact functor and an equivalence of triangulated categories. Then there is a unique (up to an isomorphism) object $\mathcal{E} \in D^b(X \times Y)$ such that the functor F is isomorphic to the functor $\Phi_{\mathcal{E}}$.*

To verify that a functor F is an equivalence, it suffices to show that F and its right (or left) adjoint functor are fully faithful. We recall that a functor F is *fully faithful* if, for any objects A and B , the natural map

$$\text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$$

is a bijection. In what follows we need a method to determine whether the functors $\Phi_{\mathcal{E}}: D^b(X) \rightarrow D^b(Y)$ are fully faithful. This is rather difficult to verify in general, but the following criterion is useful in some situations.

Theorem 1.5 [1]. *Let M, X be smooth projective varieties over an algebraically closed field of characteristic 0. Take $\mathcal{E} \in D^b(M \times X)$. Then the functor $\Phi_{\mathcal{E}}$ is fully faithful if and only if the following orthogonality conditions hold:*

- 1) $\text{Hom}_X^i(\Phi_{\mathcal{E}}(\mathcal{O}_{t_1}), \Phi_{\mathcal{E}}(\mathcal{O}_{t_2})) = 0$ for all i and $t_1 \neq t_2$;
- 2) $\text{Hom}_X^0(\Phi_{\mathcal{E}}(\mathcal{O}_t), \Phi_{\mathcal{E}}(\mathcal{O}_t)) = k$;
- 3) $\text{Hom}_X^i(\Phi_{\mathcal{E}}(\mathcal{O}_t), \Phi_{\mathcal{E}}(\mathcal{O}_t)) = 0$ for $i \notin [0, \dim M]$.

Here t, t_1, t_2 are points of M , and \mathcal{O}_{t_i} are the corresponding skyscraper sheaves.

Let now X_1, X_2, Y_1, Y_2 be smooth projective varieties. We consider objects \mathcal{E}_1 and \mathcal{E}_2 of the categories $D^b(X_1 \times Y_1)$ and $D^b(X_2 \times Y_2)$ respectively. By definition, the object

$$\mathcal{E}_1 \boxtimes \mathcal{E}_2 \in D^b((X_1 \times X_2) \times (Y_1 \times Y_2)),$$

is equal to $p_{13}^*(\mathcal{E}_1) \otimes p_{24}^*(\mathcal{E}_2)$. As before (see (1.1)), the objects $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2$ determine the functors

$$\Phi_{\mathcal{E}_1}: D^b(X_1) \rightarrow D^b(Y_1),$$

$$\Phi_{\mathcal{E}_2}: D^b(X_2) \rightarrow D^b(Y_2),$$

$$\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}: D^b(X_1 \times X_2) \rightarrow D^b(Y_1 \times Y_2)$$

We consider an object $\mathcal{G} \in D^b(X_1 \times X_2)$ and denote the object $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}(\mathcal{G}) \in D^b(Y_1 \times Y_2)$ by \mathcal{H} . By (1.1), these objects determine the functors

$$\Phi_{\mathcal{G}}: D^b(X_1) \rightarrow D^b(X_2), \quad \Phi_{\mathcal{H}}: D^b(Y_1) \rightarrow D^b(Y_2).$$

Lemma 1.6. *In the notation above, we have an isomorphism of functors $\Phi_{\mathcal{H}} \cong \Phi_{\mathcal{E}_2} \circ \Phi_{\mathcal{G}} \circ \Psi_{\mathcal{E}_1}$.*

The proof immediately follows from Proposition 1.1.

Let Z_1, Z_2 be further smooth projective varieties, and let $\mathcal{F}_1, \mathcal{F}_2$ be objects of the categories $D^b(Y_1 \times Z_1), D^b(Y_2 \times Z_2)$ respectively. Then we also have the functors $\Phi_{\mathcal{F}_1}, \Phi_{\mathcal{F}_2}, \Phi_{\mathcal{F}_1 \boxtimes \mathcal{F}_2}$. By (1.3), we can find objects \mathcal{G}_1 and \mathcal{G}_2 of the categories $D^b(X_1 \times Z_1)$ and $D^b(X_2 \times Z_2)$ such that

$$\Phi_{\mathcal{G}_1} \cong \Phi_{\mathcal{F}_1} \circ \Phi_{\mathcal{E}_1}, \quad \Phi_{\mathcal{G}_2} \cong \Phi_{\mathcal{F}_2} \circ \Phi_{\mathcal{E}_2}.$$

It is directly verified that there is a natural relation

$$\Phi_{\mathcal{F}_1 \boxtimes \mathcal{F}_2} \circ \Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \cong \Phi_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}. \quad (1.4)$$

Using this relation, we easily prove the following assertion.

Assertion 1.7. *Under the conditions above, assume that the functors $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$ are fully faithful (resp. equivalences). Then the functor*

$$\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}: D^b(X_1 \times X_2) \rightarrow D^b(Y_1 \times Y_2)$$

is also fully faithful (resp. an equivalence).

Proof. Let F be a functor having an adjoint (say, the left adjoint F^*). Then F is fully faithful if and only if the composite $F^* \circ F$ is isomorphic to the identity functor. The functors $\Phi_{\mathcal{E}_i}$ have left adjoints $\Phi_{\mathcal{E}_i}^*$, which are defined by (1.2). If they are fully faithful, then the composites $\Phi_{\mathcal{E}_i}^* \circ \Phi_{\mathcal{E}_i}$ are isomorphic to the identity functors, which are presented by the structure sheaves of the diagonals $\Delta_i \subset X_i \times X_i$. We easily verify that the sheaf $\mathcal{O}_{\Delta_1} \boxtimes \mathcal{O}_{\Delta_2}$ is isomorphic to the structure sheaf \mathcal{O}_{Δ} , where Δ is the diagonal in $(X_1 \times X_2) \times (X_1 \times X_2)$.

Using (1.4), we see that the composite $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}^* \circ \Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ is represented by the structure sheaf of the diagonal Δ . Hence it is isomorphic to the identity functor. Thus $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ is fully faithful. The assertion on the equivalences is proved similarly.

Suppose that the functor $\Phi_{\mathcal{E}}: D^b(X) \rightarrow D^b(Y)$ is an equivalence and that an object $\mathcal{F} \in D^b(X \times Y)$ satisfies $\Psi_{\mathcal{F}} \cong \Phi_{\mathcal{E}}^{-1}$. Then we denote the functor

$$\Phi_{\mathcal{F} \boxtimes \mathcal{E}}: D^b(X \times X) \rightarrow D^b(Y \times Y) \quad (1.5)$$

by $\text{Ad}_{\mathcal{E}}$. The functor $\text{Ad}_{\mathcal{E}}$ is an equivalence by Assertion 1.7. Moreover, by Lemma 1.6, for every object $\mathcal{G} \in D^b(X \times X)$ there is an isomorphism of functors

$$\Phi_{\text{Ad}_{\mathcal{E}}(\mathcal{G})} \cong \Phi_{\mathcal{E}} \circ \Phi_{\mathcal{G}} \circ \Phi_{\mathcal{E}}^{-1}. \quad (1.6)$$

§ 2. EQUIVALENCES BETWEEN THE CATEGORIES
OF COHERENT SHEAVES ON ABELIAN VARIETIES

Let A be an abelian variety of dimension n over a field k . We denote by $m: A \times A \rightarrow A$ the composition morphism (regarded as a morphism defined over k) and let e be the k -point which is the identity of the group structure.

We denote by \hat{A} the dual abelian variety, which is the moduli space of line bundles on A that belong to $\text{Pic}^0(A)$. Moreover, \hat{A} is a fine moduli space. Hence the product $A \times \hat{A}$ carries the universal line bundle P , which is called the *Poincaré bundle*. This bundle is uniquely determined by the condition that, for every k -point $\alpha \in \hat{A}$, the restriction of P to $A \times \{\alpha\}$ is isomorphic to the line bundle in $\text{Pic}^0(A)$ that corresponds to α , and the restriction $P|_{\{e\} \times \hat{A}}$ is trivial.

Definition 2.1. In what follows we denote by P_α the line bundle on A that corresponds to the k -point $\alpha \in \hat{A}$. Moreover, suppose that A_1, \dots, A_m are abelian varieties, and let $(\alpha_1, \dots, \alpha_m)$ be a k -point of $\hat{A}_1 \times \dots \times \hat{A}_m$. Then we denote by $P_{(\alpha_1, \dots, \alpha_k)}$ the line bundle $P_{\alpha_1} \boxtimes \dots \boxtimes P_{\alpha_k}$ on the product $A_1 \times \dots \times A_k$.

For any homomorphism $f: A \rightarrow B$ of abelian varieties we have the dual homomorphism $\hat{f}: \hat{B} \rightarrow \hat{A}$. It maps a point $\beta \in \hat{B}$ to a point $\alpha \in \hat{A}$ if and only if the line bundle f^*P_β coincides with the bundle P_α on A .

The second dual abelian variety $\hat{\hat{A}}$ can naturally be identified with A using the Poincaré bundles on $A \times \hat{A}$ and $\hat{A} \times \hat{\hat{A}}$. In other words, there is a unique isomorphism $\kappa_A: A \xrightarrow{\sim} \hat{\hat{A}}$ such that the lifting of the Poincaré bundle $P_{\hat{A}}$ by the isomorphism $1 \times \kappa_A: \hat{A} \times A \xrightarrow{\sim} \hat{A} \times \hat{\hat{A}}$ coincides with the Poincaré bundle P_A , that is, $(1 \times \kappa_A)^*P_{\hat{A}} \cong P_A$.

Hence the correspondence $\hat{}$ is an involution on the category of abelian varieties, that is, $\hat{}$ is a contravariant functor whose square is isomorphic to the identity functor. The isomorphism is given by the map $\kappa: \text{Id} \xrightarrow{\sim} \hat{}$.

Remark 2.2 ($k = \mathbb{C}$). Let A be an abelian variety over \mathbb{C} . We choose a basis l_1, \dots, l_{2n} in $H_1(A, \mathbb{Z})$, and let l_1^*, \dots, l_{2n}^* be the dual basis in $H_1(\hat{A}, \mathbb{Z}) \cong H_1(A, \mathbb{Z})^*$. Let $l_1^{**}, \dots, l_{2n}^{**}$ be the basis in $H_1(\hat{\hat{A}}, \mathbb{Z})$ dual to l_1^*, \dots, l_{2n}^* . The isomorphism $\kappa_A: A \xrightarrow{\sim} \hat{\hat{A}}$ induces an identification of $H_1(A, \mathbb{Z})$ with $H_1(\hat{\hat{A}}, \mathbb{Z})$ such that the elements l_i are identified with $-l_i^{**}$. The sign “minus” appears because the forms $c_1(P_A)$ and $c_1(P_{\hat{A}})$ are skew-symmetric.

Remark 2.3 ($k = \mathbb{C}$). Let $f: A \rightarrow B$ be a homomorphism of complex abelian varieties, and let $\hat{f}: \hat{B} \rightarrow \hat{A}$ be the dual homomorphism. We fix bases in the homology groups $H_1(A, \mathbb{Z})$, $H_1(B, \mathbb{Z})$ and take the dual bases in the first homology groups of \hat{A} , \hat{B} . The maps f and \hat{f} induce linear maps between the first homology groups, and we denote their matrices by F and \hat{F} . Then the matrix \hat{F} is transposed to F .

Consider a homomorphism $f: A \rightarrow \hat{A}$. By the definition of the isomorphism κ , we may assume that \hat{f} is also a homomorphism from A to \hat{A} . Hence the matrices F and \hat{F} are skew-transposed to each other, that is, $\hat{F} = -F^t$.

The Poincaré bundle P provides an example of an exact equivalence between the derived categories of coherent sheaves on two (generally non-isomorphic) varieties

A and \hat{A} . We consider the projections

$$A \xleftarrow{p} A \times \hat{A} \xrightarrow{q} \hat{A}$$

and define a functor $\Phi_P: D^b(A) \rightarrow D^b(\hat{A})$ by (1.1), that is, $\Phi_P(\cdot) = \mathbf{R}q_*(P \otimes p^*(\cdot))$.

Proposition 2.4 [9]. *Let P be the Poincaré bundle on $A \times \hat{A}$. Then the functor $\Phi_P: D^b(A) \rightarrow D^b(\hat{A})$ is an exact equivalence, and there is an isomorphism of functors*

$$\Psi_P \circ \Phi_P \cong (-1_A)^*[n],$$

where (-1_A) is the morphism of group inversion.

Remark 2.5. This assertion is proved in [9] for abelian varieties over an algebraically closed field. However, it holds over an arbitrary field since the dual variety and the Poincaré bundle are always defined over the same field (see [11], for example) while the assertion on the equivalence of the functor will follow from Lemma 2.12 (see below).

For every k -point $a \in A$ there is the shift automorphism $m(\cdot, a): A \rightarrow A$, which will be denoted by T_a . For every k -point $\alpha \in \hat{A}$, we denote by P_α the corresponding line bundle on A .

We now consider a k -point $(a, \alpha) \in A \times \hat{A}$. It determines a functor from $D^b(A)$ to itself by

$$\Phi_{(a, \alpha)}(\cdot) := T_{a*}(\cdot) \otimes P_\alpha = T_{-a}^*(\cdot) \otimes P_\alpha. \quad (2.1)$$

The functor $\Phi_{(a, \alpha)}$ is represented by the sheaf

$$S_{(a, \alpha)} = \mathcal{O}_{\Gamma_a} \otimes p_2^*(P_\alpha) \quad (2.2)$$

on the product $A \times A$, where Γ_a is the graph of the shift automorphism T_a . The functor $\Phi_{(a, \alpha)}$ is clearly an autoequivalence.

The set of functors $\Phi_{(a, \alpha)}$ (parametrized by $A \times \hat{A}$) can be combined into a single functor from $D^b(A \times \hat{A})$ to $D^b(A \times A)$, which maps the structure sheaf $\mathcal{O}_{(a, \alpha)}$ of the point to $S_{(a, \alpha)}$. (We note that this condition determines the functor non-uniquely but only up to multiplying by a line bundle lifted from $A \times \hat{A}$.)

We define a functor $\Phi_{S_A}: D^b(A \times \hat{A}) \rightarrow D^b(A \times A)$ as a composite of two other functors. To do this, we consider the object $\mathcal{P}_A = p_{14}^* \mathcal{O}_\Delta \otimes p_{23}^* P \in D^b((A \times \hat{A}) \times (A \times A))$ and denote by $\mu_A: A \times A \rightarrow A \times A$ the morphism that sends a point (a_1, a_2) to $(a_1, m(a_1, a_2))$. We get two functors:

$$\Phi_{\mathcal{P}_A}: D^b(A \times \hat{A}) \rightarrow D^b(A \times A), \quad \mathbf{R}\mu_{A*}: D^b(A \times A) \rightarrow D^b(A \times A).$$

Definition 2.6. The functor Φ_{S_A} is the composite $\mathbf{R}\mu_{A*} \circ \Phi_{\mathcal{P}_A}$.

We can explicitly describe the object S_A on the product $(A \times \hat{A}) \times (A \times A)$ that represents the functor Φ_{S_A} . Since the explicit formula is not used in what follows, we present it without proof.

Lemma 2.7. *Let S_A be the object on the product $(A \times \hat{A}) \times (A \times A)$ that represents the functor Φ_{S_A} . Then*

$$S_A = (m \cdot p_{13}, p_4)^* \mathcal{O}_\Delta \otimes p_{23}^* P_A.$$

Here $(m \cdot p_{13}, p_4)$ is the morphism onto $A \times A$ that maps a point (a_1, α, a_3, a_4) to the point $(m(a_1, a_3), a_4)$.

Assertion 2.8. *The functor Φ_{S_A} is an equivalence and, for every k -point $(a, \alpha) \in A \times \hat{A}$,*

a) Φ_{S_A} *maps the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of the point to the sheaf $S_{(a,\alpha)}$ defined by (2.2),*

b) Φ_{S_A} *maps the line bundle $P_{(\alpha,a)}$ on $A \times \hat{A}$ to the object $\mathcal{O}_{\{-a\} \times A} \otimes p_2^* P_\alpha[n]$.*

Proof. By definition, Φ_{S_A} is the composite of the functors $\mathbf{R}\mu_{A*}$ and $\Phi_{\mathcal{P}_A}$, which are equivalences. (This is obvious for the first functor, and this follows from Assertion 1.7 and Proposition 2.4 for the second one.)

The functor $\Phi_{\mathcal{P}_A}$ maps the structure sheaf $\mathcal{O}_{(a,\alpha)}$ to the sheaf $\mathcal{O}_{A \times \{a\}} \otimes p_1^* P_\alpha$, and the functor $\mathbf{R}\mu_{A*}$ maps the sheaf $\mathcal{O}_{A \times \{a\}} \otimes p_1^* P_\alpha$ to the sheaf $\mathcal{O}_{\Gamma_a} \otimes p_1^*(P_\alpha)$.

Using Proposition 2.4, we find in the same way that the functor $\Phi_{\mathcal{P}_A}$ maps the line bundle $P_{(\alpha,a)}$ to the object $\mathcal{O}_{\{-a\} \times A} \otimes p_2^* P_\alpha[n]$, and the functor $\mathbf{R}\mu_{A*}$ maps the object $\mathcal{O}_{\{-a\} \times A} \otimes p_2^* P_\alpha[n]$ to itself.

Suppose that A and B are abelian varieties whose derived categories of coherent sheaves are equivalent. We fix an equivalence. By Theorem 1.4, it is represented by an object on the product. Thus we have an object $\mathcal{E} \in D^b(A \times B)$ and the equivalence $\Phi_{\mathcal{E}}: D^b(A) \xrightarrow{\sim} D^b(B)$.

We recall that the functor

$$\text{Ad}_{\mathcal{E}}: D^b(A \times A) \xrightarrow{\sim} D^b(B \times B)$$

is defined by (1.5) and is an equivalence. We consider the composite functor $\Phi_{S_B}^{-1} \circ \text{Ad}_{\mathcal{E}} \circ \Phi_{S_A}$.

Definition 2.9. We denote by $\mathcal{J}(\mathcal{E})$ the object that represents the functor

$$\Phi_{S_B}^{-1} \circ \text{Ad}_{\mathcal{E}} \circ \Phi_{S_A}.$$

Thus there is a commutative diagram

$$\begin{array}{ccc} D^b(A \times \hat{A}) & \xrightarrow{\Phi_{S_A}} & D^b(A \times A) \\ \Phi_{\mathcal{J}(\mathcal{E})} \downarrow & & \downarrow \text{Ad}_{\mathcal{E}} \\ D^b(B \times \hat{B}) & \xrightarrow{\Phi_{S_B}} & D^b(B \times B) \end{array} \quad (2.3)$$

The following theorem enables us to describe the object $\mathcal{J}(\mathcal{E})$ and is basic for the description of abelian varieties with equivalent derived categories of coherent sheaves.

Theorem 2.10. *One can find a homomorphism $f_{\mathcal{E}}: A \times \hat{A} \rightarrow B \times \hat{B}$ of abelian varieties and a line bundle $L_{\mathcal{E}}$ on $A \times \hat{A}$ such that $f_{\mathcal{E}}$ is an isomorphism and the object $\mathcal{J}(\mathcal{E})$ is isomorphic to $i_*(L_{\mathcal{E}})$, where i is the embedding of $A \times \hat{A}$ into $(A \times \hat{A}) \times (B \times \hat{B})$ as the graph of the isomorphism $f_{\mathcal{E}}$.*

Before proving this theorem, we state two lemmas that enable us to assume that k is algebraically closed. We denote the algebraic closure of k by \bar{k} . We put $\bar{X} := X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$ and denote by $\bar{\mathcal{F}}$ the inverse image of \mathcal{F} under the morphism $\bar{X} \rightarrow X$.

Lemma 2.11. *Let \mathcal{F} be a coherent sheaf on a smooth variety X . Suppose that we have a closed subvariety $j: Z \hookrightarrow \overline{X}$ and an invertible sheaf \mathcal{L} on Z such that $\overline{\mathcal{F}} \cong j_*\mathcal{L}$. Then we can find a closed subvariety $i: Y \hookrightarrow X$ and an invertible sheaf \mathcal{M} on Y such that $\mathcal{F} \cong i_*\mathcal{M}$ and $j = \overline{i}$.*

Proof. Since the argument is local, we may assume that we have an affine variety $X = \text{Spec}(A)$ and an A -module M . Let $J \subset A$ be the annihilator of the module M , and let $J' \subset \overline{A} = A \otimes_k \overline{k}$ be the annihilator of the module $\overline{M} = M \otimes_k \overline{k}$. If $\{e_i\}$ is a basis of the field \overline{k} over k , then $\overline{M} = \bigoplus M e_i$ as a module over A . Clearly, $J \otimes_k \overline{k} \subseteq J'$. On the other hand, if an element $\sum a_i \otimes e_i \in \overline{A}$ belongs to J' , then $\sum a_i m \otimes e_i = 0$ for every $m \in M$. Hence each a_i belongs to J . Thus, $J \otimes_k \overline{k} = J'$.

The hypothesis of the lemma implies that \overline{M} is a projective module of rank 1 over the algebra $\overline{B} := \overline{A}/J' = B \otimes_k \overline{k}$, where $B = A/J$, and $\overline{M} = M \otimes_B \overline{B}$. Since \overline{B} is a strictly flat B -algebra, we see that M is a projective B -module of rank 1 (see, for example, [4], (3.1.4)).

The following lemma asserts that the property of a functor to be fully faithful (or an equivalence) is stable with respect to field extensions.

Lemma 2.12. *Let X, Y be smooth projective varieties over k , and let \mathcal{E} be an object of the derived category $D^b(X \times Y)$. Consider a field extension F/k and the varieties*

$$X' = X \times_{\text{Spec}(k)} \text{Spec}(F), \quad Y' = Y \times_{\text{Spec}(k)} \text{Spec}(F).$$

Let \mathcal{E}' be the lifting of the object \mathcal{E} to the category $D^b(X' \times Y')$. The functor $\Phi_{\mathcal{E}}: D^b(X) \rightarrow D^b(Y)$ is fully faithful (resp. an equivalence) if and only if the functor $\Phi_{\mathcal{E}'}: D^b(X') \rightarrow D^b(Y')$ is fully faithful (resp. an equivalence).

Proof. As above, we denote by $\Phi_{\mathcal{E}}^*$ the left adjoint to the functor $\Phi_{\mathcal{E}}$. If $\Phi_{\mathcal{E}}$ is fully faithful, then the composite $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}}^*$ is the identity functor $\text{id}_{D^b(X)}$, which is known to be represented by the structure sheaf \mathcal{O}_{Δ} of the diagonal in the product $X \times X$. Using Proposition 1.1 and the theorem on flat base change, we see that the composite $\Phi_{\mathcal{E}'} \circ \Phi_{\mathcal{E}'}^*$ is represented by the structure sheaf $\mathcal{O}_{\Delta'}$, where Δ' is the diagonal in $X' \times X'$. Hence the functor $\Phi_{\mathcal{E}'}$ is fully faithful.

Conversely, consider the composite $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}}^*$. It is represented by some object \mathcal{J} on $X \times X$. There is a canonical morphism $\phi: \mathcal{J} \rightarrow \mathcal{O}_{\Delta}$. Since $\Phi_{\mathcal{E}'}$ is fully faithful by the hypothesis, the morphism $\phi': \mathcal{J}' \rightarrow \mathcal{O}_{\Delta'}$ is an isomorphism. It follows immediately that ϕ is an isomorphism as well, whence the functor $\Phi_{\mathcal{E}}$ is fully faithful.

We similarly prove the assertion on equivalences, which follows since the adjoint functor is fully faithful.

Proof of Theorem 2.10. Using Lemmas 2.11 and 2.12, we may pass to the algebraic closure of k .

Step 1. We denote by e and e' the closed points of $A \times \hat{A}$ and $B \times \hat{B}$ (respectively) that are the identity elements of the group structures. We consider the skyscraper sheaf \mathcal{O}_e and calculate its image under the functor $\Phi_{\mathcal{J}(\mathcal{E})}$. By definition, we know that

$$\Phi_{\mathcal{J}(\mathcal{E})} = \Phi_{S_B}^{-1} \circ \text{Ad}_{\mathcal{E}} \circ \Phi_{S_A}.$$

By Assertion 2.8, the functor Φ_{S_A} maps the sheaf \mathcal{O}_e to the structure sheaf $\mathcal{O}_{\Delta(A)}$ of the diagonal in $A \times A$. Since the structure sheaf of the diagonal represents the

identity functor, (1.6) implies that $\text{Ad}_{\mathcal{E}}(\mathcal{O}_{\Delta(A)})$ is the structure sheaf $\mathcal{O}_{\Delta(B)}$ of the diagonal in $B \times B$. By Assertion 2.8, the functor $\Phi_{S_B}^{-1}$ maps the last sheaf to the structure sheaf $\mathcal{O}_{e'}$.

Step 2. We thus obtain that

$$\mathcal{J}(\mathcal{E}) \oplus^{\mathbf{L}} \mathcal{O}_{\{e\} \times (B \times \widehat{B})} \cong \mathcal{O}_{\{e\} \times \{e'\}}.$$

It follows that there is an affine neighbourhood $U = \text{Spec}(R)$ of e in the Zariski topology such that the object $\mathcal{J}' := \mathcal{J}(\mathcal{E})|_{U \times (B \times \widehat{B})}$ is a coherent sheaf whose support intersects the fibre $\{e\} \times (B \times \widehat{B})$ at the point $\{e\} \times \{e'\}$. We recall that the support of a coherent sheaf is a closed subset.

We now consider an affine neighbourhood $V = \text{Spec}(S)$ of e' in $B \times \widehat{B}$. The intersection of the support of \mathcal{J}' with the complement $B \times \widehat{B} \setminus V$ is a closed subset whose projection onto $A \times \widehat{A}$ is a closed subset that does not contain e .

Shrinking U if necessary, we may thus assume that it is still affine and the support of \mathcal{J}' is contained in $U \times V$. This means that there is a coherent sheaf \mathcal{F} on $U \times V$ such that $j_*(\mathcal{F}) = \mathcal{J}'$, where j is the inclusion of $U \times V$ into $U \times (B \times \widehat{B})$. We denote by M the finitely generated $R \otimes S$ -module corresponding to the sheaf \mathcal{F} , that is, $\mathcal{F} = \widetilde{M}$. Moreover, M is a finitely generated R -module since the direct image of the coherent sheaf $\mathcal{J}' = j_*\mathcal{F}$ under the projection is a coherent sheaf.

Let m be the maximal ideal in R corresponding to the point e . It is known that

$$M \otimes_R R/m \cong R/m.$$

Hence there is a homomorphism $\phi: R \rightarrow M$ of R -modules, which becomes an isomorphism after the tensor multiplication by R/m . Hence the supports of the coherent sheaves $\text{Ker } \phi$ and $\text{Coker } \phi$ do not contain e . Replacing U by a smaller affine neighbourhood of e , disjoint from the supports of $\text{Ker } \phi$ and $\text{Coker } \phi$, we see that ϕ is an isomorphism. Hence there is a subscheme $X(U) \subset U \times (B \times \widehat{B})$ such that the projection $X(U) \rightarrow U$ is an isomorphism and

$$\mathcal{J}' = \mathcal{J}(\mathcal{E})|_{U \times (B \times \widehat{B})} \cong \mathcal{O}_{X(U)}.$$

Step 3. We thus see that for every closed point $(a, \alpha) \in U$ there is a closed point $(b, \beta) \in B \times \widehat{B}$ such that

$$\Phi_{\mathcal{J}(\mathcal{E})}(\mathcal{O}_{(a, \alpha)}) \cong \mathcal{O}_{(b, \beta)}.$$

Any closed point $(a, \alpha) \in A \times \widehat{A}$ may be presented as a sum $(a, \alpha) = (a_1, \alpha_1) + (a_2, \alpha_2)$, where the points $(a_1, \alpha_1), (a_2, \alpha_2)$ belong to U . We denote by (b_1, β_1) and (b_2, β_2) the images of these points under the functor $\Phi_{\mathcal{J}(\mathcal{E})}$. It is known that the functor Φ_{S_A} maps the structure sheaf $\mathcal{O}_{(a, \alpha)}$ to the sheaf $S_{(a, \alpha)}$. We denote the object $\text{Ad}_{\mathcal{E}}(S_{(a, \alpha)})$ by \mathcal{G} . To calculate it, we use (1.6). We have

$$\Phi_{\mathcal{G}} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a, \alpha)} \circ \Phi_{\mathcal{E}}^{-1}.$$

But the functor $\Phi_{(a, \alpha)}$, which equals $T_a^*(\cdot) \otimes P_{\alpha}$ by definition (2.1), is represented as the composite $\Phi_{(a_1, \alpha_1)} \circ \Phi_{(a_2, \alpha_2)}$. We thus get a sequence of isomorphisms

$$\begin{aligned} \Phi_{\mathcal{G}} &\cong \Phi_{\mathcal{E}} \circ \Phi_{(a, \alpha)} \circ \Phi_{\mathcal{E}}^{-1} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a_1, \alpha_1)} \circ \Phi_{\mathcal{E}}^{-1} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a_2, \alpha_2)} \circ \Phi_{\mathcal{E}}^{-1} \\ &\sim \Phi_{\mathcal{E}} \circ \Phi_{(a_1, \alpha_1)} \circ \Phi_{\mathcal{E}}^{-1} \sim \Phi_{\mathcal{E}} \circ \Phi_{(a_2, \alpha_2)} \circ \Phi_{\mathcal{E}}^{-1} \end{aligned}$$

where $(b, \beta) = (b_1, \beta_1) + (b_2, \beta_2)$. Hence the object \mathcal{G} is isomorphic to $S_{(b, \beta)}$. We finally obtain that

$$\Phi_{\mathcal{J}(\mathcal{E})}(\mathcal{O}_{(a, \alpha)}) \cong \mathcal{O}_{(b, \beta)}$$

for every closed point $(a, \alpha) \in A \times \hat{A}$.

For every closed point (a, a') , we can now repeat the procedure of Step 2 to find a neighbourhood W and a subscheme $X(W) \subset W \times (B \times \hat{B})$ such that the projection $X(W) \rightarrow W$ is an isomorphism and $\mathcal{J}|_{W \times (B \times \hat{B})} \cong \mathcal{O}_{X(W)}$.

Gluing all these neighbourhoods, we find a subvariety $i: X \hookrightarrow (A \times \hat{A}) \times (B \times \hat{B})$ such that the projection $X \rightarrow A \times \hat{A}$ is an isomorphism and the sheaf $\mathcal{J}(\mathcal{E})$ is isomorphic to the sheaf i_*L , where L is a line bundle on X . The subvariety X determines a homomorphism from $A \times \hat{A}$ to $B \times \hat{B}$ which induces an equivalence of derived categories. Hence this homomorphism is an isomorphism.

In particular, Theorem 2.10 immediately implies that if abelian varieties A and B have equivalent derived categories of coherent sheaves, then the varieties $A \times \hat{A}$ and $B \times \hat{B}$ are isomorphic. Below we shall see that this isomorphism must satisfy an additional condition (see Proposition 2.18).

Corollary 2.13. *The isomorphism $f_{\mathcal{E}}$ maps a k -point $(a, \alpha) \in A \times \hat{A}$ to a point $(b, \beta) \in B \times \hat{B}$ if and only if the equivalences*

$$\Phi_{(a, \alpha)}: D^b(A) \xrightarrow{\sim} D^b(A), \quad \Phi_{(b, \beta)}: D^b(B) \xrightarrow{\sim} D^b(B)$$

defined by (2.1), satisfy the relation

$$\Phi_{(b, \beta)} \circ \Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a, \alpha)}.$$

This relation is equivalent to the following condition on the objects that represent these functors:

$$T_{b*}\mathcal{E} \otimes P_{\beta} \cong T_{-a*}\mathcal{E} \otimes P_{\alpha} = T_a^*\mathcal{E} \otimes P_{\alpha}.$$

Proof. By Theorem 2.10, the functor $\Phi_{\mathcal{J}(\mathcal{E})}$ maps the structure sheaf $\mathcal{O}_{(a, \alpha)}$ of the point (a, α) to the structure sheaf $\mathcal{O}_{(b, \beta)}$ of the point $(b, \beta) = f_{\mathcal{E}}(a, \alpha)$. Assertion 2.8 implies that Φ_{S_A} maps the sheaf $\mathcal{O}_{(a, \alpha)}$ to $S_{(a, \alpha)}$. The sheaf $S_{(a, \alpha)}$ in its turn represents the functor

$$\Phi_{(a, \alpha)} = T_{a*}(\cdot) \otimes P_{\alpha}.$$

Using diagram (2.3), we now see that $f_{\mathcal{E}}$ maps the point (a, α) to (b, β) if and only if $S_{(b, \beta)} \cong \text{Ad}_{\mathcal{E}}(S_{(a, \alpha)})$. Applying (1.6), we find that $\Phi_{(b, \beta)} \cong \Phi_{\mathcal{E}} \circ \Phi_{(a, \alpha)} \circ \Phi_{\mathcal{E}}^{-1}$.

In what follows we use an explicit formula for the object $\mathcal{J}(\mathcal{E})$ in the particular case when $A = B$ and the equivalence $\Phi_{\mathcal{E}}$ equals $\Phi_{(a, \alpha)}$, which is defined by (2.1).

Proposition 2.14. *Suppose that $A = B$. We define the equivalence $\Phi_{(a, \alpha)}$ by (2.1) and consider the object $S_{(a, \alpha)}$ on $A \times A$ that represents $\Phi_{(a, \alpha)}$. Then the sheaf $\mathcal{J}(S_{(a, \alpha)})$ is equal to $\Delta_*P_{(\alpha, -a)}$, where Δ is the diagonal embedding of $A \times \hat{A}$ into $(A \times \hat{A}) \times (A \times \hat{A})$ and $P_{(\alpha, -a)}$ is the line bundle on $A \times \hat{A}$ from Definition 2.1.*

Proof. By Assertion 2.8, the functor Φ_{S_A} maps the structure sheaf $\mathcal{O}_{(a', \alpha')}$ to the sheaf $S_{(a', \alpha')}$ on $A \times A$ (see (2.2)). The functor $\text{Ad}_{S_{(a, \alpha)}}$ maps the sheaf $S_{(a', \alpha')}$ to itself. Indeed, (1.6) implies that the object $\text{Ad}_{S_{(a, \alpha)}}(S_{(a', \alpha')})$ represents the functor

$$\Phi_{(a, \alpha)} \circ \Phi_{(a', \alpha')} \circ \Phi_{(a, \alpha)}^{-1}$$

which is in turn isomorphic to $\Phi_{(a',\alpha')}$ since all such functors commute. We thus see that the functor determined by the sheaf $\mathcal{J}(S_{(a,\alpha)})$ maps the structure sheaf of any point to itself, whence it is a line bundle L supported on the diagonal.

To find the line bundle L , we find the image of the bundle $P_{(\alpha',a')}$ under this functor. Using Assertion 2.8, we see that Φ_{S_A} maps the bundle $P_{(\alpha',a')}$ to the object $\mathcal{O}_{\{-a'\} \times A} \otimes p_2^*(P_{\alpha'})[n]$. It is easily verified that the further action of $\text{Ad}_{S_{(a,\alpha)}}$ maps this object to the object $\mathcal{O}_{\{-a'+a\} \times A} \otimes p_2^*(P_{\alpha'+\alpha})[n]$. Hence the functor determined by the sheaf $\mathcal{J}(S_{(a,\alpha)})$ maps the bundle $P_{(\alpha',a')}$ to the bundle $P_{(\alpha'+\alpha,a'-a)}$. Therefore the bundle L is isomorphic to $P_{(\alpha,-a)}$.

Given abelian varieties A and B , we denote by $\mathcal{E}q(A, B)$ the set of all exact equivalence from the category $D^b(A)$ to the category $D^b(B)$ up to an isomorphism.

We consider two groupoids \mathfrak{A} and \mathfrak{D} (that is, categories all of whose morphisms are invertible). The objects of both categories are abelian varieties. The morphisms in \mathfrak{A} are isomorphisms between the abelian varieties as algebraic groups. The morphisms in \mathfrak{D} are exact equivalences between the derived categories of coherent sheaves on abelian varieties, that is,

$$\begin{aligned} \mathcal{M}or_{\mathfrak{A}}(A, B) &:= \text{Iso}(A, B), \\ \mathcal{M}or_{\mathfrak{D}}(A, B) &:= \mathcal{E}q(A, B). \end{aligned}$$

By Theorem 2.10 there is a map of the set $\mathcal{E}q(A, B)$ to the set $\text{Iso}(A \times \hat{A}, B \times \hat{B})$, which sends each equivalence $\Phi_{\mathcal{E}}$ to the isomorphism $f_{\mathcal{E}}$. We consider the map F from \mathfrak{D} to \mathfrak{A} that sends an abelian variety A to the variety $A \times \hat{A}$ and acts on the morphisms as described above.

Proposition 2.15. *The map $F: \mathfrak{D} \rightarrow \mathfrak{A}$ is a functor.*

Proof. To prove this, we must verify that F preserves composition of morphisms. Consider abelian varieties A, B, C . Let \mathcal{E} and \mathcal{F} be objects of the categories $D^b(A \times B)$ and $D^b(B \times C)$ respectively such that the functors

$$\begin{aligned} \Phi_{\mathcal{E}}: D^b(A) &\rightarrow D^b(B), \\ \Phi_{\mathcal{F}}: D^b(B) &\rightarrow D^b(C) \end{aligned}$$

are equivalences. We denote by \mathcal{G} the object in $D^b(A \times C)$ that represents the composite of these functors.

The relation (1.4) yields an isomorphism $\text{Ad}_{\mathcal{G}} \cong \text{Ad}_{\mathcal{F}} \circ \text{Ad}_{\mathcal{E}}$, whence we get

$$\begin{aligned} \Phi_{\mathcal{J}(\mathcal{F})} \circ \Phi_{\mathcal{J}(\mathcal{E})} &\cong (\Phi_{S_A}^{-1} \circ \text{Ad}_{\mathcal{F}} \circ \Phi_{S_A}) \circ (\Phi_{S_A}^{-1} \circ \text{Ad}_{\mathcal{E}} \circ \Phi_{S_A}) \\ &\cong \Phi_{S_A}^{-1} \circ \text{Ad}_{\mathcal{G}} \circ \Phi_{S_A} \cong \Phi_{\mathcal{J}(\mathcal{G})}. \end{aligned}$$

By Theorem 2.10, the objects $\mathcal{J}(\mathcal{E}), \mathcal{J}(\mathcal{F}), \mathcal{J}(\mathcal{G})$ are line bundles supported on the graphs of the isomorphisms $f_{\mathcal{E}}, f_{\mathcal{F}}, f_{\mathcal{G}}$. We thus get the equation $f_{\mathcal{G}} = f_{\mathcal{F}} \cdot f_{\mathcal{E}}$.

Corollary 2.16. *Suppose that A is an abelian variety and $\Phi_{\mathcal{E}}$ is an autoequivalence of the derived category $D^b(A)$. Then the map $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ determines a group homomorphism*

$$\text{Aut}(D^b(A)) \rightarrow \text{Aut}(A \times \hat{A})$$

We thus have the functor $F: \mathfrak{D} \rightarrow \mathfrak{A}$. Our next purpose is to describe it. To do this, we must study which elements of $\text{Iso}(A \times \hat{A}, B \times \hat{B})$ may be realized as $f_{\mathcal{E}}$ for some \mathcal{E} and which equivalences \mathcal{E}_1 and \mathcal{E}_2 satisfy the equation $f_{\mathcal{E}_1} = f_{\mathcal{E}_2}$.

We consider an arbitrary morphism $f: A \times \hat{A} \rightarrow B \times \hat{B}$. It is convenient to write it as a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where the morphism α maps A to B , β maps \hat{A} to B , γ maps A to \hat{B} , and δ maps \hat{A} to \hat{B} . Each morphism f determines two other morphisms \hat{f} and \tilde{f} from $B \times \hat{B}$ to $A \times \hat{A}$ whose matrices are

$$\hat{f} = \begin{pmatrix} \hat{\delta} & \hat{\beta} \\ \hat{\gamma} & \hat{\alpha} \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{pmatrix}.$$

We define a set $U(A \times \hat{A}, B \times \hat{B})$ as the subset of all f in $\text{Iso}(A \times \hat{A}, B \times \hat{B})$ such that \tilde{f} coincides with the inverse to f , that is,

$$U(A \times \hat{A}, B \times \hat{B}) := \{f \in \text{Iso}(A \times \hat{A}, B \times \hat{B}) \mid \tilde{f} = f^{-1}\}.$$

If $B = A$, we denote this set by $U(A \times \hat{A})$. We note that $U(A \times \hat{A})$ is a subgroup in $\text{Aut}(A \times \hat{A})$.

Definition 2.17. An isomorphism $f: A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$ is called *isometric* if it belongs to $U(A \times \hat{A}, B \times \hat{B})$.

Proposition 2.18. For every equivalence $\Phi_{\mathcal{E}}: D^b(A) \xrightarrow{\sim} D^b(B)$, the isomorphism $f_{\mathcal{E}}$ is isometric.

Proof. Passing to the algebraic closure if necessary, we may assume that the field k is algebraically closed. To verify the equation $\tilde{f}_{\mathcal{E}} = f_{\mathcal{E}}^{-1}$, it suffices to prove that these morphisms coincide at closed points. Suppose that $f_{\mathcal{E}}$ sends a point $(a, \alpha) \in A \times \hat{A}$ to a point $(b, \beta) \in B \times \hat{B}$. We claim that $\tilde{f}_{\mathcal{E}}(b, \beta) = (a, \alpha)$ or, equivalently, that $\hat{f}(-b, \beta) = (-a, \alpha)$.

The isomorphism $f_{\mathcal{E}}$ is determined by an abelian subvariety $X \hookrightarrow A \times \hat{A} \times B \times \hat{B}$. Hence we must verify that $P_{(0,0,\beta,-b)} \otimes \mathcal{O}_X \cong P_{(\alpha,-a,0,0)} \otimes \mathcal{O}_X$ or, equivalently, that the sheaf $\mathcal{J}' := P_{(-\alpha,a,\beta,-b)} \otimes \mathcal{J}(\mathcal{E})$ is isomorphic to the sheaf $\mathcal{J}(\mathcal{E})$.

By Proposition 2.14, the functor determined by \mathcal{J}' is a composite of the functors represented by the objects $\mathcal{J}(S_{(-a,-\alpha)})$, $\mathcal{J}(\mathcal{E})$ and $\mathcal{J}(S_{(b,\beta)})$. Thus \mathcal{J}' coincides with $\mathcal{J}(\mathcal{E}')$, where \mathcal{E}' is the object in $D^b(A \times B)$ that represents the functor

$$\Phi_{(b,\beta)} \circ \Phi_{\mathcal{E}} \circ \Phi_{(-a,-\alpha)}.$$

This composite is isomorphic to the functor $\Phi_{\mathcal{E}}$ by Corollary 2.13. Hence the object \mathcal{E}' is isomorphic to \mathcal{E} , and we have $\mathcal{J}' = \mathcal{J}(\mathcal{E}') \cong \mathcal{J}(\mathcal{E})$.

As a corollary of Theorem 2.10 and Proposition 2.18, we get the following result.

Theorem 2.19. Let A, B be abelian varieties over a field k . If the derived categories of coherent sheaves $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories, then there is an isometric isomorphism between $A \times \hat{A}$ and $B \times \hat{B}$.

The converse assertion is also true for abelian varieties over an algebraically closed field of characteristic 0 [12]. Another proof of this result is given in § 4.

Corollary 2.20. *For every abelian variety A there are only finitely many non-isomorphic abelian varieties whose derived categories of coherent sheaves are equivalent to $D^b(A)$ (as triangulated categories).*

Proof. As proved in [6], for every abelian variety Z there are only finitely many (up to an isomorphism) abelian varieties that can be embedded into Z as an abelian subvariety. Applying this assertion to $Z = A \times \hat{A}$ and using Theorem 2.19, we get the required result.

In conclusion we would like to explain the term “isometric” for elements of $U(A \times \hat{A}, B \times \hat{B})$. Suppose that k is the field of complex numbers. We denote the first homology lattices $H_1(A, \mathbb{Z})$ and $H_1(B, \mathbb{Z})$ by Γ_A and Γ_B respectively. Any lattice representable as $\Gamma \oplus \Gamma^*$ carries a canonical symmetric bilinear form

$$Q((x, l), (y, m)) = l(y) + m(x).$$

We denote by Q_A and Q_B the corresponding symmetric bilinear forms on $\Lambda_A := H_1(A \times \hat{A}, \mathbb{Z})$ and $\Lambda_B := H_1(B \times \hat{B}, \mathbb{Z})$.

The set of homomorphisms from $A \times \hat{A}$ to $B \times \hat{B}$ is a subset of $\text{Hom}_{\mathbb{Z}}(\Lambda_A, \Lambda_B)$. The elements of $U(A \times \hat{A}, B \times \hat{B})$ can be described in these terms as follows.

Proposition 2.21. *An isomorphism $f: A \times \hat{A} \rightarrow B \times \hat{B}$ belongs to $U(A \times \hat{A}, B \times \hat{B})$ if and only if it determines an isometry of the lattices (Λ_A, Q_A) and (Λ_B, Q_B) , that is, $F^t Q_B F = Q_A$, where $F: \Lambda_A \rightarrow \Lambda_B$ is the map induced by f on the first homology.*

The proof is obtained by a direct matrix computation using Remark 2.3.

§ 3. OBJECTS THAT REPRESENT EQUIVALENCES, AND THE AUTOEQUIVALENCE GROUPS

Propositions 2.15, 2.18 imply that there is a homomorphism from the group $\text{Auteq } D^b(A)$ of exact autoequivalences to the group $U(A \times \hat{A})$ of isometric isomorphisms. In this section we describe the kernel of this homomorphism. By Proposition 2.14 we know that all equivalences $\Phi_{(a, \alpha)}[n]$ (see (2.1)) belong to the kernel. We shall show that they exhaust the kernel. To prove this, we use an assertion of independent interest: for abelian varieties, if the functor $\Phi_{\mathcal{E}}$ is an equivalence, then the object \mathcal{E} is a sheaf on the product (up to a shift in the derived category). This assertion is specific to abelian varieties and breaks in other cases, say, for K3-surfaces.

Lemma 3.1. *Let \mathcal{E} be an object on the product $A \times B$ such that $\Phi_{\mathcal{E}}: D^b(A) \rightarrow D^b(B)$ is an equivalence. We consider the projection $q: (A \times \hat{A}) \times (B \times \hat{B}) \rightarrow A \times B$ and denote by K the direct image $\mathbf{R}q_* \mathcal{J}(\mathcal{E})$, where $\mathcal{J}(\mathcal{E})$ is the object from Definition 2.9. Then K is isomorphic to the object $\mathcal{E} \otimes (\mathcal{E}^\vee|_{(0,0)})$, where $\mathcal{E}^\vee|_{(0,0)}$ denotes the complex of vector spaces which is the inverse image of the object $\mathbf{R}\text{Hom}(\mathcal{E}, \mathcal{O}_{A \times B})$ under the embedding of the point $(0, 0)$ to the abelian variety $A \times B$.*

Proof. We consider the abelian variety

$$Z = (A \times \hat{A}) \times (A \times \hat{A}) \times (B \times \hat{B}) \times (B \times \hat{B})$$

and the object

$$H = p_{1234}^* S_A \otimes p_{35}^* \mathcal{E}^\vee[n] \otimes p_{46}^* \mathcal{E} \otimes p_{5678}^* S_B^\vee[2n].$$

Using Proposition 1.1 on the composition of functors and diagram (2.3), we see that $\mathcal{J}(\mathcal{E}) \cong p_{1278*} H$. Hence, the object K equals $p_{17*} H$. To compute the last object, we consider the projection of Z onto

$$V = A \times (A \times A) \times (B \times B) \times B$$

and denote it by v . To calculate $v_* H$, we use the fact that the functor Φ_{S_A} is the composite of $\Phi_{\mathcal{P}_A}$ and $\mathbf{R}\mu_{A*}$, where

$$\mathcal{P}_A = p_{14}^* \mathcal{O}_\Delta \otimes p_{23}^* P \in D^b((A \times \hat{A}) \times (A \times A)).$$

It is easy to see that $p_{134*} \mathcal{P}_A \cong \mathcal{O}_{T_A}[-n]$, where $T \subset A \times A \times A$ is the subvariety isomorphic to A and consisting of the points $(a, 0, a)$. Taking into account that $\mu_A(a_1, a_2) = (a_1, m(a_1, a_2))$, we see that $p_{134*} S_A$ is also isomorphic to $\mathcal{O}_{T_A}[-n]$. We similarly verify that $p_{134*} S_B^\vee[2n] = \mathcal{O}_{T_B}$.

Thus we have

$$v_* H \cong p_{123}^* \mathcal{O}_{T_A} \otimes p_{24}^* \mathcal{E}^\vee \otimes p_{35}^* \mathcal{E} \otimes p_{456}^* \mathcal{O}_{T_B}$$

on V . We consider the embedding

$$j: A \times A \times B \times B \rightarrow V, \quad (a_1, a_2, b_1, b_2) \mapsto (a_1, 0, a_2, 0, b_1, b_2).$$

The object $v_* H$ is isomorphic to $j_* M$, where

$$M = (\mathcal{E}^\vee|_{(0,0)}) \otimes p_{12}^* \mathcal{O}_{\Delta_A} \otimes p_{23}^* \mathcal{E} \otimes p_{34}^* \mathcal{O}_{\Delta_B}.$$

We finally obtain that $K \cong p_{14*} M \cong (\mathcal{E}^\vee|_{(0,0)}) \otimes \mathcal{E}$.

Proposition 3.2. *Let A, B be abelian varieties, and let \mathcal{E} be an object of $D^b(A \times B)$ such that the functor $\Phi_{\mathcal{E}}: D^b(A) \xrightarrow{\sim} D^b(B)$ is an exact equivalence. Then \mathcal{E} has only one non-trivial cohomology, that is, \mathcal{E} is isomorphic to the object $\mathcal{F}[n]$, where \mathcal{F} is a sheaf on $A \times B$.*

Proof. Consider the projection

$$q: (A \times \hat{A}) \times (B \times \hat{B}) \rightarrow A \times B$$

and denote by q' its restriction to the abelian subvariety X which is the support of the sheaf $\mathcal{J}(\mathcal{E})$ and the graph of the isomorphism $f_{\mathcal{E}}$. By Theorem 2.10, the sheaf $\mathcal{J}(\mathcal{E})$ equals $i_*(L)$, where L is a line bundle on X .

We denote the object $\mathbf{R}q_* \mathcal{J}(\mathcal{E}) = \mathbf{R}q'_* L$ by K . The morphism q' is a homomorphism of abelian varieties. Let d be the dimension of $\text{Ker}(q')$. Then $\dim \text{Im}(q') = 2n - d$, whence the cohomology sheaves $H^j(K)$ are trivial for $j \notin [0, d]$.

On the other hand, the object K is isomorphic to $\mathcal{E} \otimes (\mathcal{E}^\vee|_{(0,0)})$ by Lemma 3.1.

Shifting the object \mathcal{E} in the derived category if necessary, we may assume that the highest non-zero cohomology of \mathcal{E} is $U^0(\mathcal{E})$. Let $U^{-i}(\mathcal{E})$ be the lowest non-zero

cohomology of \mathcal{E} (here $i \geq 0$), and let $H^k(\mathcal{E}^\vee)$ be the highest non-zero cohomology of \mathcal{E}^\vee . Replacing \mathcal{E} by $T_{(a,b)}^* \mathcal{E}$ if necessary, we may assume that the point $(0,0)$ belongs to the support of the sheaf $H^k(\mathcal{E}^\vee)$. Since the supports of \mathcal{E} and K coincide, the supports of all cohomology sheaves of \mathcal{E} belong to $\text{Im}(q')$. In particular, we have $\text{codim Supp } H^{-i}(\mathcal{E}) \geq d$. Hence the object $(H^{-i}(\mathcal{E}))^\vee[-i]$ has trivial cohomologies in all degrees less than $i+d$.

The canonical morphism $H^{-i}(\mathcal{E})[i] \rightarrow \mathcal{E}$ induces a non-trivial morphism

$$\mathcal{E}^\vee \rightarrow (H^{-i}(\mathcal{E}))^\vee[-i].$$

Since the degrees of non-trivial cohomologies of the second object belong to $[i+d, \infty)$, we see that $k \geq i+d$, where $H^k(\mathcal{E}^\vee)$ is the highest non-zero cohomology of \mathcal{E}^\vee . Thus we obtain that the object

$$K = \mathcal{E}^\vee|_{(0,0)} \otimes \mathcal{E} \quad (3.1)$$

has a non-trivial cohomology at the same degree $k \geq i+d$. On the other hand, it is known that the cohomology sheaves $H^j(K)$ are trivial for $j \notin [0, d]$. This is possible only when $i=0$. Hence the object \mathcal{E} has only one non-trivial cohomology at degree 0 and, therefore, \mathcal{E} is isomorphic to a sheaf.

We consider the case $B \cong A$. Let \mathcal{E} be a sheaf on $A \times A$ such that $\Phi_{\mathcal{E}}$ is an autoequivalence. Let us describe all \mathcal{E} such that $f_{\mathcal{E}}$ is the identity map and, therefore, the graph X of this map is the diagonal in $(A \times \hat{A}) \times (A \times \hat{A})$. We thus obtain that the object

$$K = \mathcal{E}^\vee|_{(0,0)} \otimes \mathcal{E} = \mathbf{R}q_* \mathcal{J}(\mathcal{E})$$

takes the form $\Delta_*(M)$, where M is an object on A and $\Delta: A \rightarrow A \times A$ is the diagonal embedding.

We suppose that the point $(0,0)$ belongs to the support of the sheaf \mathcal{E} . Hence $\mathcal{E}^\vee|_{(0,0)}$ is a non-trivial complex of vector spaces. Then the assumption $K = \Delta_*(M)$ implies that there is a sheaf E on A such that $\mathcal{E} \cong \Delta_*(E)$. Therefore $\Phi_{\mathcal{E}}(\cdot) \cong E \otimes (\cdot)$. Since $\Phi_{\mathcal{E}}$ is an autoequivalence, E is a line bundle. We easily verify that the condition $f_{\mathcal{E}} = \text{id}$ may hold only when $E \in \text{Pic}^0(A)$.

If the point $(0,0)$ does not belong to $\text{Supp } \mathcal{E}$, then we can replace \mathcal{E} by a sheaf $\mathcal{E}' := T_{(a_1, a_2)*} \mathcal{E}$ whose support contains $(0,0)$. Proposition 2.14 implies that $f_{\mathcal{E}'} = f_{\mathcal{E}}$. As shown above, there is an isomorphism $\mathcal{E}' \cong \Delta_*(E')$, where $E' \in \text{Pic}^0(A)$. Hence $\mathcal{E} \cong T_{(a_1 - a_2, 0)*} \Delta_*(E')$. Thus we get the following proposition.

Proposition 3.3. *Let A be an abelian variety. Then the kernel of the homomorphism*

$$\gamma_A: \text{Auteq } D^b(A) \rightarrow U(A \times \hat{A})$$

consists of the autoequivalences $\Phi_{(a,\alpha)}[i] = T_{a}(\cdot) \otimes P_\alpha[i]$ and is thus isomorphic to the group $\mathbb{Z} \oplus (A \times \hat{A})_k$, where $(A \times \hat{A})_k$ is the group of k -points of the abelian variety $A \times \hat{A}$.*

Corollary 3.4. *Let A, B be abelian varieties, and let $\mathcal{E}_1, \mathcal{E}_2$ be objects on the product $A \times B$ that determine equivalences between the derived categories of coherent sheaves. If $f_{\mathcal{E}_1} = f_{\mathcal{E}_2}$, then*

$$\mathcal{E}_2 \cong T_{a*} \mathcal{E}_1 \otimes P_\alpha[i]$$

for some k -point $(a, \alpha) \in A \times \hat{A}$

§ 4. SEMIHOMOGENEOUS VECTOR BUNDLES

As shown in previous sections, an equivalence $\Phi_{\mathcal{E}}$ from $D^b(A)$ to $D^b(B)$ induces an isometric isomorphism of the varieties $A \times \hat{A}$ and $B \times \hat{B}$. In this section we suppose that the field k is algebraically closed and $\text{char}(k) = 0$. Using the technique of [10] and the results of [1], we shall show under these assumptions that every isometric isomorphism $f: A \times \hat{A} \rightarrow B \times \hat{B}$ is induced by some equivalence $\Phi_{\mathcal{E}}$. The fact that the existence of an isometric isomorphism between $A \times \hat{A}$ and $B \times \hat{B}$ implies equivalence of the derived categories $D^b(A)$ and $D^b(B)$ was proved in [13]. We shall give another proof of this fact.

We recall that any line bundle L on an abelian variety D determines a map ϕ_L from D to \hat{D} , which sends a point d to the point corresponding to the bundle $T_d^*L \otimes L^{-1} \in \text{Pic}^0(D)$. This correspondence yields an embedding of $\text{NS}(D)$ into $\text{Hom}(D, \hat{D})$. Moreover, it is known that a map $\phi: D \rightarrow \hat{D}$ belongs to the image of $\text{NS}(D)$ if and only if $\hat{\phi} = \phi$.

Semihomogeneous bundles on abelian varieties enable us to generalize this phenomenon as follows. Every element of $\text{NS}(D) \otimes \mathbb{Q}$ determines some correspondence on $D \times \hat{D}$, and every such correspondence is obtained from some semihomogeneous bundle (see Proposition 4.6 below and Lemma 2.13). We recall some results on homogeneous and semihomogeneous bundles on abelian varieties.

Definition 4.1. A vector bundle E on an abelian variety D is called *homogeneous* if $T_d^*(E) \cong E$ for every point $d \in D$.

Definition 4.2. A vector bundle F on an abelian variety D is called *unipotent* if there is a filtration

$$0 = F_0 \subset F_1 \subset \dots \subset F_n = F$$

such that $F_i/F_{i-1} \cong \mathcal{O}_D$ for all $i = 1, \dots, n$.

The following proposition characterizes homogeneous vector bundles.

Proposition 4.3 [8], [10]. *Let E be a vector bundle on an abelian variety D . Then the following conditions are equivalent.*

- (i) *The vector bundle E is homogeneous.*
- (ii) *There are line bundles $P_i \in \text{Pic}^0(D)$ and unipotent bundles F_i such that $E \cong \bigoplus_i (F_i \otimes P_i)$.*

Definition 4.4. A vector bundle E on an abelian variety D is called *semihomogeneous* if, for every point $d \in D$, there is a line bundle L on D such that $T_d^*(E) \cong E \otimes L$. (We note that then L belongs to $\text{Pic}^0(D)$.)

We recall that a vector bundle on a variety is called *simple* if its automorphism algebra coincides with the field k .

Proposition 4.5 ([10], **Theorem 5.8**). *Let E be a simple vector bundle on an abelian variety D . Then the following conditions are equivalent.*

- 1) $\dim H^j(D, \text{End}(E)) = \binom{n}{j}$ for every $j = 0, \dots, n$.
- 2) E is a semihomogeneous bundle.
- 3) $\text{End}(E)$ is a homogeneous bundle.
- 4) $E \cong \tau^*(L)$ for some isogeny $\tau: Y \rightarrow D$ and some line bundle L on Y .

Let E be a vector bundle on an abelian variety D . We denote by $\mu(E)$ the equivalence class of $\frac{\det(E)}{r(E)}$ in $\mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$. Every element $\mu = \frac{[L]}{l} \in \mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ (hence every bundle E) determines a correspondence $\Phi_{\mu} \subset D \times \widehat{D}$ by

$$\Phi_{\mu} = \mathrm{Im}[D \xrightarrow{(l, \phi_L)} D \times \widehat{D}].$$

Here ϕ_L is the well-known map from D to \widehat{D} which sends a point d to the point corresponding to the bundle $T_d^*L \otimes L^{-1} \in \mathrm{Pic}^0(D)$. We denote the projections of Φ_{μ} onto D and \widehat{D} by q_1 and q_2 respectively. In the particular case when the bundle is a line bundle L , we get the graph of the map $\phi_L: D \rightarrow \widehat{D}$.

A complete description of all simple semihomogeneous bundles is given in [10].

Proposition 4.6 ([10], **Theorem 7.10**). *Suppose that $\mu = \frac{[L]}{l}$, where $[L]$ is the equivalence class of the bundle L in $\mathrm{NS}(D)$ and l is a positive integer. Then*

- 1) *there is a simple semihomogeneous vector bundle E of slope $\mu(E) = \mu$;*
- 2) *every simple semihomogeneous vector bundle E' of slope $\mu(E') = \mu$ coincides with $E \otimes M$ for some line bundle $M \in \mathrm{Pic}^0(D)$;*
- 3) *we have equations $r(E)^2 = \deg(q_1)$ and $\chi(E)^2 = \deg(q_2)$.*

The following assertion enables us to characterize all semihomogeneous vector bundles in terms of simple bundles.

Proposition 4.7 ([10], **Propositions 6.15, 6.16**). *Every semihomogeneous vector bundle F of slope μ admits a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

such that $E_i = F_i/F_{i-1}$ are simple semihomogeneous vector bundles of the same slope μ . Every simple semihomogeneous bundle is stable.

We shall use the following two lemmas on semihomogeneous bundles. These lemmas are direct corollaries of the previous assertions.

Lemma 4.8. *Any two simple semihomogeneous bundles E_1, E_2 of the same slope μ are either isomorphic or orthogonal to each other, that is, either $E_1 = E_2$ or*

$$\mathrm{Ext}^i(E_1, E_2) = 0, \quad \mathrm{Ext}^i(E_2, E_1) = 0$$

for all i .

Proof. Proposition 4.6 implies that $E_2 \cong E_1 \otimes M$, whence $\mathcal{H}om(E_1, E_2)$ is a homogeneous bundle. By Proposition 4.3, every homogeneous bundle is equal to a direct sum of unipotent bundles twisted by line bundles belonging to $\mathrm{Pic}^0(D)$. Therefore, either all cohomologies of $\mathcal{H}om(E_1, E_2)$ are equal to zero (and hence the bundles E_1, E_2 are orthogonal), or the bundle $\mathcal{H}om(E_1, E_2)$ has a non-zero section. In the last case we get a non-zero homomorphism from E_1 to E_2 . But these two bundles are stable and have the same slope. Hence every non-zero homomorphism is actually an isomorphism.

Lemma 4.9. *Let E be a simple semihomogeneous vector bundle on an abelian variety D . We have $T_d^*(E) \cong E \otimes P_\delta$ if and only if $(d, \delta) \in \Phi_\mu$.*

Proof. Let us prove that $T_d^*(E) \cong E \otimes P_\delta$ for every point $(d, \delta) \in \Phi_\mu$. Indeed, we put $l = r(E)$ and $L = \det(E)$. By definition of Φ_μ we can write $(d, \delta) = (lx, \phi_L(x))$ for some point $x \in D$. Since E is semihomogeneous, there is a line bundle $M \in \text{Pic}^0(D)$ such that

$$T_x^*(E) \cong E \otimes M. \quad (4.1)$$

Comparing the determinants, we get $T_x^*(L) \cong L \otimes M^{\otimes l}$. By definition of the map ϕ_L , this means that $P_{\phi_L(x)} = M^{\otimes l}$. On the other hand, iterating the equation (4.1) l times, we get

$$T_{lx}^*(E) \cong E \otimes M^{\otimes l} = E \otimes P_{\phi_L(x)}.$$

Therefore $T_d^*(E) \cong E \otimes P_\delta$ because $(d, \delta) = (lx, \phi_L(x))$.

Conversely, define a subgroup $\Sigma^0(E) \subset \widehat{D}$ by the condition

$$\Sigma^0(E) := \{\delta \in \widehat{D} \mid E \otimes P_\delta \cong E\}. \quad (4.2)$$

Since E is semihomogeneous, the bundle $\mathcal{E}nd(E)$ is homogeneous by Proposition 4.5. Thus $\mathcal{E}nd(E)$ may be presented as a sum $\bigoplus_i (F_i \otimes P_i)$, where all F_i are unipotent. Hence $H^0(\mathcal{E}nd(E) \otimes P) \neq 0$ for at most r^2 line bundles $P \in \text{Pic}^0(D)$, that is, the order of the group $\Sigma^0(E)$ does not exceed r^2 . On the other hand, it is known that $q_2(\text{Ker}(q_1)) \subset \Sigma^0(E)$. Hence we obtain that $\text{ord } \Sigma^0(E) = r^2$ and $q_2(\text{Ker}(q_1)) = \Sigma^0(E)$.

We now suppose that $T_d^*(E) \cong E \otimes P_\delta$ for some point $(d, \delta) \in D \times \widehat{D}$. Consider the point $\delta' \in \widehat{D}$ such that $(d, \delta') \in \Phi_\mu$. It was shown that then we have an isomorphism $T_d^*(E) \cong E \otimes P_{\delta'}$. Hence $E \otimes P_{(\delta - \delta')} \cong E$ and, therefore, $(\delta - \delta') \in \Sigma^0(E)$. Since $\Sigma^0(E) = q_2(\text{Ker}(q_1))$, the point $(0, \delta - \delta')$ belongs to Φ_μ . Hence the point (d, δ) also belongs to Φ_μ .

We now present a construction that starts from an isometric isomorphism f and produces an object \mathcal{E} on the product such that \mathcal{E} determines an equivalence of the derived categories and $f_{\mathcal{E}}$ coincides with f .

Construction 4.10. We fix an isometric isomorphism $f: A \times \hat{A} \rightarrow B \times \hat{B}$ and denote its graph by Γ . As above, we write f in the matrix form:

$$f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Suppose that $y: \hat{A} \rightarrow B$ is an isogeny. Then the map f determines an element $g \in \text{Hom}(A \times B, \hat{A} \times \hat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the formula

$$g = \begin{pmatrix} y^{-1}x & -y^{-1} \\ -\hat{y}^{-1} & w y^{-1} \end{pmatrix}.$$

The element g determines a correspondence on $(A \times B) \times (\hat{A} \times \hat{B})$. Since f is isometric, we easily verify that $\hat{g} = g$. This means that g actually belongs to the image of $\text{NS}(A \times B) \otimes \mathbb{Q}$ under the canonical embedding into $\text{Hom}(A \times B, \hat{A} \times \hat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

$\widehat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (see, for example, [11]). Hence there is $\mu = \frac{[L]}{t} \in \text{NS}(A \times B)$ such that Φ_{μ} coincides with the graph of the correspondence g . Proposition 4.6 asserts that for every μ one can construct a simple semihomogeneous bundle \mathcal{E} on $A \times B$ of slope $\mu(\mathcal{E}) = \mu$.

Below, we shall see that the functor $\Phi_{\mathcal{E}}$ from $D^b(A)$ to $D^b(B)$ is an equivalence and $f_{\mathcal{E}} = f$. But now we compare the graphs Γ and Φ_{μ} . If a point (a, α, b, β) belongs to Γ , then

$$b = x(a) + y(\alpha), \quad \beta = z(a) + w(\alpha)$$

and, therefore,

$$\begin{aligned} \alpha &= -y^{-1}x(a) + y^{-1}(b), \\ \beta &= (z - wy^{-1}x)(a) + wy^{-1}(b). \end{aligned}$$

Since f is isometric, we have $(z - wy^{-1}x) = -\hat{y}^{-1}$. Hence the point (a, α, b, β) belongs to Γ if and only if $(a, -\alpha, b, \beta)$ belongs to Φ_{μ} . We thus find that $\Phi_{\mu} = (1_A, -1_{\widehat{A}}, 1_B, 1_{\widehat{B}})\Gamma$.

In particular, since f is an isomorphism, the projections of Φ_{μ} onto $A \times \widehat{A}$ and $B \times \widehat{B}$ are also isomorphisms.

Proposition 4.11. *Let \mathcal{E} be the semihomogeneous bundle on $A \times B$ constructed from an isometric isomorphism f in the way described above. Then the functor $\Phi_{\mathcal{E}}: D^b(A) \rightarrow D^b(B)$ is an equivalence.*

Proof. We denote by \mathcal{E}_a the restriction of the bundle \mathcal{E} to the fibre $\{a\} \times B$. By Theorem 1.5, to prove that $\Phi_{\mathcal{E}}$ is fully faithful, we must verify that all the bundles \mathcal{E}_a are simple and orthogonal to each other for different points.

We first note that, by Proposition 4.6, the rank of the bundle \mathcal{E} equals the square root of the degree of the map $\Phi_{\mu} \rightarrow A \times B$, that is, $\sqrt{\deg(\beta)}$.

It follows from the semihomogeneity of \mathcal{E} that all bundles \mathcal{E}_a are semihomogeneous. Moreover, the slope $\mu(\mathcal{E}_a)$ of the restriction is equal to $\delta\beta^{-1} \in \text{NS}(B) \otimes \mathbb{Q} \subset \text{Hom}(B, \widehat{B}) \otimes \mathbb{Q}$. For brevity, we denote the element $\delta\beta^{-1}$ by ν and regard it as an element of $\text{NS}(B) \otimes \mathbb{Q}$. By Proposition 4.6 there is a simple semihomogeneous bundle F on B of slope $\mu(F) = \nu$. Then Φ_{ν} is clearly equal to $\text{Im}[\widehat{A} \xrightarrow{(\beta, \delta)} B \times \widehat{B}]$.

Since f is an isomorphism, the map $\widehat{A} \xrightarrow{(\beta, \delta)} B \times \widehat{B}$ is an embedding. Using Proposition 4.6 again, we get the equations $r(F) = \sqrt{\deg(\beta)} = r(\mathcal{E}_a)$. Thus the bundles F and \mathcal{E}_a are semihomogeneous, of the same slope and rank. Moreover, F is simple. It follows from Propositions 4.7 and 4.6, 2) that the bundle \mathcal{E}_a is also simple.

Lemma 4.8 now implies that for any points $a_1, a_2 \in A$ the bundles \mathcal{E}_{a_1} and \mathcal{E}_{a_2} are either orthogonal or isomorphic. Suppose that they are isomorphic. Since \mathcal{E} is semihomogeneous, we have

$$T_{(a_2 - a_1, 0)}^* \mathcal{E} \cong \mathcal{E} \otimes P_{(\alpha, \beta)} \tag{4.3}$$

for some point $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. In particular,

$$\mathcal{E} \otimes P_{(\alpha, \beta)} \cong \mathcal{E} \otimes P_{(\alpha, \beta)}$$

Therefore $P_\beta \in \Sigma^0(\mathcal{E}_a)$ (see (4.2)).

By Lemma 4.9 and Proposition 4.6, the orders of $\Sigma^0(E)$ and $\Sigma^0(E_a)$ are equal to r^2 . We claim that the natural map $\sigma: \Sigma^0(\mathcal{E}) \rightarrow \Sigma^0(\mathcal{E}_a)$ is an isomorphism. Indeed, otherwise there is a point $\alpha' \in \hat{A}$ such that $\mathcal{E} \otimes P_{\alpha'} \cong \mathcal{E}$. Then Lemma 4.9 yields that $(0, \alpha', 0, 0) \in \Phi_\mu$, contrary to the fact that the projection $\Phi_\mu \rightarrow B \times \hat{B}$ is an isomorphism.

Since σ is an isomorphism, there is a point $\alpha' \in \hat{A}$ such that $\mathcal{E} \otimes P_{(\alpha', \beta)} \cong \mathcal{E}$. It follows from (4.3) that

$$T_{(a_2 - a_1, 0)}^* \mathcal{E} \cong \mathcal{E} \otimes P_{(\alpha - \alpha', 0)}.$$

By Lemma 4.9, this means that the point $(a_2 - a_1, \alpha - \alpha', 0, 0)$ belongs to Φ_μ . Since the projection $\Phi_\mu \rightarrow B \times \hat{B}$ is an isomorphism, we see that $a_2 - a_1 = 0$. Thus the bundles \mathcal{E}_{a_1} and \mathcal{E}_{a_2} are orthogonal for different points a_1, a_2 . Hence the functor $\Phi_\mathcal{E}: D^b(A) \rightarrow D^b(B)$ is fully faithful. The same arguments show that the adjoint functor $\Psi_{\mathcal{E}^\vee}$ is also fully faithful. Hence $\Phi_\mathcal{E}$ is an equivalence.

Proposition 4.12. *Let \mathcal{E} be the semihomogeneous bundle constructed from an isometric isomorphism $f: A \times \hat{A} \rightarrow B \times \hat{B}$ in the way described above. Then $f_\mathcal{E} = f$.*

Proof. We denote by X the graph of the morphism $f_\mathcal{E}$. It follows from Corollary 2.13 that a point (a, α, b, β) belongs to X if and only if

$$T_{b*} \mathcal{E} \otimes P_\beta \cong T_a^* \mathcal{E} \otimes P_\alpha,$$

which is equivalent to the equation $T_{(a,b)}^* \mathcal{E} \cong E \otimes P_{(-\alpha, \beta)}$. Hence Lemma 4.9 implies that $X = (1_A, -1_{\hat{A}}, 1_B, 1_{\hat{B}}) \Phi_\mu$, where $\mu = \mu(\mathcal{E})$ is the slope of \mathcal{E} . On the other hand, the graph Γ of the map f is also equal to $(1_A, -1_{\hat{A}}, 1_B, 1_{\hat{B}}) \Phi_\mu$ by Construction 4.10. Hence the isomorphisms $f_\mathcal{E}$ and f coincide.

In the construction of \mathcal{E} from f , we assumed that the map $y: \hat{A} \rightarrow B$ is an isogeny. If this is not the case, we present f as a composite of two maps $f_1 \in U(A \times \hat{A}, B \times \hat{B})$ and $f_2 \in U(A \times \hat{A})$ such that y_1 and y_2 are isogenies. We easily see that this can always be done. For every f_i , we find the corresponding object \mathcal{E}_i . Then we consider the composite of the functors $\Phi_{\mathcal{E}_i}$ and take the object that represents it.

Assertions proved in this and previous sections can be joint into the following theorems.

Theorem 4.13. *Let A, B be abelian varieties over an algebraically closed field of characteristic 0. Then the bounded derived categories of coherent sheaves $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories if and only if there is an isometric isomorphism $f: A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$.*

Theorem 4.14. *Let A be an abelian variety over an algebraically closed field of characteristic 0. Then the group $\text{Auteq } D^b(A)$ of exact autoequivalences of the derived category may be included into the following short exact sequence of groups:*

$$0 \rightarrow \mathbb{Z} \oplus (A \times \hat{A})_k \rightarrow \text{Auteq } D^b(A) \rightarrow U(A \times \hat{A}) \rightarrow 1.$$

Let us study the group $\text{Auteq } D^b(A)$ in more detail. It has a normal subgroup $(A \times \hat{A})_k$ consisting of the functors $T_{(a, \alpha)} \otimes P_\alpha$ where $(a, \alpha) \in A \times \hat{A}$. The quotient

with respect to this subgroup is a central extension of the group $U(A \times \hat{A})$ by \mathbb{Z} . We denote this central extension by $\tilde{U}(A \times \hat{A})$. There are short exact sequences

$$0 \rightarrow (A \times \hat{A})_k \rightarrow \text{Auteq } D^b(A) \rightarrow \tilde{U}(A \times \hat{A}) \rightarrow 1, \quad (4.4)$$

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{U}(A \times \hat{A}) \rightarrow U(A \times \hat{A}) \rightarrow 1. \quad (4.5)$$

To describe the central extension (4.5), it suffices to present the corresponding 2-cocycle $\lambda(g_1, g_2) \in \mathbb{Z}$, where g_1, g_2 are elements of $U(A \times \hat{A})$.

By Proposition 3.2, if $\Phi_{\mathcal{E}}$ is an equivalence, then the object $\mathcal{E} \in D^b(A \times A)$ is isomorphic to $E[k]$ for some sheaf E on $A \times A$. Let E, F be sheaves on $A \times A$ which determine autoequivalences Φ_E, Φ_F . Then the composite $\Phi_F \circ \Phi_E$ is presented by some object $G[k]$, where G is a sheaf. We put $\lambda(f_F, f_E) = k$. It is clear that λ is a 2-cocycle, and it determines the central extension (4.5).

Let us calculate λ in terms of elements of $U(A \times \hat{A})$. We put $k = \mathbb{C}$ for simplicity. Then $A \cong \mathbb{C}^n / \mathbb{Z}^{2n}$, and every line bundle L on A determines a Hermitian form $H(L)$ on \mathbb{C}^n (see [11]). We denote by $p(H)$ the number of positive eigenvalues of H . This yields a function $p: \text{NS}(A) \rightarrow \mathbb{Z}$. It may be extended to $\text{NS}(A) \otimes \mathbb{R}$ by

$$p\left(\sum_i r_i [L_i]\right) = p\left(\sum_i r_i H(L_i)\right).$$

We thus get a lower semicontinuous function p on the whole of $\text{NS}(A) \otimes \mathbb{R}$. (It is easy to define the function p for any algebraically closed field. Indeed, take $p(L)$ to be equal to the number of negative roots of the polynomial $P(n) = \chi(L \otimes M^n)$, where M is some ample line bundle. Then extend p to the whole of $\text{NS}(A) \otimes \mathbb{R}$ in the way described above.)

We now consider any two elements of $U(A \times \hat{A})$,

$$g_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix} \quad (4.6)$$

such that y_1 and y_2 are isogenies. This means that there are inverse elements $y_1^{-1}, y_2^{-1} \in \text{Hom}(A, \hat{A}) \otimes \mathbb{Q}$. We introduce the notation

$$g_1 g_2 = \begin{pmatrix} x_3 & y_3 \\ z_3 & w_3 \end{pmatrix}.$$

Consider the element $y_1^{-1} y_3 y_2^{-1} = y_1^{-1} x_1 + w_2 y_2^{-1}$ of the group $\text{Hom}(A, \hat{A}) \otimes \mathbb{Q}$. We easily see that it belongs to $\text{NS}(A) \otimes \mathbb{Q}$. It is easy to prove the equation

$$\lambda(g_1, g_2) = p(y_1^{-1} y_3 y_2^{-1}) - n.$$

This yields a formula for $\lambda(g_1, g_2)$ in the case when y_1 and y_2 are invertible. Since λ is a cocycle, it is uniquely determined by this formula and is uniquely extended to the whole group $U(A \times \hat{A}) \otimes \mathbb{R} \subset \text{End}(A \times \hat{A}) \otimes \mathbb{R}$.

Example 4.15. Consider the example $A = E^n$, where E is an elliptic curve without complex multiplication. Then the group $U(A \times \hat{A})$ is isomorphic to $\mathrm{Sp}_{2n}(\mathbb{Z})$. It is well known that the fundamental group of the real symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{R})$ is isomorphic to \mathbb{Z} and there is a universal central extension \tilde{G} . Moreover, the symplectic group carries a \mathbb{Z} -valued 2-cocycle

$$\mu(g_1, g_2) = \tau(l, g_1 l, g_1 g_2 l),$$

where τ is the Maslov index and l is a Lagrangian subspace (see, for example, [7]). There is a formula for the cocycle μ in the matrix notation similar to (4.6). In the previous notation it takes the form

$$\mu(g_1, g_2) = \mathrm{sign}(y_1^{-1} y_3 y_2^{-1}),$$

the right-hand side being the signature of the symmetric matrix $y_1^{-1} y_3 y_2^{-1}$.

Comparing the cocycles λ and μ , we easily see that the cocycle $(2\lambda - \mu)$ is trivial as an element of the second cohomology group. Moreover, it is known that the second cohomology group of the symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{R})$ is equal to \mathbb{Z} , and the generator determines the universal extension \tilde{G} . It is also known that the Maslov cocycle μ equals four times the generator. Hence \tilde{G}_λ is included into an exact sequence $1 \rightarrow \tilde{G} \rightarrow \tilde{G}_\lambda \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$, which actually splits, that is, \tilde{G}_λ is isomorphic to $\tilde{G} \times \mathbb{Z}/2\mathbb{Z}$.

Example 4.16. Consider an abelian variety A with the endomorphism ring $\mathrm{End}(A) = \mathbb{Z}$. Then the Néron–Severi group $\mathrm{NS}(A)$ is isomorphic to \mathbb{Z} . We denote by L and M the generators of $\mathrm{NS}(A)$ and $\mathrm{NS}(\hat{A})$ respectively. The composite $\phi_M \circ \phi_L$ is equal to $N \mathrm{id}_A$ for some $N > 0$. The group $U(A \times \hat{A})$ coincides with the congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$. Furthermore, let B be another abelian variety such that $B \times \hat{B}$ is isomorphic to $A \times \hat{A}$. It is easily verified that any such isomorphism is isometric. The abelian variety B can be presented as the image of some morphism $A \xrightarrow{(k \cdot \mathrm{id}, m \phi_L)} A \times \hat{A}$, and we may assume that $\mathrm{GCD}(k, m) = 1$. We denote this morphism from A to B by ψ . The kernel of ψ equals $\mathrm{Ker}(m \phi_L) \cap A_k$. Since $\mathrm{GCD}(k, m) = 1$, we actually have $\mathrm{Ker}(\psi) = \mathrm{Ker}(\phi_L) \cap A_k$. On the other hand, we know that $\mathrm{Ker}(\phi) \subset A_N$. Thus we may assume without loss of generality that k divides N . Each divisor k of N induces an abelian variety $B := A / (\mathrm{Ker}(\phi_L) \cap A_k)$, and it is clear that different divisors of N induce non-isomorphic abelian varieties. Moreover, it is easy to verify that the embedding of B into $A \times \hat{A}$ splits if and only if $\mathrm{GCD}(k, N/k) = 1$. Hence the number of abelian varieties B with $D^b(B) \simeq D^b(A)$ is equal to 2^s , where s is the number of prime divisors of N .

We now additionally assume that the abelian variety A is principally polarized, that is, we have $N = 1$. In this case, if $D^b(A) \simeq D^b(B)$, then $B \cong A$. Moreover, the group $U(A \times \hat{A})$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. As proved in [14], the sequence (4.4) splits for a principally polarized abelian variety. Hence we get a description of $\mathrm{Auteq} D^b(A)$ as a semidirect product of the normal subgroup $(A \times \hat{A})_k$ and the group $\tilde{U}(A \times \hat{A})$:

$$\mathrm{Auteq} D^b(A) \cong \tilde{U}(A \times \hat{A}) \ltimes (A \times \hat{A})_k,$$

and $\tilde{U}(A \times \hat{A})$ is the central extension of $\mathrm{SL}(2, \mathbb{Z})$ given by a cocycle λ . One can show that there is a sequence

$$1 \rightarrow B \rightarrow \tilde{U}(A \times \hat{A}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

where B_3 is the braid group of 3 strings. (We recall that B_3 is the universal central extension of $\mathrm{SL}(2, \mathbb{Z})$, which is also induced by the universal covering $\mathrm{SL}(2, \mathbb{R})$.)

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