

DERIVED CATEGORIES OF COHERENT SHEAVES AND MOTIVES.

DMITRI ORLOV

The bounded derived category of coherent sheaves $\mathbf{D}^b(X)$ is a natural triangulated category which can be associated with an algebraic variety X . It happens sometimes that two different varieties have equivalent derived categories of coherent sheaves $\mathbf{D}^b(X) \simeq \mathbf{D}^b(Y)$. There arises a natural question: can one say anything about motives of X and Y in that case? The first such example (see [4]) – abelian variety A and its dual \widehat{A} – shows us that the motives of such varieties are not necessary isomorphic. However, it seems that the motives with rational coefficients are isomorphic

Recall a definition of the category of effective Chow motives $\text{CH}^{\text{eff}}(\mathbf{k})$ over a field \mathbf{k} . The category $\text{CH}^{\text{eff}}(\mathbf{k})$ can be obtained as the pseudo-abelian envelope (i.e. as formal adding of cokernels of all projectors) of a category, whose objects are smooth projective schemes over \mathbf{k} , and the group of morphisms from X to Y is the sum $\bigoplus_{X_i} A^m(X_i \times Y)$ (over all connected components X_i) of the groups of cycles of codimension $m = \dim Y$ on $X_i \times Y$ modulo rational equivalence (see [3, 1]). In [7] Voevodsky introduced a triangulated category of geometric motives $\text{DM}_{\text{gm}}^{\text{eff}}(\mathbf{k})$. He started with an additive category $\text{SmCor}(\mathbf{k})$, objects of which are smooth schemes of finite type over \mathbf{k} , and the group of morphisms from X to Y is the free abelian group generated by integral closed subschemes $Z \subset X \times Y$ such that the projection on X is finite and surjective onto a connected component of X . There is a natural embedding $[-] : \text{Sm}(\mathbf{k}) \rightarrow \text{SmCor}(\mathbf{k})$ of the category $\text{Sm}(\mathbf{k})$ of smooth schemes of finite type over \mathbf{k} . The category $\text{SmCor}(\mathbf{k})$ is additive and one has $[X \amalg Y] = [X] \oplus [Y]$. Further, he considered the quotient of the homotopy category $\mathcal{H}^b(\text{SmCor}(\mathbf{k}))$ of bounded complexes by minimal thick triangulated subcategory T , which contains all objects of the form $[X \times \mathbb{A}^1] \rightarrow [X]$ and $[U \cap V] \rightarrow [U] \oplus [V] \rightarrow [X]$ for any open covering $U \cup V = X$. Triangulated category $\text{DM}_{\text{gm}}^{\text{eff}}(\mathbf{k})$ is defined as the pseudo-abelian envelope of the quotient category $\mathcal{H}^b(\text{SmCor}(\mathbf{k}))/T$ (see [7, 1]).

There exists a canonical functor $\text{CH}^{\text{eff}}(\mathbf{k}) \rightarrow \text{DM}_{\text{gm}}^{\text{eff}}(\mathbf{k})$, which is a full embedding if \mathbf{k} admits resolution of singularities ([7, 4.2.6]). Thus, it doesn't matter in which category (in $\text{CH}^{\text{eff}}(\mathbf{k})$ or in $\text{DM}_{\text{gm}}^{\text{eff}}(\mathbf{k})$) motives of smooth projective varieties are considered. Denote the

This work was done with a partial financial support from grant RFFI 05-01-01034, from the President of RF young Russian scientists award MD-2731.2004.1, from grant CRDF Award No RUM1-2661-MO-05, and from the Russian Science Support Foundation.

motive of a variety X by $M(X)$, and its motive in the category of motives with rational coefficients $DM_{\text{gm}}^{\text{eff}}(\mathbf{k}) \otimes \mathbb{Q}$ (and in $CH^{\text{eff}}(\mathbf{k}) \otimes \mathbb{Q}$) by $M(X)_{\mathbb{Q}}$.

Conjecture 1. *Let X and Y be smooth projective varieties, and let $\mathbf{D}^b(X) \simeq \mathbf{D}^b(Y)$. Then the motives $M(X)_{\mathbb{Q}}$ and $M(Y)_{\mathbb{Q}}$ are isomorphic in $CH^{\text{eff}}(\mathbf{k}) \otimes \mathbb{Q}$ (and in $DM_{\text{gm}}^{\text{eff}}(\mathbf{k}) \otimes \mathbb{Q}$)*

Conjecture 2. *Let X and Y be smooth projective varieties and let $F : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ be a fully faithful functor. Then the motive $M(X)_{\mathbb{Q}}$ is a direct summand of the motive $M(Y)_{\mathbb{Q}}$.*

The category $DM_{\text{gm}}^{\text{eff}}(\mathbf{k})$ has a tensor structure, and $M(X) \otimes M(Y) = M(X \times Y)$. One defines the Tate object $\mathbb{Z}(1)$ to be the image of the complex $[\mathbb{P}^1] \rightarrow [\text{Spec}(\mathbf{k})]$ placed in degree 2 and 3 and put $M(p) = M \otimes \mathbb{Z}(1)^{\otimes p}$ for any motive $M \in DM_{\text{gm}}^{\text{eff}}(\mathbf{k})$ and $p \in \mathbb{N}$. The triangulated category of geometric motives $DM_{\text{gm}}(\mathbf{k})$ is defined by formally inverting the functor $- \otimes \mathbb{Z}(1)$ on $DM_{\text{gm}}^{\text{eff}}(\mathbf{k})$. The important and nontrivial fact here is the statement that the canonical functor $DM_{\text{gm}}^{\text{eff}}(\mathbf{k}) \rightarrow DM_{\text{gm}}(\mathbf{k})$ is a full embedding [7, 4.3.1]. Therefore, we can work in the category $DM_{\text{gm}}(\mathbf{k})$. Moreover (see [7]), for any smooth projective varieties X, Y and for any integer i there is an isomorphism

$$\text{Hom}_{DM_{\text{gm}}(\mathbf{k})}(M(X), M(Y)(i)[2i]) \cong A^{m+i}(X \times Y), \quad \text{where } m = \dim Y.$$

Suppose, one has a fully faithful functor $F : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ between derived categories of coherent sheaves of two smooth projective varieties X and Y of dimension n and m respectively. Any such functor has a right adjoint F^* by [2], and by Theorem 2.2 from [5] (see also [6, 3.2.1]) the functor F can be represented by an object on the product $X \times Y$, i.e. $F \cong \Phi_{\mathcal{A}}$, where $\Phi_{\mathcal{A}} = \mathbf{R}p_{2*}(p_1^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{A})$ for some $\mathcal{A} \in \mathbf{D}^b(X \times Y)$. With any functor of the form $\Phi_{\mathcal{A}} : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ one can associate an element $a \in A^*(X \times Y, \mathbb{Q})$ by the following rule

$$(1) \quad a = p_1^* \sqrt{\text{td}_X} \cdot \text{ch}(\mathcal{A}) \cdot p_2^* \sqrt{\text{td}_Y},$$

where td_X and td_Y are Todd classes of the varieties X and Y . The cycle a has a mixed type. Let us consider its decomposition on the components $a = a_0 + \cdots + a_{n+m}$, where index is the codimension of a cycle on $X \times Y$. Each component a_q induces a map of motives

$$\alpha_q : M(X)_{\mathbb{Q}} \rightarrow M(Y)_{\mathbb{Q}}(q-m)[2(q-m)].$$

Thus the total cycle a gives a map $\alpha : M(X)_{\mathbb{Q}} \rightarrow \bigoplus_{i=-m}^n M(Y)_{\mathbb{Q}}(i)[2i]$. Now consider the object $\mathcal{B} \in \mathbf{D}^b(X \times Y)$ which represents the (left) adjoint functor F^* , i.e. $F^* \cong \Psi_{\mathcal{B}}$, where $\Psi_{\mathcal{B}} = \mathbf{R}p_{1*}(p_2^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{B})$. One attaches to the object \mathcal{B} a cycle $b = b_0 + \cdots + b_{n+m}$ defined by the same formula (1). The cycle b induces a map $\beta : \bigoplus_{i=-m}^n M(Y)_{\mathbb{Q}}(i)[2i] \rightarrow M(X)_{\mathbb{Q}}$.

Since the functor $\Phi_{\mathcal{A}}$ is fully faithful, the composition $\Psi_{\mathcal{B}} \circ \Phi_{\mathcal{A}}$ is isomorphic to the identity functor. Applying the Riemann-Roch-Grothendieck theorem, we obtain that the composition

$$\mathrm{M}(X)_{\mathbb{Q}} \xrightarrow{\alpha} \bigoplus_{i=-m}^n \mathrm{M}(Y)_{\mathbb{Q}}(i)[2i] \xrightarrow{\beta} \mathrm{M}(X)_{\mathbb{Q}}$$

is the identity map, i.e. $\mathrm{M}(X)_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{i=-m}^n \mathrm{M}(Y)_{\mathbb{Q}}(i)[2i]$.

Assume now that $\dim X = \dim Y = n$ and, moreover, suppose that the support of the object \mathcal{A} also has the dimension n . Therefore, $a_q = 0$ when $q = 0, \dots, n-1$, i.e. $a = a_n + \dots + a_{2n}$. It is easily to see that in this case $b = b_n + \dots + b_{2n}$ as well. This implies that the composition $\beta \cdot \alpha : \mathrm{M}(X)_{\mathbb{Q}} \rightarrow \mathrm{M}(X)_{\mathbb{Q}}$, which is the identity, coincides with $\beta_n \cdot \alpha_n$. Hence, $\mathrm{M}(X)_{\mathbb{Q}}$ is a direct summand of $\mathrm{M}(Y)_{\mathbb{Q}}$. Furthermore, since the cycles a_n and b_n are integral in this case we get the same result for integral motives, i.e. the integral motive $\mathrm{M}(X)$ is a direct summand of the motive $\mathrm{M}(Y)$ as well. Thus, we obtain

Theorem 1. *Let X and Y be smooth projective varieties of dimension n , and let $F : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ be a fully faithful functor such that the dimension of the support of an object \mathcal{A} on $X \times Y$, which represents F , is equal to n . Then the motive $\mathrm{M}(X)$ is a direct summand of the motive $\mathrm{M}(Y)$. If, in addition, the functor F is an equivalence, then the motives $\mathrm{M}(X)$ and $\mathrm{M}(Y)$ are isomorphic.*

Examples of such functors are known, they come from birational geometry (see e.g. [6]). In these examples one of the connected components of $\mathrm{supp}(\mathcal{A})$ gives a birational map $X \dashrightarrow Y$. Blow ups and antflips induce fully faithful functors, and flops induce equivalences. Note that an isomorphism of motives implies an isomorphism of any realization (singular cohomologies, l-adic cohomologies, Hodge structures and so on).

For arbitrary equivalence $\Phi_{\mathcal{A}} : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ the map of motives $\alpha_n : \mathrm{M}(X)_{\mathbb{Q}} \rightarrow \mathrm{M}(Y)_{\mathbb{Q}}$, induced by the cycle $a_n \in A^n(X \times Y, \mathbb{Q})$, is not necessary an isomorphism (e.g. Poincare line bundle \mathcal{P} on the product of abelian variety A and its dual \widehat{A}). However, the following conjecture, which specifies Conjecture 1, may be true.

Conjecture 3. *Let \mathcal{A} be an object of $\mathbf{D}^b(X \times Y)$, for which $\Phi_{\mathcal{A}} : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ is an equivalence. Then there exist line bundles L and M on X and on Y respectively such that the component a'_n of the object $\mathcal{A}' := p_1^*L \otimes \mathcal{A} \otimes p_2^*M$ gives an isomorphism between motives $\mathrm{M}(X)_{\mathbb{Q}}$ and $\mathrm{M}(Y)_{\mathbb{Q}}$.*

I am grateful to Yu. I. Manin for very useful discussions.

REFERENCES

- [1] BLOCH, S. Lectures on mixed motives. *Algebraic geometry—Santa Cruz 1995*, 329–359, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.
- [2] BONDAL, A., AND VAN DEN BERGH, M. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.* 3, 1 (2003), 1–36.
- [3] MANIN, YU. Correspondences, motifs and monoidal transformations. *Matem. Sb.* 77, 119 (1968), 475–507.
- [4] MUKAI, S. Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves. *Nagoya Math. J.* 81 (1981), 153–175.
- [5] ORLOV, D. Equivalences of derived categories and K3 surfaces. *Journal of Math. Sciences, Alg. geom.-7* 84, 5 (1997), 1361–1381.
- [6] ORLOV, D. Derived categories of coherent sheaves and equivalences between them. *Uspekhi Matem. Nauk* 58, 3(351) (2003), 89–172.
- [7] VOEVODSKY, V. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, vol. 143 of *Ann. of Math. Stud.*, pp. 188–238.

STEKLOV MATHEMATICAL INSTITUTE RAN

E-mail address: orlov@mi.ras.ru