MIRROR SYMMETRY FOR DEL PEZZO SURFACES: VANISHING CYCLES AND COHERENT SHEAVES

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Abstract. We study homological mirror symmetry for Del Pezzo surfaces and their mirror Landau-Ginzburg models. In particular, we show that the derived category of coherent sheaves on a Del Pezzo surface X_k obtained by blowing up $\mathbb{CP}^2$ at k points is equivalent to the derived category of vanishing cycles of a certain elliptic fibration $W_k : M_k \to \mathbb{C}$ with $k+3$ singular fibers, equipped with a suitable symplectic form. Moreover, we also show that this mirror correspondence between derived categories can be extended to noncommutative deformations of X_k, and give an explicit correspondence between the deformation parameters for X_k and the cohomology class $[B+i\omega] \in H^2(M_k, \mathbb{C})$.

1. Introduction

The phenomenon of mirror symmetry has been studied extensively in the case of Calabi-Yau manifolds (where it corresponds to a duality between $N = 2$ superconformal sigma models), but also manifests itself in more general situations. For example, a sigma model whose target space is a Fano variety is expected to admit a mirror, not necessarily among sigma models, but in the more general context of *Landau-Ginzburg models*.

For us, a Landau-Ginzburg model is simply a pair (M, W), where M is a non-compact manifold (carrying a symplectic structure and/or a complex structure), and W is a complex-valued function on M called *superpotential*. The general philosophy is that, when a Landau-Ginzburg model (M, W) is mirror to a Fano variety X, the complex (resp. symplectic) geometry of X corresponds to the symplectic (resp. complex) geometry of the *critical points* of W.

We place ourselves in the context of *homological mirror symmetry*, where mirror symmetry is interpreted as an equivalence between certain triangulated categories naturally associated to a mirror pair [11]. In our case, B-branes on a Fano variety are described by its derived category of coherent sheaves, and under mirror symmetry they correspond to the A-branes of a mirror Landau-Ginzburg model. These A-branes are described by a suitable analogue of the Fukaya category for a symplectic fibration, namely the *derived category of Lagrangian vanishing cycles*. A rigorous definition of this category has been proposed by Seidel [16] in the case where the critical points of the superpotential are isolated and non-degenerate, following ideas of Kontsevich [12] and Hori, Iqbal, Vafa [9].

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Therefore, for a Fano variety X and a mirror Landau-Ginzburg model $W : M \to \mathbb{C}$, the homological mirror symmetry conjecture can be formulated as follows:

Conjecture 1.1. *The derived category of Lagrangian vanishing cycles* $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ *is equivalent to the derived category of coherent sheaves* $\mathbf{D}^b(\mathrm{coh}(X))$.

Remark 1.2. Homological mirror symmetry also predicts another equivalence of derived categories. Namely, viewing now X as a symplectic manifold and M as a complex manifold, the derived category of B-branes of the Landau-Ginzburg model $W : M \to \mathbb{C}$, which was defined algebraically in [10, 15] following ideas of Kontsevich, should be equivalent to the derived Fukaya category of X. This aspect of mirror symmetry will be addressed in a further paper; for now, we focus exclusively on Conjecture 1.1.

One of the first examples for which Conjecture 1.1 has been verified is that of $\mathbb{CP}^2$ and its mirror Landau-Ginzburg model which is the elliptic fibration with three singular fibers determined by the superpotential $W_0 = x+y+1/xy$ on $(\mathbb{C}^*)^2$ (or rather a fiberwise compactification of this fibration), see [17, 3]. Other examples of surfaces for which the derived category of coherent sheaves has been shown to be equivalent to the derived category of Lagrangian vanishing cycles of a mirror Landau-Ginzburg model include weighted projective planes, Hirzebruch surfaces [3], and toric blow-ups of $\mathbb{CP}^2$ [19]. For all these examples, the toric structure plays a crucial role in determining the geometry of the mirror Landau-Ginzburg model.

Our goal in this paper is to consider the case of a Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at a set K of $k \leq 8$ points (this is never toric as soon as $k \geq 4$). Our proposal is that a mirror of X_K can be constructed in the following manner. Observe that the elliptic fibration with three singular fibers determined by the superpotential $W_0 = x + y + 1/xy$ on $(\mathbb{C}^*)^2$ (i.e., the mirror of $\mathbb{CP}^2$) admits a natural compactification to an elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$ in which the fiber above infinity consists of nine rational components (see §3.1 for details). Consider a deformation of $\overline{W_0}$ to another elliptic fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$, such that k of the 9 critical points in the fiber $\overline{W_0}^{-1}(\infty)$ are displaced towards finite values of the superpotential. Let

$$M_k = \overline{M} \setminus \overline{W_k}^{-1}(\infty),$$

and denote by $W_k : M_k \to \mathbb{C}$ the restriction of $\overline{W_k}$ to M_k. In the generic case, W_k is an elliptic fibration with $k+3$ nodal fibers, while $\overline{W_k}^{-1}(\infty)$ is a singular fiber with $9-k$ rational components. Although we will focus on the Del Pezzo case, this construction also provides a mirror in some borderline situations. For example, it can be applied without modification to the case where $\mathbb{CP}^2$ is blown up at $k = 9$ points which lie at the intersection of two elliptic curves (the fiber $\overline{W_k}^{-1}(\infty)$ is then a smooth elliptic curve).

There are two aspects to the geometry of M_k. Viewing M_k as a *complex* manifold (a Zariski open subset of a rational elliptic surface), its complex structure is closely related to the set of critical values of W_k, which has to be chosen in accordance with a given symplectic structure on

X_K. A generic choice of the symplectic structure on X_K (for which there are no homologically nontrivial Lagrangian submanifolds) determines a complex structure on M_k for which the $k+3$ critical values of W_k are all distinct (leading to a very simple category of B-branes). In the opposite situation, which we will not consider here, if we equip X_K with a symplectic form for which there are homologically nontrivial Lagrangian submanifolds, then some of the critical values of W_k become equal, and the topology of the singular fibers may become more complicated.

The *symplectic* geometry of M_k is more important to us. Since $H^2(M_k,\mathbb{C}) \simeq \mathbb{C}^{k+2}$, the symplectic form ω on M_k, or rather its complexified variant $B+i\omega$, depends on $k+2$ moduli parameters. As we will see in §4, these parameters completely determine the derived category of Lagrangian vanishing cycles of W_k; the actual positions of the critical values are of no importance, as long as the critical points of W_k remain isolated and non-degenerate (see Lemma 3.2). This means that we shall not concern ourselves with the complex structure on M_k; in fact, a compatible almost-complex structure is sufficient for our purposes, which makes the problem of deforming the elliptic fibration $\overline{W_0}$ in the prescribed manner a non-issue.

To summarize, we have:

Construction 1.3. Given a Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at k points, the mirror Landau-Ginzburg model is an elliptic fibration $W_k : M_k \to \mathbb{C}$ with $k+3$ nodal singular fibers, which has the following properties:

(*i*) the fibration W_k compactifies to an elliptic fibration $\overline{W_k}$ over $\mathbb{CP}^1$ in which the fiber above infinity consists of $9-k$ rational components;

(*ii*) the compactified fibration $\overline{W_k}$ can be obtained as a deformation of the elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$ which compactifies the mirror to $\mathbb{CP}^2$.

Moreover, the manifold M_k is equipped with a symplectic form ω and a B-field B, whose cohomology classes are determined by the set of points K in an explicit manner as discussed in §5.

Our main result is the following:

Theorem 1.4. *Given any Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at k points, there exists a complexified symplectic form $B+i\omega$ on M_k for which* $\mathbf{D}^b(\mathrm{coh}(X_K)) \cong \mathbf{D}^b(\mathrm{Lag_{vc}}(W_k))$.

The *mirror map*, i.e. the relation between the cohomology class $[B+i\omega] \in H^2(M_k,\mathbb{C})$ and the positions of the blown up points in $\mathbb{CP}^2$, can be described explicitly (see Proposition 5.1).

On the other hand, not every choice of $[B+i\omega] \in H^2(M_k,\mathbb{C})$ yields a category equivalent to the derived category of coherent sheaves on a Del Pezzo surface. There are two reasons for this. First, certain specific choices of $[B+i\omega]$ correspond to deformations of the complex structure of X_K for which the surface contains a -2-curve, which causes the anticanonical class to no longer be ample. There are many ways in which this can occur, but perhaps the simplest one corresponds to the case where a same point is blown up twice, i.e. we first blow up $\mathbb{CP}^2$ at $k-1$ generic points and then blow up a point on one of the exceptional curves. We then say that X_K is obtained from $\mathbb{CP}^2$

by blowing up k points, two of which are infinitely close, and call this a "simple degeneration" of a Del Pezzo surface. In this case again we have:

Theorem 1.5. *If X_K is a blowup of $\mathbb{CP}^2$ at k points, two of which are infinitely close, and a simple degeneration of a Del Pezzo surface, then there exists a complexified symplectic form $B+i\omega$ on M_k for which $\mathbf{D}^b(\mathrm{coh}(X_K)) \cong \mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$.*

More importantly, deformations of the symplectic structure on M_k need not always correspond to deformations of the complex structure on X_K (observe that $H^2(M_k, \mathbb{C})$ is larger than $H^1(X_K, TX_K)$). The additional deformation parameters on the mirror side can however be interpreted in terms of *noncommutative deformations* of the Del Pezzo surface X_K (i.e., deformations of the derived category $\mathbf{D}^b(\mathrm{coh}(X_K))$). In this context we have the following theorem, which generalizes the result obtained in [3] for the case of $\mathbb{CP}^2$:

Theorem 1.6. *Given any noncommutative deformation of the Del Pezzo surface X_K, there exists a complexified symplectic form $B+i\omega$ on M_k for which the deformed derived category $\mathbf{D}^b(\mathrm{coh}(X_{K,\mu}))$ is equivalent to $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$. Conversely, for a generic choice of $[B+i\omega] \in H^2(M_k, \mathbb{C})$, the derived category of Lagrangian vanishing cycles $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ is equivalent to the derived category of coherent sheaves of a noncommutative deformation of a Del Pezzo surface.*

The mirror map is again explicit, i.e. the parameters which determine the noncommutative Del Pezzo surface can be read off in a simple manner from the cohomology class $[B+i\omega]$.

Remark 1.7. The key point in the determination of the mirror map is that the parameters which determine the composition tensors in $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ can be expressed explicitly in terms of the cohomology class $[B+i\omega]$ (see §4.3). A remarkable feature of these formulas is that they can be interpreted in terms of *theta functions* on a certain elliptic curve (see §4.5). As a consequence, our description of the mirror map also involves theta functions (see §5).

The rest of the paper is organized as follows. In §2 we describe the bounded derived categories of coherent sheaves on Del Pezzo surfaces, their simple degenerations, and their noncommutative deformations. In §3 we describe the topology of the elliptic fibration M_k and its vanishing cycles. In §4.1 we recall Seidel's definition of the derived category of Lagrangian vanishing cycles of a symplectic fibration, and in the rest of §4 we determine $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$. Finally in §5 we compare the two viewpoints, describe the mirror map, and prove the main theorems.

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2. Derived categories of coherent sheaves on blowups of $\mathbb{CP}^2$

The purpose of this section is to give a description of the bounded derived categories of coherent sheaves on Del Pezzo surfaces, their simple degenerations, and their noncommutative deformations. We always work over the field of complex numbers $\mathbb{C}$.

2.1. Del Pezzo surfaces and blowups of the projective plane at distinct points.

Definition 2.1. *A smooth projective surface S is called a Del Pezzo surface if the anticanonical sheaf $\mathcal{O}_S(-K_S)$ is ample (i.e., a Del Pezzo surface is a Fano variety of dimension 2).*

The Kodaira vanishing theorem and Serre duality give us immediately that for any Del Pezzo surface

$$H^1(S,\mathcal{O}(-mK_S)) = 0 \qquad \text{for all } \ m \in \mathbb{Z},$$
$$H^2(S,\mathcal{O}(-mK_S)) = 0 \qquad \text{for all } \ m \geq 0,$$
$$H^2(S,\mathcal{O}(-mK_S)) = H^0(S,\mathcal{O}((m+1)K_S)) \qquad \text{for all } \ m \in \mathbb{Z}.$$

In particular, we obtain that $H^1(S,\mathcal{O}_S) \cong H^2(S,\mathcal{O}_S) = 0$, and $H^0(S,\mathcal{O}(mK_S)) = 0$ for all $m > 0$. By the Castelnuovo-Enriques criterion any Del Pezzo surface is rational.

Let S be a Del Pezzo surface. The integer K_S^2 is called the *degree* of S and will be denoted by d. The Noether formula gives a relation between the degree and the rank of the Picard group of a Del Pezzo surface: $d = K_S^2 = 10 - \operatorname{rk}\operatorname{Pic} S \leq 9$.

We can also introduce another integer number which is called the *index* of S. This is the maximal $r > 0$ such that $\mathcal{O}(-K_S) = \mathcal{O}(rH)$ for some divisor H. The inequality $d \leq 9$ implies that $r \leq 3$.

Now recall the classification of Del Pezzo surfaces.

If $r = 3$, then $S \cong \mathbb{P}^2$ is the projective plane and $d = 9$. If $r = 2$, then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the quadric and $d = 8$. The other Del Pezzo surfaces are not minimal and can be obtained by blowing up the projective plane $\mathbb{P}^2$. More precisely, if S is a Del Pezzo surface of index $r = 1$, then it has degree $1 \leq d \leq 8$ and S is a blowup of the projective plane $\mathbb{P}^2$ at $k = 9 - d$ distinct points. The ampleness of the anticanonical class requires that in this set no three points lie on a line, and no six points lie on a conic; moreover, if $k = 8$ the eight points are not allowed to lie on an irreducible cubic which has a double point at one of these points. Conversely, any surface which is a blowup of the projective plane at a set of $k \leq 8$ different points satisfying these constraints is a Del Pezzo surface of degree $d = 9 - k$. All these facts are well-known and can be found in any textbook on surfaces (see e.g. [8]).

Denote by $\mathbf{D}^b(\operatorname{coh}(S))$ the bounded derived category of coherent sheaves on S. It is known that the bounded derived category of coherent sheaves on any Del Pezzo surface has a full exceptional collection, which makes it possible to establish an equivalence between the category $\mathbf{D}^b(\operatorname{coh}(S))$ and the bounded derived category of finitely generated modules over the algebra of the exceptional collection ([14], see also [13]). This is a particular case of a more general statement about derived categories of blowups.

First, recall the notion of exceptional collection.

Definition 2.2. *An object E of a $\mathbb{C}$-linear triangulated category $\mathcal{D}$ is said to be* exceptional *if* $\mathrm{Hom}(E, E[k]) = 0$ *for all* $k \neq 0$, *and* $\mathrm{Hom}(E, E) = \mathbb{C}$. *An ordered set of exceptional objects* $\sigma = (E_0, \dots E_n)$ *is called an* exceptional collection *if* $\mathrm{Hom}(E_j, E_i[k]) = 0$ *for* $j > i$ *and all* k. *The exceptional collection* σ *is said to be* strong *if it satisfies the additional condition* $\mathrm{Hom}(E_j, E_i[k]) = 0$ *for all* i, j *and for* $k \neq 0$.

Definition 2.3. *An exceptional collection* $(E_0, \dots, E_n)$ *in a category* $\mathcal{D}$ *is called* full *if it generates the category* $\mathcal{D}$, *i.e. the minimal triangulated subcategory of* $\mathcal{D}$ *containing all objects* E_i *coincides with* $\mathcal{D}$. *In this case we say that* $\mathcal{D}$ *has a semiorthogonal decomposition of the form*

$$\mathcal{D} = \langle E_0, \dots, E_n \rangle .$$

The most studied example of an exceptional collection is the sequence of invertible sheaves $\langle \mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$ on the projective space $\mathbb{P}^n$ ([4]). In particular, this exceptional collection on the projective plane $\mathbb{P}^2$ has length 3.

Definition 2.4. *The algebra of a strong exceptional collection* $\sigma = (E_0, \dots, E_n)$ *is the algebra of endomorphisms* $B(\sigma) = \mathrm{End}(\mathcal{E})$ *of the object* $\mathcal{E} = \bigoplus_{i=0}^{n} E_i$.

Assume that the triangulated category $\mathcal{D}$ has a full strong exceptional collection $(E_0, \dots, E_n)$ and B is the corresponding algebra. Denote by mod–B the category of finitely generated right modules over B. There is a theorem according to which if $\mathcal{D}$ is an *enhanced triangulated category* in the sense of Bondal and Kapranov [5], then it is equivalent to the bounded derived category $\mathbf{D}^b(\text{mod–}B)$. This equivalence is given by the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ (see [5]).

For example, if $\mathcal{D} \cong \mathbf{D}^b(\mathrm{coh}(X))$ is the bounded derived category of coherent sheaves on a projective variety X, then it is enhanced. Actually, the category of quasi-coherent sheaves Qcoh has enough injectives, and $\mathbf{D}^b(\mathrm{coh}(X))$ is equivalent to the full subcategory $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(X)) \subset \mathbf{D}^b(\mathrm{Qcoh}(X))$ whose objects are complexes with cohomologies in $\mathrm{coh}(X)$.

Assume that X is smooth and $(E_0, \dots, E_n)$ is a strong exceptional collection on X. The object $\mathcal{E} = \bigoplus_{i=0}^n E_i$ defines the derived functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \mathbf{D}^+(\mathrm{Qcoh}(X)) \longrightarrow \mathbf{D}^+(\text{Mod–}B),$$

where Mod–B is the category of all right modules over B. Moreover, the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ sends objects of $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(X))$ to objects of the subcategory $\mathbf{D}^b_{\mathrm{mod}}(\text{Mod–}B)$, which is also equivalent to $\mathbf{D}^b(\text{mod–}B)$. This gives us a functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\text{mod–}B).$$

The objects E_i for $i = 0, \dots, n$ are mapped to the projective modules $P_i = \mathrm{Hom}(\mathcal{E}, E_i)$. Moreover, $B = \bigoplus_{i=0}^n P_i$. The algebra B has $n+1$ primitive idempotents $e_i,\ i = 0, \dots, n$ such that $1_B = e_0 + \cdots + e_n$ and $e_i e_j = 0$ if $i \neq j$. The right projective modules P_i coincide with $e_i B$. The morphisms between them can be easily described since

$$\mathrm{Hom}(P_i, P_j) = \mathrm{Hom}(e_i B, e_j B) \cong e_j B e_i \cong \mathrm{Hom}(E_i, E_j).$$

This yields an equivalence between the triangulated subcategory of $\mathbf{D}^b(\mathrm{coh}(X))$ generated by the collection $\langle E_0, \dots, E_n \rangle$ and the derived category $\mathbf{D}^b(\text{mod}–B)$. Here we use the fact that the algebra B has a finite global dimension and any right (and left) module M has a finite projective resolution consisting of the projective modules P_i with $i = 0, \dots, n$. Finally, if the collection $(E_0, \dots, E_n)$ is full, then we obtain an equivalence between $\mathbf{D}^b(\mathrm{coh}(X))$ and $\mathbf{D}^b(\text{mod}–B)$.

Sometimes it is useful to represent the algebra B as a category $\mathfrak{B}$ which has $n+1$ objects, say $v_0, \dots, v_n$, and morphisms defined by the rule $\mathrm{Hom}(v_i, v_j) \cong \mathrm{Hom}(E_i, E_j)$ with the natural composition law. Thus $B = \bigoplus\limits_{0 \le i,j \le n} \mathrm{Hom}(v_i, v_j)$.

Theorem 2.5. [14, 13] *Let $\pi : X_K \to \mathbb{P}^2$ be a blowup of the projective plane $\mathbb{P}^2$ at a set $K = \{p_1, \dots, p_k\}$ of any k distinct points, and let $l_1, \dots, l_k$ be the exceptional curves of the blowup. Let (F_0, F_1, F_2) be a full strong exceptional collection of vector bundles on $\mathbb{P}^2$. Then the sequence*

$$(\pi^* F_0, \pi^* F_1, \pi^* F_2, \mathcal{O}_{l_1}, \dots, \mathcal{O}_{l_k}), \tag{2.1}$$

where the $\mathcal{O}_{l_i}$ are the structure sheaves of the exceptional -1-curves l_i, is a full strong exceptional collection on X_K. Moreover, the sheaves $\mathcal{O}_{l_i}$ and $\mathcal{O}_{l_j}$ are mutually orthogonal for all $i \neq j$.

In particular, there is an equivalence

$$\mathbf{D}^b(\mathrm{coh}(X_K)) \cong \mathbf{D}^b(\text{mod}–B_K), \tag{2.2}$$

where B_K is the algebra of homomorphisms of the exceptional collection (2.1).

There are no restrictions on the set of points $K = \{p_1, \dots, p_k\}$ in this theorem and, in particular, we do not need to assume that X_K is a Del Pezzo surface.

We can easily describe the space of morphisms from $\pi^* F_i$ to the sheaf $\mathcal{O}_{l_j}$, since it is naturally identified with the space that is dual to the fiber of the vector bundle F_i at the point $p_j \in \mathbb{P}^2$, i.e.

$$\mathrm{Hom}_{X_K}(\pi^* F_i, \mathcal{O}_{l_j}) \cong \mathrm{Hom}_{\mathbb{P}^2}(F_i, \mathcal{O}_{p_j}).$$

There are various standard exceptional collections on the projective plane. One of them is the collection of line bundles $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$, another is the collection $(\mathcal{O}, \mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{O}(1))$, where $\mathcal{T}_{\mathbb{P}^2}$ is the tangent bundle on $\mathbb{P}^2$. The latter choice is the most convenient for us. It is easy to see that

$$\mathrm{Hom}(\mathcal{O}, \mathcal{T}_{\mathbb{P}^2}(-1)) \cong \mathrm{Hom}(\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{O}(1)) \cong V \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \cong \Lambda^2 V \cong V^*,$$

where V is the 3-dimensional vector space whose projectivization $\mathbb{P}(V)$ is the given projective plane $\mathbb{P}^2$.

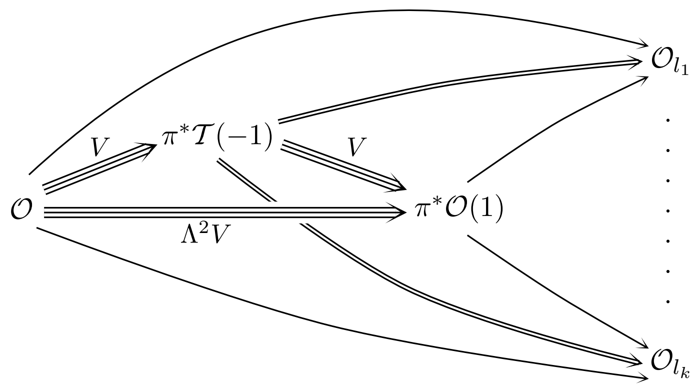

FIGURE 1. The quiver $\mathfrak{B}_K$ for a blowup of $\mathbb{P}^2$ at k distinct points.

Let us consider the blowup X_K of the projective plane $\mathbb{P}(V)$ at a set $K = \{p_1, \ldots, p_k\}$ of k distinct points, and the exceptional collection

$$\sigma = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_1}, \ldots, \mathcal{O}_{l_k}). \tag{2.3}$$

Let $\mathfrak{B}_K(\sigma)$ be the category of homomorphisms of this exceptional collection (see Figure 1). Then the surface X_K can be recovered from the category $\mathfrak{B}_K(\sigma)$ by means of the following procedure.

Denote by S_j the 2-dimensional space of homomorphisms from $\pi^*\mathcal{T}_{\mathbb{P}^2}(-1)$ to $\mathcal{O}_{l_j}$ and denote by U_j the 1-dimensional space of homomorphisms from $\mathcal{O}(1)$ to $\mathcal{O}_{l_j}$. The composition law in the category $\mathfrak{B}_K(\sigma)$ gives a map from $U_j \otimes V$ to S_j. The kernel of this map is a 1-dimensional subspace $V_j \subset V$, which defines a point $p_j \in \mathbb{P}(V)$. In this way, we can determine all the points $p_1, \ldots, p_k \in \mathbb{P}(V)$ and completely recover the surface X_K starting from the category $\mathfrak{B}_K(\sigma)$.

Remark 2.6. Exceptional objects and exceptional collections on Del Pezzo surfaces are well-studied objects. First, any exceptional object of the derived category is isomorphic to a sheaf up to translation. Second, any exceptional sheaf can be included in a full exceptional collection. Third, any full exceptional collection can be obtained from a given one by a sequence of natural operations on exceptional collections called *mutations*. All these facts can be found in the paper [13].

2.2. **Simple degenerations of Del Pezzo surfaces.** We now look at some simple degenerations of the situation considered above, namely when two points, for example p_1 and p_2, converge to each other and finally coincide. More precisely, this means that we first blow up a point p and after that we blow up some point p' on the -1-curve which is the pre-image of p under the first blowup. This operation is sometimes called a blowup at two "infinitely close" points; more precisely, it corresponds to blowing up a subscheme of length 2 supported at p. In this case, the pre-image $\pi^{-1}(p)$ consists of two rational curves meeting at one point. One of them is a -1-curve which we denote by l', and the other is a -2-curve which we denote by l. The curve l is the proper transform of the exceptional curve of the first blow up performed at the point $p \in \mathbb{P}^2$.

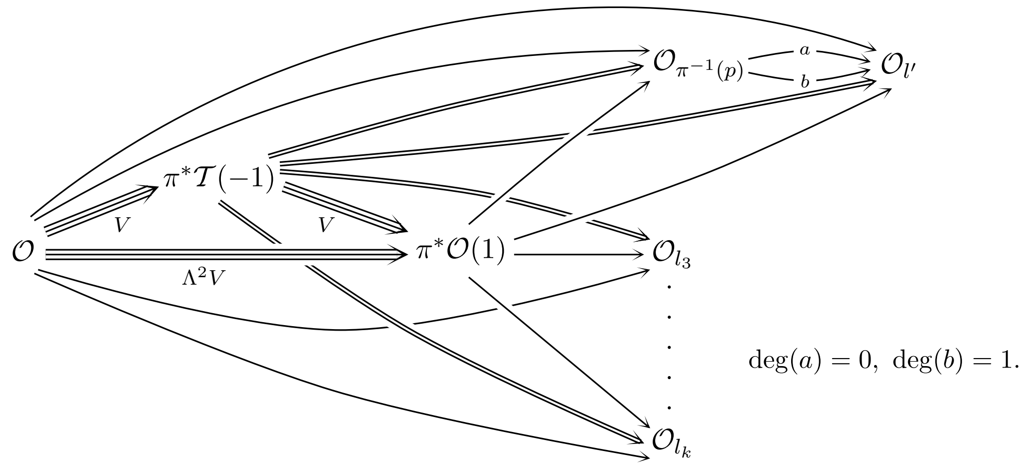

FIGURE 2. The cohomology algebra of the DG-quiver $\mathfrak{B}_K(\tau)$ for a blowup of $\mathbb{P}^2$ with two infinitely close points.

In this paragraph, we consider the situation where the surface X_K is the blowup of the projective plane $\mathbb{P}^2$ at a subscheme K which is supported at a set of $k-1$ points $\{p, p_3, \ldots, p_k\}$ and has length 2 at the point p. In this case the surface X_K is not a Del Pezzo surface, because it possesses a -2-curve l. However, it follows from general results about blowups that $\mathbf{D}^b(\mathrm{coh}(X_K))$ still possesses a full exceptional collection [14].

Proposition 2.7. *Let X_K be the blowup of $\mathbb{P}^2$ at a subscheme K supported at a set of $k-1$ points $\{p, p_3, \ldots, p_k\}$ and with length 2 at the point p. Then the sequence*

$$\tau = \big(\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}, \mathcal{O}_{l_3}, \ldots, \mathcal{O}_{l_k}\big) \tag{2.4}$$

is a full exceptional collection on X_K.

As before we can see that the sheaves $\mathcal{O}_{l_i}$ and $\mathcal{O}_{l_j}$ are mutually orthogonal for all $i \neq j$, and each $\mathcal{O}_{l_i}$ is orthogonal to both $\mathcal{O}_{l'}$ and $\mathcal{O}_{\pi^{-1}(p)}$. However, the collection τ is not strong, because there are non-trivial morphisms from $\mathcal{O}_{\pi^{-1}(p)}$ to $\mathcal{O}_{l'}$ in degrees 0 and 1. More precisely,

$$\mathrm{Hom}(\mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}) \cong \mathbb{C} \qquad \text{and} \qquad \mathrm{Ext}^1(\mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}) \cong \mathbb{C}.$$

Denote by a and b two morphisms from $\mathcal{O}_{\pi^{-1}(p)}$ to $\mathcal{O}_{l'}$ of degrees 0 and 1 respectively. It is easy to see that composition with the morphism a gives isomorphisms between the spaces $\mathrm{Hom}(F, \mathcal{O}_{\pi^{-1}(p)})$ and $\mathrm{Hom}(F, \mathcal{O}_{l'})$ for any element F of the exceptional collection τ (see Figure 2).

Two approaches can be used to obtain an analogue of equivalence (2.2) for this situation. The first possibility is to associate to the non-strong exceptional collection τ a differential graded algebra of homomorphisms, and obtain an equivalence between the derived category of coherent sheaves and the derived category of finitely generated (right) DG-modules over the DG-algebra of homomorphisms of the exceptional collection. (One could also try to work in the framework of

A_∞-algebras, which might be more appropriate here considering that the mirror situation involves an A_∞-category with non-zero m_3, see §4.4).

Another approach is to change the exceptional collection τ to another one which is strong. There are natural operations on exceptional collections which are called mutations and which allow us to obtain new exceptional collections starting from a given one.

We omit the definition of mutations, which is classical and can be found in many places. However, we note that the left mutation of the exceptional collection (2.4) in the pair $(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\pi^{-1}(p)})$ gives us a new exceptional collection

$$\tau' = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l'}, \mathcal{O}_{l_3}, \dots, \mathcal{O}_{l_k}) \tag{2.5}$$

where $\mathcal{M}$ is the line bundle on X_K which is the kernel of the surjection $\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\pi^{-1}(p)}$. This new exceptional collection τ' is strong.

In fact, we can also consider the same left mutation when the blown up points are all distinct, and obtain in that case as well a strong exceptional collection

$$\sigma' = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_2}, \mathcal{O}_{l_3}, \dots, \mathcal{O}_{l_k}),$$

which behaves very much like τ'. The distinguishing feature of the case where we blow up the point p twice is that in the exceptional collection τ' the composition map

$$\mathrm{Hom}(\mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathrm{Hom}(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l'}) \longrightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{O}_{l'}) \tag{2.6}$$

is identically zero, whereas for σ' (i.e., when the points of K are distinct) the corresponding composition is non-trivial. In this sense, the mutation allows us to give a simple description of the behaviour of the category under the degeneration process where two points of K converge to each other. Namely, the algebra $B_K(\tau')$ of homomorphisms of the exceptional collection τ' is obtained

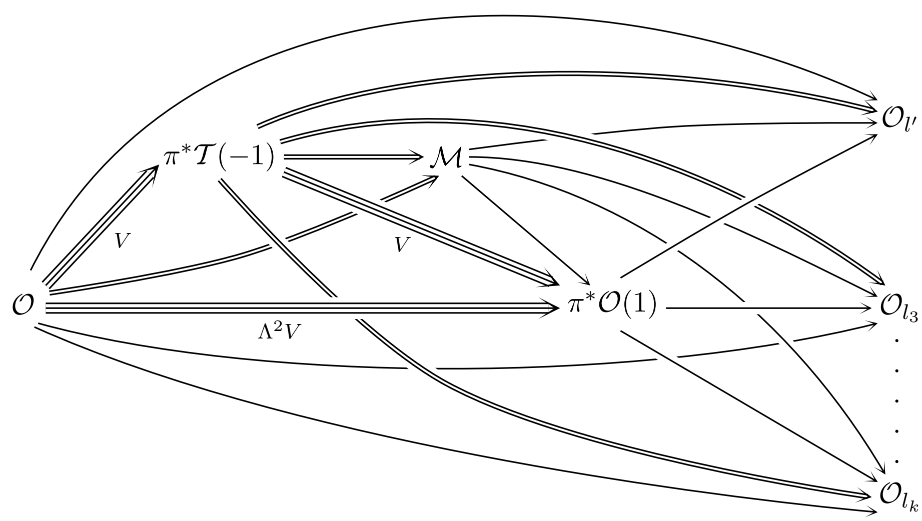

FIGURE 3. The quiver $\mathfrak{B}_K(\tau')$ for a blowup of $\mathbb{P}^2$ with two infinitely close points.

as a degeneration of the algebra of homomorphisms of the exceptional collection σ' in which the composition (2.6) becomes zero.

Proposition 2.8. *Let X_K be the blowup of $\mathbb{P}^2$ at a subscheme K supported at a set of $k-1$ points $\{p, p_3, \dots, p_k\}$ and with length 2 at the point p. Then there is an equivalence $\mathbf{D}^b(\mathrm{coh}(X_K)) \cong \mathbf{D}^b(\mathrm{mod}\text{–}B_K(\tau'))$, where $B_K(\tau')$ is the algebra of homomorphisms of the exceptional collection τ'.*

In this context, the surface X_K can again be recovered from the category $\mathfrak{B}_K(\tau')$. Namely, the points $p, p_3, \dots, p_k$ can be determined by the same method as above. To recover X_K, we also have to determine the position of the point p' on the exceptional curve of the blowup of the point p. This is equivalent to finding a tangent direction at the point p. Consider the kernel of the composition map

$$\mathrm{Hom}(\mathcal{O}, \mathcal{M}) \otimes \mathrm{Hom}(\mathcal{M}, \mathcal{O}_{l'}) \longrightarrow \mathrm{Hom}(\mathcal{O}, \mathcal{O}_{l'}).$$

It is a one-dimensional subspace of $\mathrm{Hom}(\mathcal{O}, \mathcal{M})$. The image of this subspace in the space $V^* = \mathrm{Hom}(\mathcal{O}, \pi^*\mathcal{O}(1))$ determines a line in the projective space $\mathbb{P}(V)$ which passes through the point p and hence a tangent direction at p.

2.3. **Noncommutative deformations of Del Pezzo surfaces.** As before, let X_K be the blowup of the projective plane at a set K of k distinct points. Consider the strong exceptional collection

$$\sigma = (\mathcal{O}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_1}, \dots, \mathcal{O}_{l_k}).$$

By the discussion in §2.1, the derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(X_K))$ is equivalent to the category of finitely generated (right) modules over the algebra B_K of homomorphisms of σ. The algebra B_K can also be represented by the category $\mathfrak{B}_K$ associated to the exceptional collection σ (see Figure 1).

The category $\mathfrak{B}_K$ has strictly more deformations than the surface X_K. We saw above that the surface X_K can be reconstructed from the category $\mathfrak{B}_K$, and that the deformation of the surface X_K is controlled by the variation of the set $K \subset \mathbb{P}^2$.

A general deformation of the category $\mathfrak{B}_K$ can be viewed as the category of an exceptional collection on a noncommutative deformation of the surface X_K. In other words, if $\mathfrak{B}_{K,\mu}$ is a deformation of the category $\mathfrak{B}_K$ then the bounded derived category $\mathbf{D}^b(\mathrm{mod}\text{–}B_{K,\mu})$ of finitely generated (right) modules over the algebra $B_{K,\mu}$ will be viewed as the derived category of coherent sheaves on a noncommutative surface $X_{K,\mu}$. Any such noncommutative surface can be represented as the blowup of a noncommutative plane $\mathbb{P}^2_\mu$ at some set K consisting of k of its "points". This procedure is discussed in detail in [20].

In the rest of this section, we describe the deformations of the category $\mathfrak{B}_K$. Recall that a deformation of a category is, by definition, a deformation of its composition law. We proceed in two steps. The first step is to describe the deformations of the subcategory $\mathfrak{B}(\sigma_0)$ associated to the subcollection $\sigma_0 = (\mathcal{O}, \pi^*\mathcal{T}(-1), \pi^*\mathcal{O}(1))$. This subcategory $\mathfrak{B}(\sigma_0)$ is the category of homomorphisms of the full strong exceptional collection $(\mathcal{O}, \mathcal{T}(-1), \mathcal{O}(1))$ on $\mathbb{P}^2$. Therefore, considering a

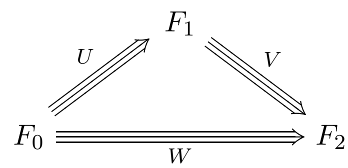

FIGURE 4. The quiver $\mathfrak{B}_\mu$ for a noncommutative $\mathbb{P}^2_\mu$.

deformation of the subcategory $\mathfrak{B}(\sigma_0)$ we obtain a noncommutative deformation $\mathbb{P}^2_\mu$ of the projective plane. The second step is to describe the deformations of all other compositions in the category $\mathfrak{B}_K$. These deformations correspond to variations of the set of "points" K on the noncommutative projective plane $\mathbb{P}^2_\mu$.

Noncommutative deformations of the projective plane have been described in [2, 6]. Any deformation of the category $\mathfrak{B}(\sigma_0)$ is a category with three ordered objects F_0, F_1, F_2 and with three-dimensional spaces of homomorphisms from F_i to F_j when $i < j$ (see Figure 4). Any such category $\mathfrak{B}_\mu$ is determined by the composition tensor $\mu : V \otimes U \to W$. We will consider only the nondegenerate (geometric) case, where the restrictions $\mu_u = \mu(\cdot, u) : V \to W$ and $\mu_v = \mu(v, \cdot) : U \to W$ have rank at least two for all nonzero elements $u \in U$ and $v \in V$, and the composition of μ with any nonzero linear form on W is a bilinear form of rank at least two on $V \otimes U$. The equations $\det \mu_u = 0$ and $\det \mu_v = 0$ define closed subschemes $\Gamma_U \subset \mathbb{P}(U)$ and $\Gamma_V \subset \mathbb{P}(V)$. Namely, up to projectivization the set of points of Γ_U (resp. Γ_V) consists of all $u \in U$ (resp. $v \in V$) for which the rank of μ_u (resp. μ_v) is equal to 2. It is easy to see that the correspondence which associates to a vector $v \in V$ the kernel of the map $\mu_v : U \to W$ defines an isomorphism between Γ_V and Γ_U. Moreover, under these circumstances Γ_V is either the entire projective plane $\mathbb{P}(V)$ or a cubic in $\mathbb{P}(V)$. If $\Gamma_V = \mathbb{P}(V)$ then μ is isomorphic to the tensor $V \otimes V \to \Lambda^2 V$, i.e. we get the usual projective plane $\mathbb{P}^2$.

Thus, the non-trivial case is the situation where Γ_V is a cubic, i.e. an elliptic curve which we now denote by E. This elliptic curve comes equipped with two embeddings into the projective planes $\mathbb{P}(U)$ and $\mathbb{P}(V)$ respectively; by restriction of $\mathcal{O}(1)$ these embeddings determine two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ of degree 3 over E, and it can be checked that $\mathcal{L}_1 \neq \mathcal{L}_2$. This construction has a converse:

Construction 2.9. The tensor μ can be reconstructed from the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$. Namely, the spaces U, V are isomorphic to $H^0(E, \mathcal{L}_1)^*$ and $H^0(E, \mathcal{L}_2)^*$ respectively, and the space W is dual to the kernel of the canonical map

$$H^0(E, \mathcal{L}_1) \otimes H^0(E, \mathcal{L}_2) \longrightarrow H^0(E, \mathcal{L}_1 \otimes \mathcal{L}_2),$$

which induces the tensor $\mu : V \otimes U \longrightarrow W$.

The details of these constructions and statements can be found in [2, 6].

Remark 2.10. Note that we can also consider a triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ such that $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the procedure described above produces a tensor μ with $\Gamma_V \cong \mathbb{P}(V)$, which defines the usual commutative projective plane. Thus, in this particular case the tensor μ does not depend on the curve E.

Now we describe the deformations of the other compositions in the category $\mathfrak{B}_K$. Given a category $\mathfrak{B}_\mu$ of the form described above, corresponding to a noncommutative projective plane $\mathbb{P}^2_\mu$, and given a set $K = \{p_1, \dots, p_k\}$ of k points on the elliptic curve E, we can construct a category $\mathfrak{B}_{K,\mu}$ in the following manner. A point $p_j \in E \subset \mathbb{P}(U)$ determines a one-dimensional subspace of U, generated by a vector $u_j \in U$. The map $\mu_{u_j} : V \to W$ has rank 2; denote by v_j a non-zero vector in its kernel. The category $\mathfrak{B}_{K,\mu}$ is then constructed from the category $\mathfrak{B}_\mu$ by adding k mutually orthogonal objects $\mathcal{O}_{l_j}$ for $j = 1, \dots, k$, and defining the spaces of morphisms by the rule

$$\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) = \mathbb{C}, \qquad \mathrm{Hom}(F_1, \mathcal{O}_{l_j}) = V/\operatorname{Ker}\mu_{u_j} = V/\langle v_j\rangle, \qquad \mathrm{Hom}(F_0, \mathcal{O}_{l_j}) = W/\operatorname{Im}\mu_{v_j}.$$

The two composition tensors involving $\mathrm{Hom}(F_2, \mathcal{O}_{l_j})$ are defined in the obvious manner as suggested by the notation. The only non-obvious composition is the map $V/\langle v_j\rangle \otimes U \to W/\operatorname{Im}\mu_{v_j}$, which is by definition induced by the tensor $\mu : V \otimes U \longrightarrow W$.

Conversely, if we consider a category $\mathfrak{B}_{K,\mu}$ which is a deformation of $\mathfrak{B}_K$ and an extension of the category $\mathfrak{B}_\mu$, then the kernel of the composition map

$$\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) \otimes V \longrightarrow \mathrm{Hom}(F_1, \mathcal{O}_{l_j})$$

defines a one-dimensional subspace $\langle v_j\rangle \subset V$. The map μ_{v_j} must have rank 2, since otherwise μ_{v_j} would be an isomorphism and the composition map $\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) \otimes W \longrightarrow \mathrm{Hom}(F_0, \mathcal{O}_{l_j})$ would vanish identically, which by assumption is not the case. Therefore, the objects $\mathcal{O}_{l_j}$ correspond to points on the curve E.

Thus, any category $\mathfrak{B}_{K,\mu}$ is defined by the data $(E, \mathcal{L}_1, \mathcal{L}_2, p_1, \dots, p_k)$, where E is a cubic, $\mathcal{L}_1, \mathcal{L}_2$ are line bundles of degree 3 on E, and $p_1, \dots, p_k$ is a set of distinct points on E. If $\mathcal{L}_1 \cong \mathcal{L}_2$, then we obtain the category $\mathfrak{B}_K$ related to a blowup of the usual commutative projective plane. In the general case, the bounded derived category $\mathbf{D}^b(\text{mod–}B_{K,\mu})$ of finite rank modules over the algebra $B_{K,\mu}$ is viewed as the derived category of coherent sheaves on the non-commutative surface $X_{K,\mu}$, which is a blowup of k points on the non-commutative projective plane $\mathbb{P}^2_\mu$.

A standard approach to noncommutative geometry is to determine a noncommutative variety either by an abelian category of (quasi)coherent sheaves on it or by a noncommutative (graded) algebra which is considered as its (homogeneous) coordinate ring. The question of how to define the abelian category of coherent sheaves on Del Pezzo surfaces and on other blowups of surfaces is discussed in the paper [20]. We briefly describe one of the possible approaches. It is very important to note that the category $\mathbf{D}^b(\text{mod–}B_{K,\mu})$ possess a Serre functor S, i.e. an additive autoequivalence for which there are bi-functorial isomorphisms

$$\mathrm{Hom}(X, SY) \xrightarrow{\sim} \mathrm{Hom}(Y, X)^*$$

for any $X, Y \in \mathbf{D}^b(\text{mod–}B_{K,\mu})$. In the case of the bounded derived category of finite rank modules over a finite dimensional algebra of finite homological dimension, the Serre functor is the functor which takes a complex of modules $M^\bullet$ to the complex $\mathbf{R}\operatorname{Hom}_{B_{K,\mu}}(M, B_{K,\mu})^*$. The Serre functor is an exact autoequivalence.

Now we can take the projective module P_0 (corresponding to $\mathcal{O}$, see the discussion after Definition 2.4) and consider the sequence of objects $R_m = S^m[-2m]P_0$ for all $m \in \mathbb{Z}$. Let us consider the subcategory $\mathcal{A} \subset \mathbf{D}^b(\text{mod–}B_{K,\mu})$ consisting of all objects F such that

$$\operatorname{Hom}(R_m, F[i]) = 0 \quad \text{for all } i \neq 0 \text{ and sufficiently large } m \gg 0.$$

If the category $\mathcal{A}$ is abelian and its bounded derived category $\mathbf{D}^b(\mathcal{A})$ is equivalent to $\mathbf{D}^b(\text{mod–}B_{K,\mu})$ then $\mathcal{A}$ can be considered as the category of coherent sheaves on the noncommutative surface $X_{K,\mu}$, and $X_{K,\mu}$ can be called a noncommutative Del Pezzo surface.

The reason of such a definition of the abelian category of coherent sheaves on a noncommutative Del Pezzo surface is inspired by the commutative case. In the commutative case the Serre functor is isomorphic to the functor $\otimes\mathcal{O}(K)[2]$, where $\mathcal{O}(K)$ is the canonical line bundle. Hence, for usual commutative surfaces the objects R_m are isomorphic to the invertible sheaves $\mathcal{O}(mK)$. Since for a Del Pezzo surface X the anticanonical sheaf $\mathcal{O}(-K)$ is ample, we have $H^i(X, F(-mK)) = 0$ for all $i \neq 0$ and any coherent sheaf F when m is sufficiently large. This property makes it possible to separate pure coherent sheaves from other complexes of coherent sheaves.

We can also consider the graded space $A = \bigoplus_{p=0}^{\infty} \operatorname{Hom}(R_0, R_{-p})$ and can endow it with the structure of a graded algebra using the isomorphisms $\operatorname{Hom}(R_0, R_{-p}) \cong \operatorname{Hom}(R_i, R_{i-p})$ given by the functors $S^i[-2i]$ for all $i \in \mathbb{Z}$. This algebra can be considered as the homogeneous coordinate ring of a noncommutative Del Pezzo surface. It seems that such rings are noncommutative deformations of homogeneous commutative coordinate rings of usual Del Pezzo surfaces.

In any case, these remarks about abelian categories of coherent sheaves on noncommutative Del Pezzo surfaces will not be needed in the rest of the argument. We will only use the description of the bounded derived category of coherent sheaves on the noncommutative blowup $X_{K,\mu}$ in terms of finite rank modules over the algebra $B_{K,\mu}$, i.e. we state an equivalence of triangulated categories

$$\mathbf{D}^b(\operatorname{coh}(X_{K,\mu})) \cong \mathbf{D}^b(\text{mod–}B_{K,\mu}). \tag{2.7}$$

3. The mirror Landau-Ginzburg models

3.1. Compactification of the mirror of $\mathbb{CP}^2$. As mentioned in the introduction, the mirror of $\mathbb{CP}^2$ is an elliptic fibration with 3 singular fibers, determined by (a fiberwise compactification of) the superpotential $W_0 = x + y + 1/xy$ on $(\mathbb{C}^*)^2$. This Landau-Ginzburg model compactifies naturally to an elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$, which we now describe.

Compactifying $(\mathbb{C}^*)^2$ to $\mathbb{CP}^2$, we can view W_0 as the quotient of the two homogeneous degree 3 polynomials $P_0 = X^2Y + XY^2 + Z^3$ and $P_\infty = XYZ$, which define a pencil of cubics with three base points of multiplicities respectively 4, 4, and 1. Namely, the cubic C_0 defined by P_0 intersects

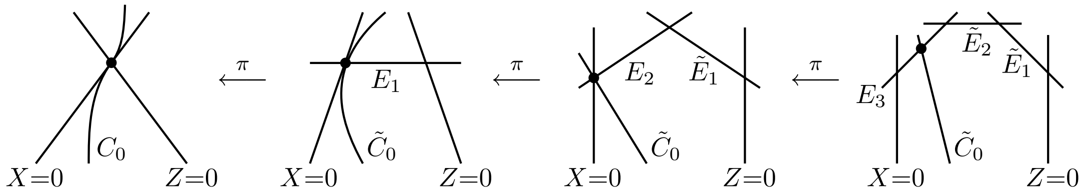

FIGURE 5. The successive blowups at $(0:1:0)$.

the line $X=0$ at $(0:1:0)$ (with multiplicity 3), the line $Y=0$ at $(1:0:0)$ (with multiplicity 3), and the line $Z=0$ at $(0:1:0)$, $(1:0:0)$ and $(1:-1:0)$. Blow up $\mathbb{CP}^2$ three times successively at the point where the cubic C_0 and the line $X=0$ (or their proper transforms) intersect each other, i.e. first at the point $(0:1:0)$, and then twice at suitable points of the exceptional divisors (see Figure 5). Similarly, blow up three times the intersection of the cubic C_0 with the line $Y=0$.

Let $\tilde{C}_0$ be the proper transform of C_0 under these blowups, and let $\tilde{C}_\infty$ be the configuration of 9 rational curves formed by the proper transforms of the three coordinate lines and the exceptional divisors of the six blowups (so, in Figure 5, all components other than $\tilde{C}_0$ are eventually part of $\tilde{C}_\infty$). Then $\tilde{C}_0$ and $\tilde{C}_\infty$ intersect transversely at three smooth points, and define a pencil of elliptic curves representing the anticanonical class in $\mathbb{CP}^2$ blown up six times. The complement of $\tilde{C}_\infty$ identifies with $(\mathbb{C}^*)^2$, and the restriction of the $\mathbb{CP}^1$-valued map defined by the pencil to this open subset coincides with W_0. Blowing up the three points where $\tilde{C}_0$ and $\tilde{C}_\infty$ intersect, we obtain a rational elliptic surface $\overline{M}$, and the pencil becomes an elliptic fibration $\overline{W_0}:\overline{M}\to\mathbb{CP}^1$, which provides a natural compactification of $W_0:(\mathbb{C}^*)^2\to\mathbb{C}$.

The meromorphic function $\overline{W_0}$ has 12 isolated non-degenerate critical points. Three of them are the pre-images of the points $(1:1:1)$, $(j:j:1)$, and $(j^2:j^2:1)$ ($j=e^{2i\pi/3}$), and correspond to the three critical points of W_0 in $(\mathbb{C}^*)^2$ (with associated critical values 3, $3j$, and $3j^2$). The nine other critical points all lie in the fiber above infinity: they are the nodes of the reducible configuration $\tilde{C}_\infty$ (see Figure 6).

This compactification process can also be described in a more symmetric manner by viewing $(\mathbb{C}^*)^2$ as the surface $\{xyz=1\}\subset(\mathbb{C}^*)^3$, and $W_0=x+y+z$. Compactifying $(\mathbb{C}^*)^3$ to $\mathbb{CP}^3$ leads one to consider the cubic surface $\{XYZ=T^3\}\subset\mathbb{CP}^3$, which presents A_2 singularities at the three points $(1:0:0:0)$, $(0:1:0:0)$, and $(0:0:1:0)$. After blowing up $\mathbb{CP}^3$ at these three points,

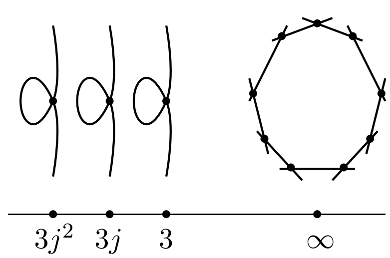

FIGURE 6. The singular fibers of $\overline{W_0}$.

we obtain a smooth cubic surface, in which the hyperplane sections $\tilde{C}_0 = \{X+Y+Z=0\}$ and $\tilde{C}_\infty = \{T=0\}$ define a pencil of elliptic curves with three base points. As before, $\tilde{C}_0$ is a smooth elliptic curve, and $\tilde{C}_\infty$ is a configuration of 9 rational curves (the proper transforms of the three coordinate lines where the singular cubic surface intersects the plane $T=0$, and the six -2-curves arising from the resolution of the singularities). Blowing up the three points of $\tilde{C}_0 \cap \tilde{C}_\infty$, we again obtain a rational elliptic surface, and an elliptic fibration with 12 isolated critical points, 9 of which lie in the fiber above infinity (as in Figure 6).

3.2. **The vanishing cycles of $\overline{W_0}$.** Each singular fiber of $\overline{W_0}$ is obtained from the regular fiber by collapsing a certain number of *vanishing cycles*, and the monodromy of the fibration around a singular fiber is given by a product of Dehn twists along these vanishing cycles. In this section, we determine the homology classes of the vanishing cycles associated to the critical points of $\overline{W_0}$.

More precisely, consider the fiber $\Sigma_0 = \overline{W_0}^{-1}(0)$, which is a smooth elliptic curve (in fact, the proper transform of the curve called $\tilde{C}_0$ in §3.1), and consider the following ordered collection of arcs $(\gamma_i)_{0\le i\le 3}$ joining the origin to the various critical values of $\overline{W_0}$: γ_0, γ_1, γ_2 are straight line segments joining the origin to $\lambda_0 = 3$, $\lambda_1 = 3j^2$, and $\lambda_2 = 3j$ respectively, and γ_3 is the straight line $e^{i\pi/3}\mathbb{R}_+$ joining the origin to $\lambda_3 = \infty$.

Using parallel transport (with respect to an arbitrary horizontal distribution) along the arc γ_i, we can associate a vanishing cycle to each critical point $p \in \overline{W_0}^{-1}(\lambda_i)$; this vanishing cycle is well-defined up to isotopy, and in particular we can consider its homology class in $H_1(\Sigma_0,\mathbb{Z}) \simeq \mathbb{Z}^2$ (well-defined up to a choice of orientation). If we fix a symplectic structure on $\overline{M}$ for which the fibers of $\overline{W_0}$ are symplectic submanifolds, then we have a canonical horizontal distribution (given by the symplectic orthogonal to the fiber), which allows us to consider the vanishing cycles as Lagrangian submanifolds of Σ_0, well-defined up to *Hamiltonian isotopy*; in §4 this will be of utmost importance, but for now we ignore the symplectic structure and only view $\overline{W_0}$ as a topological fibration.

Lemma 3.1. *In terms of a suitable basis $\{a,b\}$ of $H_1(\Sigma_0,\mathbb{Z})$, the vanishing cycles L_0, L_1, L_2 associated to the critical values $\lambda_0, \lambda_1, \lambda_2$ (and the arcs $\gamma_0, \gamma_1, \gamma_2$) represent the classes $[L_0] = -2a-b$, $[L_1] = -a+b$, and $[L_2] = a+2b$, respectively; and the vanishing cycles $L_3, \dots, L_{11}$ associated to the nine critical points in the fiber at infinity represent the class $[L_3] = \cdots = [L_{11}] = a+b$.*

Proof. The vanishing cycles L_0, L_1, L_2 are exactly those of the mirror of $\mathbb{CP}^2$, and are well-known (cf. e.g. [17] or [3]). In particular it is known that, choosing a suitable homology basis $\{a,b\}$ for $H_1(\Sigma_0,\mathbb{Z})$, and fixing appropriate orientations of L_0, L_1, L_2, we have $[L_0] = -2a-b$, $[L_1] = -a+b$, and $[L_2] = a+2b$ (cf. e.g. Figure 14 in [3]).

We now consider the 9 critical points in the fiber at infinity. It is clear that $L_3, \dots, L_{11}$ admit disjoint representatives, and hence are all homologous. Their homology class can be determined by considering the monodromy of the elliptic fibration $\overline{W_0}$, which is given by the product of the positive Dehn twists along the vanishing cycles. Considering the action on $H_1(\Sigma_0, \mathbb{Z})$, and still using the basis $\{a, b\}$ considered above, the monodromies around the critical values $\lambda_0, \lambda_1, \lambda_2$ are given by

$$\tau_0 = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and } \tau_2 = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix},$$

while the monodromy around the fiber at infinity is given by τ^9, where τ is the positive Dehn twist along $[L_3] = \cdots = [L_{11}]$. On the other hand, because the arcs $\gamma_0, \dots, \gamma_3$ are ordered clockwise around the origin, we have $\tau_0\tau_1\tau_2\tau^9 = 1$. Therefore,

$$\tau^9 = \begin{pmatrix} -8 & 9 \\ -9 & 10 \end{pmatrix},$$

and considering $\mathrm{Ker}(\tau^9 - 1)$ we obtain $[L_3] = \cdots = [L_{11}] = a + b$. □

Alternative proof. (compare with §4.2 of [3]). Recall that $\overline{M}$ is obtained from $\mathbb{CP}^2$ by successive blowups of the base points of the pencil of cubics defined by $P_0 = X^2Y + XY^2 + Z^3$ and $P_\infty = XYZ$. Consider the ruled surface F obtained by blowing up $\mathbb{CP}^2$ just once at the point $(0:1:0)$: the projection $(X:Y:Z) \mapsto (X:Z)$ naturally extends into a fibration $\pi_x : F \to \mathbb{CP}^1$, of which the exceptional divisor is a section. For $\lambda \in \mathbb{CP}^1$, denote by $\hat{C}_\lambda$ the proper transform of the plane cubic C_λ defined by $P_0 - \lambda P_\infty$, which is also the image of $\overline{W_0}^{-1}(\lambda)$ under the natural projection $p : \overline{M} \to F$.

The restriction $\pi_{x,\lambda}$ of π_x to $\hat{C}_\lambda$ has degree two, and for $\lambda \notin \mathrm{crit}(\overline{W_0})$ its four branch points are associated to distinct critical values in $\mathbb{CP}^1$, namely zero and the three roots of the equation $x(\lambda - x)^2 = 4$. Indeed, since C_λ always has an order 3 tangency with the line $X = 0$ at $(0:1:0)$, $\hat{C}_\lambda$ is always tangent to the fiber $\pi_x^{-1}(0)$. The three other branch points are the critical points of the projection to the first coordinate on $(\mathbb{C}^*)^2 \cap C_\lambda = \{(x, y) \in (\mathbb{C}^*)^2,\ xy(\lambda - x - y) = 1\}$; viewing $xy(\lambda - x - y) = 1$ as a quadratic equation in the variable y, the discriminant is $x(\lambda - x)^2 - 4$.

As λ tends to λ_i ($i \in \{0, 1, 2\}$), two of the critical values of $\pi_{x,\lambda}$ converge to each other; keeping track of the manner in which these critical values coalesce when λ varies from 0 to λ_i along the arc γ_i, we obtain an arc $\delta_i \subset \mathbb{CP}^1$, with end points in $\mathrm{crit}(\pi_{x,0})$ (see Figure 7). The lift of δ_i under the double cover $\pi_{x,0}$ is (up to homotopy) the vanishing cycle L_i (note that the projection $p : \overline{M} \to F$ allows us to implicitly identify $\hat{C}_\lambda$ with $\overline{W_0}^{-1}(\lambda)$ for $\lambda \neq \infty$).

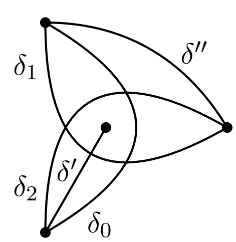

FIGURE 7. The projections of the vanishing cycles of $\overline{W_0}$

Similarly, the behavior of the critical values of $\pi_{x,\lambda}$ as λ tends to infinity describes the degeneration of $\hat{C}_\lambda$ to the singular configuration $\hat{C}_\infty$, which consists of two sections and two fibers of $\pi_x : F \to \mathbb{CP}^1$ (the fibers above 0 and ∞, the exceptional section, and the pre-image of the line $Y = 0$). Namely, as λ tends to infinity along the arc γ_3, the critical value with argument $-2\pi/3$ approaches zero, while the two other roots of $x(\lambda - x)^2 - 4$ tend to infinity. The manner in which pairs of critical values coalesce is encoded by the arcs δ' and δ'' in Figure 7, and the four vanishing cycles associated to the degeneration are essentially the lifts under $\pi_{x,0}$ of closed loops which bound regular neighborhoods of the arcs δ' and δ''; they all represent the same homotopy class inside $\hat{C}_0$.

Recall that $\overline{W_0}^{-1}(\infty) \simeq \tilde{C}_\infty$ is obtained from $\hat{C}_\infty$ by repeatedly blowing up two of the nodes. Taking pre-images under these blowup operations, the vanishing cycles associated to the two other nodes of $\hat{C}_\infty$ are naturally identified with two of the nine vanishing cycles $L_3, \dots, L_{11}$ associated to the fiber at infinity of $\overline{W_0}$. In particular, these vanishing cycles represent the same homology class in $H_1(\Sigma_0, \mathbb{Z}) \simeq H_1(\hat{C}_0, \mathbb{Z})$ as the lifts of δ' and δ''.

It is then easy to check that, for suitable choices of orientations, we have $[L_0]\cdot[L_1] = [L_0]\cdot[L_2] = [L_1]\cdot[L_2] = -3$, $[L_0]\cdot[L_{3+i}] = [L_2]\cdot[L_{3+i}] = -1$, and $[L_1]\cdot[L_{3+i}] = -2$, which completes the proof of Lemma 3.1. □

3.3. **The vanishing cycles of** (M_k, W_k)**.** Recall from the introduction that our proposal for the mirror of a Del Pezzo surface X_K obtained from $\mathbb{CP}^2$ by blowing up $k \le 8$ generic points is an elliptic fibration $W_k : M_k \to \mathbb{C}$, obtained by deforming the fibration $\overline{W_0}$ to another elliptic fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$, and considering the restriction to $M_k = \overline{M} \setminus \overline{W_k}^{-1}(\infty)$. More precisely, remember that $\overline{W_k}$ has $3 + k$ irreducible nodal fibers corresponding to critical values $\lambda_0, \dots, \lambda_{k+2} \in \mathbb{C}$, of which the first three correspond naturally to the irreducible nodal fibers of $\overline{W_0}$, while the k other finite critical values correspond to the deformation of critical points in $\overline{W_0}^{-1}(\infty)$ towards finite values of the superpotential. Meanwhile, $\overline{W_k}^{-1}(\infty)$ is a singular fiber with $9 - k$ components.

While the precise locations of the critical values λ_i are closely related to the complex structure on M_k, they are essentially irrelevant from the point of view of symplectic topology and categories of vanishing cycles. Indeed, if we consider a family $(M_{k,t}, W_{k,t})$ of fibrations indexed by a real parameter t, with the property that for all t the critical points of $W_{k,t}$ are isolated and non-degenerate, then the vanishing cycles remain the same for all values of t, up to smooth isotopies inside the reference fiber. For this reason, we do not need to make a specific choice of λ_i. To fix ideas, let us say that λ_0 is close to 3, λ_1 is close to $3j^2$, λ_2 is close to $3j$, and λ_i is close to infinity for $i \ge 3$; we again choose the origin as base point, and note that the smooth elliptic curve $W_k^{-1}(0)$ is diffeomorphic to $\overline{W_0}^{-1}(0)$, so we implicitly identify them and again call our reference fiber Σ_0. We also choose an ordered collection of arcs γ_i joining the origin to λ_i which lie close to those considered in §3.2, thus ensuring that the homology classes $[L_0], \dots, [L_{k+2}] \in H_1(\Sigma_0, \mathbb{Z})$ of the corresponding vanishing cycles remain those given by Lemma 3.1.

Fixing a symplectic form ω_k on M_k (compatible with W_k, i.e. restricting positively to the fibers), the vanishing cycles $L_0, \dots, L_{k+2}$ associated to the arcs $\gamma_0, \dots, \gamma_{k+2}$ naturally become *Lagrangian submanifolds* of the reference fiber $(\Sigma_0, \omega_{k|\Sigma_0})$ (cf. e.g. [1, 16, 18]). Indeed, the symplectic form defines a natural horizontal distribution outside of the critical points of W_k, given by the symplectic orthogonal to the fiber. Using this horizontal distribution, parallel transport induces symplectomorphisms between the smooth fibers, and the vanishing cycle L_i is by definition the set of points in the reference fiber Σ_0 for which parallel transport along γ_i converges to the critical point in the fiber $W_k^{-1}(\lambda_i)$. It is also useful to consider the *Lefschetz thimble* D_i, which is the set of points swept out by parallel transport of L_i above γ_i; by construction, D_i is a Lagrangian disk in (M_k, ω_k), fibered above the arc γ_i, and $\partial D_i = L_i$.

We recall the following classical result (we provide a proof for completeness):

Lemma 3.2. *A deformation of the system of arcs $\{\gamma_i\}$ by an isotopy in $\mathrm{Diff}(\mathbb{C}, \mathrm{crit}(W_k))$ affects the vanishing cycles L_i by Hamiltonian isotopies; moreover, the same property holds if the symplectic fibration (M_k, ω_k, W_k) is deformed in a manner such that the cohomology class $[\omega_k]$ remains constant and the critical points of W_k remain isolated and non-degenerate.*

Proof. We first consider a deformation of the system of arcs $\{\gamma_i\}$, based at a regular value $\lambda_* \in \mathbb{C} \setminus \mathrm{crit}(W_k)$ (in our case the origin), to an isotopic system of arcs $\{\gamma_i'\}$ based at a regular value λ_*'. This means that we are given an arc $\delta : [0,1] \to \mathbb{C} \setminus \mathrm{crit}(W_k)$ joining λ_* to λ_*', and continuous families of arcs $\{\gamma_{i,t}\}$, $0 \le t \le 1$, with $\gamma_{i,0} = \gamma_i$ and $\gamma_{i,1} = \gamma_i'$, such that $\gamma_{i,t}$ joins the regular value $\delta(t)$ to the critical value λ_i, and $\{\gamma_{i,t}\}_{0 \le i \le k+2}$ is an ordered collection of arcs for all $t \in [0,1]$. The vanishing cycles L_i' associated to the arcs γ_i' live inside $\Sigma_*' = W_k^{-1}(\lambda_*')$, while the original vanishing cycles L_i associated to γ_i are submanifolds of $\Sigma_* = W_k^{-1}(\lambda_*)$. However, we claim that the isotopy induces a symplectomorphism $\phi : \Sigma_* \to \Sigma_*'$ with the property that $\phi(L_i)$ and L_i' are mutually Hamiltonian isotopic for all i; this is the meaning of the statement of the lemma.

Namely, parallel transport along the arc δ (using the horizontal distribution described above) induces a symplectomorphism ϕ from $\Sigma_* = W_k^{-1}(\lambda_*)$ to $\Sigma_*' = W_k^{-1}(\lambda_*')$. For all $t \in [0,1]$ we can consider the vanishing cycle $L_{i,t} \subset W_k^{-1}(\delta(t))$ associated to the arc $\gamma_{i,t}$, and its image $L_{i,t}' \subset \Sigma_*'$ under the symplectomorphism induced by parallel transport along $\delta([t,1])$. The family $L_{i,t}'$, $t \in [0,1]$ defines a smooth isotopy from $L_{i,0}' = \phi(L_i)$ to $L_{i,1}' = L_i'$ through Lagrangian submanifolds of Σ_*'. Moreover, each vanishing cycle $L_{i,t} \subset W_k^{-1}(\delta(t))$ bounds a Lagrangian thimble $D_{i,t}$, and the cylinder $C_{i,t}$ swept by $L_{i,t}$ under parallel transport along $\delta([t,1])$ is also Lagrangian. By continuity, the relative cycles $D_{i,t} \cup C_{i,t}$ (with boundary $L_{i,t}'$) all represent the same relative homotopy class in $\pi_2(M_k, \Sigma_*')$, and since $D_{i,t}$ and $C_{i,t}$ are Lagrangian they all have zero symplectic area. This implies that the Lagrangian submanifolds $L_{i,t}'$, $t \in [0,1]$ are all Hamiltonian isotopic inside Σ_*'; in particular, $\phi(L_i)$ and L_i' are Hamiltonian isotopic.

We now consider a symplectic fibration $W_k' : (M_k, \omega_k') \to \mathbb{C}$ which is isotopic to W_k through an isotopy $W_{k,t} : (M_k, \omega_{k,t}) \to \mathbb{C}$ that preserves the cohomology class of the symplectic form (i.e.,

$[\omega_{k,t}] = [\omega_k]$ for all $t \in [0,1]$). We assume that each $W_{k,t}$ has isolated non-degenerate critical points. This allows us to deform the system of arcs $\{\gamma_i\}$ through a family $\{\gamma_{i,t}\}$ with end points at the critical values of $W_{k,t}$; for $t = 1$ we obtain a system of arcs $\{\gamma'_i\}$ based at a regular value λ'_* of W'_k. By Moser's theorem, there exists a continuous family of symplectomorphisms ϕ_t from $(M_k, \omega_{k,t})$ to (M_k, ω'_k), or rather, since these are non-compact manifolds, from open subsets of $(M_k, \omega_{k,t})$ to an open subset of (M_k, ω'_k); however, after "enlarging" (M_k, ω'_k) by adding to ω'_k the pullback of a suitable area form on $\mathbb{C}$, which affects neither the symplectic structure on the fibers nor the parallel transport symplectomorphisms, we can ensure that ϕ_t is defined over an arbitrarily large open subset of M_k, which is good enough for our purposes. Moreover, by a relative version of Moser's argument, we can also ensure that ϕ_t maps the reference fiber of $W_{k,t}$ to the reference fiber of W'_k, and in particular that $\phi = \phi_0$ maps $\Sigma_* = W_k^{-1}(\lambda_*)$ to $\Sigma'_* = {W'_k}^{-1}(\lambda'_*)$.

We now claim that $\phi(L_i) \subset \Sigma'_*$ is Hamiltonian isotopic to the vanishing cycle L'_i of W'_k associated to the arc γ'_i. Indeed, by considering the images under ϕ_t of the vanishing cycles $L_{i,t}$ associated to the arcs $\gamma_{i,t}$, we obtain a smooth isotopy from $\phi(L_i)$ to L'_i through Lagrangian submanifolds of Σ'_*. Moreover, the thimbles D'_i and $\phi(D_i)$ represent the same relative homotopy class (as can be seen by considering the images by ϕ_t of the thimbles $D_{i,t}$ associated to $\gamma_{i,t}$), and both are Lagrangian with respect to ω'_k, which again implies that $\phi(L_i)$ and L'_i are Hamiltonian isotopic. □

3.4. **A basis of $H_2(M_k)$.** The manifold M_k is simply connected, and its second Betti number is equal to $k+2$. A $\mathbb{Q}$-basis of $H_2(M_k)$ is given by considering the homology class of the fiber of W_k, $[\Sigma_0]$, and $k+1$ classes $[\bar{C}], [\bar{C}_0], \dots, [\bar{C}_{k-1}]$ constructed from the vanishing cycles L_i and Lefschetz thimbles D_i in the following manner.

By Lemma 3.1 we have $[L_1] = [L_0] + [L_2]$ in $H_1(\Sigma_0, \mathbb{Z})$, so there exists a 2-chain C in Σ_0 such that $\partial C = -L_0 + L_1 - L_2$. Then

$$\bar{C} = C + D_0 - D_1 + D_2$$

is a 2-cycle in M_k. Note that $[\bar{C}]$ is in fact the image of the generator of $H_2((\mathbb{C}^*)^2, \mathbb{Z}) \simeq \mathbb{Z}$ under the inclusion map (see the proof of Lemma 4.9 in [3]).

Similarly, for $0 \le i < k$ we have $3\,[L_{3+i}] = [L_2] - [L_0]$ in $H_1(\Sigma_0, \mathbb{Z})$, so there exists a 2-chain C_i in Σ_0 such that $\partial C_i = 3\,L_{3+i} + L_0 - L_2$, and we can consider the 2-cycle

$$\bar{C}_i = C_i - 3\,D_{3+i} - D_0 + D_2$$

in M_k. We also introduce 2-chains $\Delta_{i,j}$ $(i, j \in \{0, \dots, k-1\})$ in Σ_0 such that $\partial \Delta_{i,j} = L_{3+j} - L_{3+i}$, and the corresponding 2-cycles

$$\bar{\Delta}_{i,j} = \Delta_{i,j} + D_{3+i} - D_{3+j}.$$

We can choose C_i and $\Delta_{i,j}$ in such a way that $C_j - C_i = 3\,\Delta_{i,j}$ (and hence $[\bar{C}_j] - [\bar{C}_i] = 3\,[\bar{\Delta}_{i,j}]$ in $H_2(M_k)$).

To summarize the discussion, the vanishing cycles L_i and the 2-chains C, C_i, $\Delta_{i,j}$ are represented on Figures 8–9 (compare with Figure 2 in [17] and with [19]).

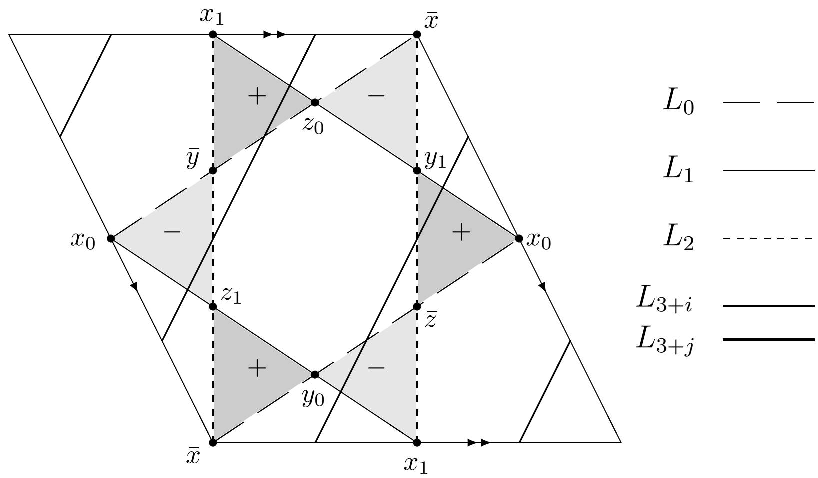

FIGURE 8. The vanishing cycles of W_k and the chain C

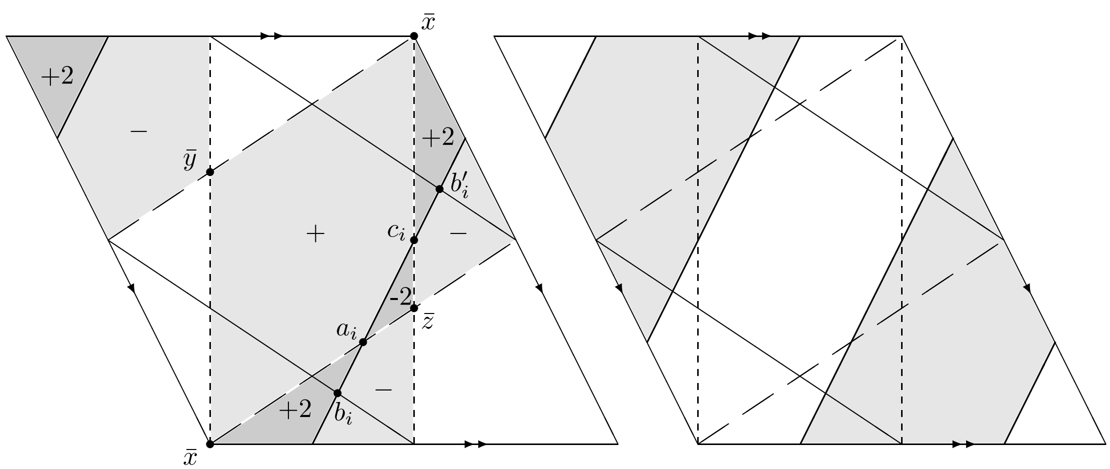

FIGURE 9. The chains C_i (left) and $\Delta_{i,j}$ (right)

4. Categories of vanishing cycles

4.1. **Definition.** As proposed by Kontsevich [12] and Hori-Iqbal-Vafa [9], the category of A-branes associated to a Landau-Ginzburg model $W : (M, \omega) \to \mathbb{C}$ is a Fukaya-type category which contains not only compact Lagrangian submanifolds of M but also certain non-compact Lagrangians whose ends fiber in a specific way above half-lines in $\mathbb{C}$. In the case where the critical points of W are isolated and non-degenerate, this category admits an exceptional collection whose objects are Lagrangian thimbles associated to the critical points. Following the formalism introduced by Seidel [16, 18], we view it as the derived category of a finite directed A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ associated to an ordered collection of arcs $\{\gamma_i\}$. We briefly recall the definition; the reader is referred to [16, 18] and to §3.1 of [3] for details.

Consider a symplectic fibration $W : (M, \omega) \to \mathbb{C}$ with isolated non-degenerate critical points, and assume for simplicity that the critical values $\lambda_0, \ldots, \lambda_r$ of W are distinct. Pick a regular value λ_* of W, and choose a collection of arcs $\gamma_0, \ldots, \gamma_r \subset \mathbb{C}$ joining λ_* to the various critical values of

W, intersecting each other only at λ_*, and ordered in the clockwise direction around λ_*. Consider the horizontal distribution defined by the symplectic form: by parallel transport along the arc γ_i, we obtain a Lagrangian thimble D_i and a vanishing cycle $L_i = \partial D_i \subset \Sigma_*$ (where $\Sigma_* = W^{-1}(\lambda_*)$). After a small perturbation we can always assume that the vanishing cycles L_i intersect each other transversely inside Σ_*. The following definition is motivated by the observation that the intersection theory of the Lagrangian thimbles $D_i \subset M$ is closely related to that of the vanishing cycles L_i inside Σ_* [16]:

Definition 4.1 (Seidel). *The directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is an* A_∞-*category (over a coefficient ring* R*) with objects* $L_0, \dots, L_r$ *corresponding to the vanishing cycles (or more accurately to the thimbles); the morphisms between the objects are given by*

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot id & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

and the differential m_1*, composition* m_2 *and higher order products* m_k *are defined in terms of Lagrangian Floer homology inside* Σ_**. More precisely,*

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \cdots < i_k$ *fails to hold (i.e. it is always zero in this case, except for* m_2 *where composition with an identity morphism is given by the obvious formula). When* $i_0 < \cdots < i_k$*,* m_k *is defined by fixing a generic* ω*-compatible almost-complex structure on* Σ_* *and counting pseudo-holomorphic maps from a disk with* $k+1$ *cyclically ordered marked points on its boundary to* Σ_**, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to* $L_{i_0}, \dots, L_{i_k}$ *respectively.*

This definition calls for several clarifications. First of all, in our case Σ_* is a smooth elliptic curve and the vanishing cycles are homotopically non-trivial closed loops, we have $\pi_2(\Sigma_*) = 0$ and $\pi_2(\Sigma_*, L_i) = 0$; hence, we need not be concerned by bubbling issues in the definition of the Floer differential and products. In fact, the pseudo-holomorphic disks in Σ_* that we have to consider are nothing but immersed polygonal regions bounded by the vanishing cycles, satisfying a local convexity condition at each corner point.

Also, the Maslov class vanishes identically, so we have a well-defined $\mathbb{Z}$-grading by Maslov index on the Floer complexes $CF^*(L_i, L_j; R)$ once we choose graded Lagrangian lifts of the vanishing cycles. Since in our case $c_1(\Sigma_*) = 0$, we can do this by considering a nowhere vanishing 1-form $\Omega \in \Omega^1(\Sigma_*, \mathbb{C})$ and choosing a real lift of the phase function $\phi_i = \arg(\Omega_{|L_i}) : L_i \to S^1$ for each vanishing cycle. The degree of a given intersection point $p \in L_i \cap L_j$ is then determined by the difference between the phases of L_i and L_j at p.

Our next remark is that the pseudo-holomorphic disks appearing in Definition 4.1 are counted with appropriate weights, and with signs determined by choices of orientations of the relevant moduli spaces. The orientation is determined by the choice of a spin structure for each vanishing cycle L_i; in our case this spin structure must extend to the thimble, so it is necessarily the non-trivial one. In the one-dimensional case there is a convenient recipe for determining the correct sign factors, due to Seidel [18]. As will be clear from the discussion in §4.2 below, we will only be interested in the specific case where all morphisms have even degree and all spin structures are non-trivial. The sign rule can then be summarized as follows: pick a marked point on each L_i, distinct from the intersections with the other vanishing cycles; then the sign associated to a pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ is $(-1)^{\nu(u)}$, where $\nu(u)$ is the number of marked points that the boundary of u passes through ([18], see also §4.6 of [3]).

Finally, the weight attributed to each pseudo-holomorphic map u keeps track of its relative homology class, which makes it possible to avoid convergence problems. The usual approach favored by mathematicians is to work over a Novikov ring, which keeps track of the relative homology class by introducing suitable formal variables. To remain closer to the physics, we use $\mathbb{C}$ as our coefficient ring, and assign weights according to the symplectic areas; this is in fact equivalent to working over the Novikov ring and specializing at the cohomology class of the symplectic form.

The weight formula is simplest when there is no B-field; in that case, we consider untwisted Floer theory, since any flat unitary bundle over the thimble D_i is trivial and hence restricts to L_i as the trivial bundle. We then count each map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ with a coefficient $(-1)^{\nu(u)} \exp(-2\pi \int_{D^2} u^*\omega)$. (The normalization factor 2π is purely a matter of conventions, and is sometimes omitted in the literature; here we include it for convenience). Hence, given two intersection points $p \in L_i \cap L_j$, $q \in L_j \cap L_k$ $(i < j < k)$, we have by definition

$$m_2(p,q) = \sum_{\substack{r \in L_i \cap L_k \\ \deg r = \deg p + \deg q}} \left(\sum_{[u] \in \mathcal{M}(p,q,r)} (-1)^{\nu(u)} \exp(-2\pi \int_{D^2} u^*\omega) \right) r$$

where $\mathcal{M}(p,q,r)$ is the moduli space of pseudo-holomorphic maps u from the unit disk to Σ_* (equipped with a generic ω-compatible almost-complex structure) such that $u(1) = p$, $u(\mathrm{j}) = q$, $u(\mathrm{j}^2) = r$ (where $\mathrm{j} = \exp(\frac{2i\pi}{3})$), and mapping the portions of unit circle $[1,\mathrm{j}]$, $[\mathrm{j},\mathrm{j}^2]$, $[\mathrm{j}^2,1]$ to L_i, L_j and L_k respectively. The other products are defined similarly. (Observe that Seidel's definition [16] does not involve any weights; this is because he only considers exact Lagrangian submanifolds in exact symplectic manifolds, in which case the symplectic areas are entirely determined by the primitives of the Liouville form and can be eliminated by considering suitably rescaled bases of the Floer complexes.)

In presence of a B-field, the weights are modified by the fact that we now consider *twisted* Floer homology. Indeed, each thimble D_i now comes equipped with a trivial complex line bundle $E_i = \underline{\mathbb{C}}$ and a connection ∇_i with curvature $-2\pi i B$, so its boundary L_i is equipped with the restricted bundle and the restricted connection, whose holonomy is $\mathrm{hol}_{\nabla_i}(L_i) = \exp(-2\pi i \int_{D_i} B)$ by Stokes' theorem. Since this property characterizes the connection ∇_i uniquely up to gauge, we can drop the line bundle and the connection from the notation when considering the objects (L_i, E_i, ∇_i) of $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$. Nonetheless, the holonomy of ∇_i modifies the weights attributed to the pseudo-holomorphic disks in the definition of the twisted Floer differentials and compositions. Namely, the weight attributed to a given pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ is modified by a factor corresponding to the holonomy along its boundary, and becomes

$$(-1)^{\nu(u)}\,\mathrm{hol}(u(\partial D^2))\,\exp(2\pi i \int_{D^2} u^*(B + i\omega)).$$

More precisely, we fix trivializations of the line bundles E_i, so that for each intersection point $p \in L_i \cap L_j$ we have a preferred isomorphism between the fibers $(E_i)_{|p}$ and $(E_j)_{|p}$; then it becomes possible to define the holonomy along the closed loop $u(\partial D^2)$ using the parallel transport induced by ∇_i from one "corner" of u to the next one, and the chosen isomorphism at each corner.

Although the category $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ depends on the chosen ordered collection of arcs $\{\gamma_i\}$, Seidel has obtained the following result [16] (for the exact case, but the proof extends to our situation):

Theorem 4.2 (Seidel)**.** *If the ordered collection $\{\gamma_i\}$ is replaced by another one $\{\gamma'_i\}$, then the categories $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ and $\mathrm{Lag_{vc}}(W, \{\gamma'_i\})$ differ by a sequence of mutations.*

Hence, the category naturally associated to the fibration W is not the finite A_∞-category defined above, but rather a (bounded) *derived* category, obtained from $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ by considering twisted complexes of formal direct sums of Lagrangian vanishing cycles, and adding idempotent splittings and formal inverses of quasi-isomorphisms (see [12] and §5 of [16]). If two categories differ by mutations, then their derived categories are equivalent; hence the derived category $\mathbf{D}^b(\mathrm{Lag_{vc}}(W))$ depends only on the symplectic topology of W and not on the choice of an ordered system of arcs [16].

For the examples we consider, the A_∞-category $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ will in fact be an honest category (see below); the bounded derived category $\mathbf{D}^b(\mathrm{Lag_{vc}}(W))$ is then by definition the bounded derived category of finite rank modules over the algebra associated to this category.

4.2. **Objects and morphisms.** We now determine the categories $\mathrm{Lag_{vc}}(W_k, \{\gamma_i\})$ associated to the Landau-Ginzburg models (M_k, W_k) mirror to Del Pezzo surfaces and the systems of arcs $\{\gamma_i\}$ introduced in §3.3. We start with the objects and morphisms.

Recall that W_k has $k+3$ isolated critical points, giving rise to $k+3$ vanishing cycles $L_0, \dots, L_{k+2}$ in the reference fiber $\Sigma_0 \simeq W_k^{-1}(0)$. The homology classes of these vanishing cycles have been determined in §3 and are given by Lemma 3.1; these determine the vanishing cycles up to Lagrangian isotopy.

The derived category of vanishing cycles is not affected if we modify some of the vanishing cycles by Hamiltonian isotopies (more precisely, a Hamiltonian isotopy induces a chain map on the Floer complexes, which yields a quasi-isomorphism between the finite A_∞-categories of vanishing cycles).

Hence, equipping the elliptic curve Σ_0 with a compatible flat metric, we can identify Σ_0 with the quotient of $\mathbb{C}$ by a lattice, and represent the vanishing cycles L_i by closed geodesics parallel to those represented in Figure 8.

Assume that the cohomology class of the symplectic form ω_k on M_k is generic (or more precisely, with the notations of §3.4, that $[\omega_k]\cdot[\bar{\Delta}_{i,j}]$ is never an integer multiple of $[\omega_k]\cdot[\Sigma_0]$). Then the geodesics L_i are all distinct, and their intersections are as pictured in Figure 8, so we have:

Lemma 4.3. *The geometric intersection numbers between the vanishing cycles are:*

- $|L_0\cap L_1| = |L_0\cap L_2| = |L_1\cap L_2| = 3$*;*
- *for* $0\le i<k$, $|L_0\cap L_{3+i}| = |L_2\cap L_{3+i}| = 1$ *and* $|L_1\cap L_{3+i}| = 2$*;*
- *for* $0\le i<j<k$, $|L_{3+i}\cap L_{3+j}| = 0$ *as soon as* $[\omega_k]\cdot[\Sigma_0] \not| \, [\omega_k]\cdot[\bar{\Delta}_{i,j}]$.

In the rest of this section, unless otherwise specified we always assume that the vanishing cycles L_i are represented by distinct closed geodesics.

As in [3], we denote by x_0, y_0, z_0 (resp. x_1, y_1, z_1 and $\bar{x}, \bar{y}, \bar{z}$) the generators of $\mathrm{Hom}(L_0, L_1)$ (resp. $\mathrm{Hom}(L_1, L_2)$ and $\mathrm{Hom}(L_0, L_2)$) corresponding to the intersection points represented in Figure 8. Moreover, we denote by a_i (resp. b_i, b'_i and c_i) the generators of $\mathrm{Hom}(L_0, L_{3+i})$ (resp. $\mathrm{Hom}(L_1, L_{3+i})$ and $\mathrm{Hom}(L_2, L_{3+i})$) corresponding to the intersection points between these vanishing cycles (see Figure 9).

Lemma 4.4. *For suitable choices of graded lifts of the vanishing cycles, all the morphisms in* $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ *have degree* 0*.*

Proof. Equip Σ_0 with a compatible flat metric and with a constant holomorphic 1-form Ω. Taking geodesic representatives of the vanishing cycles, the phase functions $\phi_i = \arg(\Omega_{|L_i}) : L_i \to \mathbb{R}/2\pi\mathbb{Z}$ are constant, and we can normalize Ω so that it takes real negative values on the oriented tangent space to L_0, i.e. $\phi_0 = \pi$. Then it is possible to choose real lifts $\tilde{\phi}_i \in \mathbb{R}$ of the phases in such a way that $\pi = \tilde{\phi}_0 > \tilde{\phi}_1 > \tilde{\phi}_2 > \tilde{\phi}_3 = \cdots = \tilde{\phi}_{k+2} > 0$ (see Figure 8 and recall the orientations chosen in Lemma 3.1). In the 1-dimensional case, the relationship between Maslov index and phase is very simple: given a transverse intersection point p between two graded Lagrangians $L, L' \subset \Sigma_0$, the Maslov index of $p \in CF^*(L, L')$ is equal to the smallest integer greater than $\frac{1}{\pi}(\tilde{\phi}_{L'}(p) - \tilde{\phi}_L(p))$. Since we only consider the Floer complexes $CF^*(L_i, L_j)$ for $i<j$, which implies that $\tilde{\phi}_j - \tilde{\phi}_i \in (-\pi, 0)$ at every intersection point, for these choices of graded Lagrangian lifts of the vanishing cycles all morphisms in $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ have degree 0. □

Since each product m_j shifts degree by $2-j$, it follows immediately that the A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ is actually an honest category:

Corollary 4.5. $m_j = 0$ *for all* $j \neq 2$*.*

Hence, the final step of the argument is a careful study of the various immersed triangular regions bounded by the vanishing cycles in Σ_0 and their contributions to m_2.

4.3. **Compositions.** As before we assume that the Lagrangian vanishing cycles are realized by distinct closed geodesics in the flat torus Σ_0, and we determine the contributions to m_2 of the various immersed triangular regions in $(\Sigma_0, \cup L_i)$. We use the notations introduced in §4.2 for the intersection points, and those introduced in §3.4 for various 2-chains in Σ_0 and the corresponding 2-cycles in M_k. We also introduce the following notations:

Definition 4.6. *Let* $q_C = \exp(2\pi i\,[B+i\omega]\cdot[\bar{C}])$ *and* $q_F = \exp(2\pi i\,[B+i\omega]\cdot[\Sigma_0])$, *and define*

$$\zeta_+ = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n+1)/2}, \quad \zeta_- = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2}, \quad \zeta_0 = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{3n(n-1)/2}.$$

Since ω is a symplectic form on Σ_0, we have $|q_F| = \exp(-2\pi\,[\omega]\cdot[\Sigma_0]) < 1$, which ensures the convergence of the series ζ_+, ζ_- and ζ_0.

Proposition 4.7. *There exist constants* $\alpha_{xy}, \alpha_{yx}, \alpha_{yz}, \alpha_{zy}, \alpha_{zx}, \alpha_{xz} \in \mathbb{C}$ *such that*

$$\begin{aligned} m_2(x_0,y_1) &= \alpha_{xy}\bar{z}, \qquad & m_2(y_0,x_1) &= \alpha_{yx}\bar{z},\\ m_2(y_0,z_1) &= \alpha_{yz}\bar{x}, \qquad & m_2(z_0,y_1) &= \alpha_{zy}\bar{x},\\ m_2(z_0,x_1) &= \alpha_{zx}\bar{y}, \qquad & m_2(x_0,z_1) &= \alpha_{xz}\bar{y}, \end{aligned}$$

and these constants satisfy the relation

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -q_C \left(\frac{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n+1)/2}}{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2}} \right)^3 = -q_C \left(\frac{\zeta_+}{\zeta_-}\right)^3. \tag{4.1}$$

Remark 4.8. The quantity appearing in the right-hand side of (4.1) can be understood in terms of certain theta functions; see §4.5 for details.

Before giving the proof, we make an observation which will be useful throughout this section. The geodesics L_i are not necessarily those pictured in Figure 8, but they are parallel to them. So we can deform (in a non-Hamiltonian manner) the configuration of vanishing cycles to that of Figure 8, and all intersection points and relative 2-cycles in $(\Sigma_0, \cup L_i)$ can be followed through the deformation. Hence, immersed triangular regions in $(\Sigma_0, \cup L_i)$ are in one to one correspondence with those in the configuration of Figure 8 (but of course the deformation does not preserve areas). Moreover, we can choose the deformation in such a way that the relative 2-cycles C and C_i in Σ_0 deform to those represented on Figures 8–9 (rather than to 2-cycles which differ by a multiple of the fundamental cycle of Σ_0).

Proof of Proposition 4.7. The composition $m_2(x_0, y_1)$ is the sum of an infinite series of contributions, corresponding to all immersed triangular regions in Σ_0 with corners at the intersection points x_0, y_1, and one of the points in $L_0 \cap L_2$. By the above remark we can enumerate these regions by looking at Figure 8. Considering the side which lies on L_1, it is then easy to see that for every homotopy class of arc joining x_0 to y_1 inside L_1 there is a unique such immersed triangular region, and the third vertex is always $\bar{z}$.

These various regions can be labelled by integers $n \in \mathbb{Z}$ in such a way that, denoting by $T_{xy,n}$ the corresponding 2-chains in Σ_0, we have $\partial T_{xy,n} - \partial T_{xy,n'} = (n-n')(-L_0 + L_1 - L_2)$ for all $n, n' \in \mathbb{Z}$. We can choose the integer labels in such a way that, after deforming to the configuration in Figure 8, $T_{xy,0}$ becomes the smallest triangle with vertices $x_0, y_1, \bar{z}$. (So, in Figure 8, $T_{xy,-1}$ is the immersed region bounded by the portions of $L_0 \cup L_1 \cup L_2$ which do not belong to $\partial T_{xy,0}$; and all the other $T_{xy,n}$ have edges which wrap more than once around the vanishing cycles).

By comparing $\partial T_{xy,n}$ and $\partial T_{xy,0}$, it is clear that the 2-chain represented by $T_{xy,n}$ can be expressed in the form $T_{xy,n} = T_{xy,0} + nC + \phi(n)\Sigma_0$ for some $\phi(n) \in \mathbb{Z}$. Moreover, by looking at Figure 8 one easily checks that $\phi(n) = \frac{1}{2}n(3n+1)$. (So e.g. $T_{xy,-1} = T_{xy,0} - C + \Sigma_0$, and $T_{xy,1} = T_{xy,0} + C + 2\Sigma_0$). Let $\psi_{xy} \in \mathbb{C}$ be the coefficient of the contribution of $T_{xy,0}$ to $m_2(x_0, y_1)$. Then, by comparing the symplectic areas and boundary holonomies for $T_{xy,n}$ and $T_{xy,0}$, one easily checks that the contribution of $T_{xy,n}$ is equal to

$$(-1)^n \exp\left(2\pi i\, [B + i\omega] \cdot \left(n[\bar{C}] + \frac{n(3n+1)}{2}[\Sigma_0]\right)\right) \psi_{xy} = (-1)^n\, q_C^n\, q_F^{n(3n+1)/2}\, \psi_{xy}.$$

In this expression the sign factor $(-1)^n$ is due to the non-triviality of the spin structures (observe that $\partial C = -L_0 + L_1 - L_2$ passes once through each of the three marked points on L_0, L_1, L_2); the total holonomy of the flat connections ∇_i along $\partial T_{xy,n} - \partial T_{xy,0} = n\,\partial C$ is $\exp(2\pi i\, n \int_{D_0 - D_1 + D_2} B)$ by Stokes' theorem; and the integral of $B + i\omega$ over $T_{xy,n}$ differs from that over $T_{xy,0}$ by the amount $n\int_C (B + i\omega) + \frac{1}{2}n(3n+1)\,[B + i\omega]\cdot[\Sigma_0]$.

Summing over $n \in \mathbb{Z}$, and using the notation introduced in Definition 4.6, we obtain

$$\alpha_{xy} = \zeta_+\, \psi_{xy}.$$

The calculations of $m_2(y_0, z_1)$ and $m_2(z_0, x_1)$ are exactly identical, and lead to similar expressions. Namely, denote by ψ_{yz} (resp. ψ_{zx}) the contribution of the triangular region $T_{yz,0}$ (resp. $T_{zx,0}$) which, after deforming to the configuration in Figure 8, corresponds to the smallest triangle with vertices y_0, z_1, $\bar{x}$ (resp. z_0, x_1, $\bar{y}$). Then one easily checks by the same argument as above that $\alpha_{yz} = \zeta_+\, \psi_{yz}$ and $\alpha_{zx} = \zeta_+\, \psi_{zx}$.

Next we consider the composition $m_2(y_0, x_1)$, which is again the sum of an infinite series of contributions from triangular regions $T_{yx,n}$, $n \in \mathbb{Z}$, which all have vertices y_0, x_1, $\bar{z}$. We can choose the labels in such a way that, after deforming to the configuration in Figure 8, $T_{yx,0}$ becomes the smallest such triangle, and $T_{yx,n} = T_{yx,0} + nC + \frac{1}{2}n(3n-1)\Sigma_0$. Denoting by ψ_{yx} the coefficient associated to $T_{yx,0}$, it is easy to check by the same argument as above that the contribution of $T_{yx,n}$ is equal to $(-1)^n\, q_C^n\, q_F^{n(3n-1)/2}\, \psi_{yx}$, so that

$$\alpha_{yx} = \zeta_- \psi_{yx}.$$

Similarly, with the obvious notations we have $\alpha_{zy} = \zeta_- \psi_{zy}$ and $\alpha_{xz} = \zeta_- \psi_{xz}$. Finally, observe that

$$\frac{\psi_{xy}\psi_{yz}\psi_{zx}}{\psi_{yx}\psi_{zy}\psi_{xz}} = -q_C.$$

Indeed, $T_{xy,0}+T_{yz,0}+T_{zx,0}-T_{yx,0}-T_{zy,0}-T_{xz,0}=C$ (cf. Figure 8). Therefore, comparing the weights associated to these various triangles, the weighting by area gives a factor of $\exp(2\pi i\int_C B+i\omega)$, while the holonomy along the boundary $\partial C=-L_0+L_1-L_2$ is equal to $\exp(2\pi i\int_{D_0-D_1+D_2}B)$, and finally the minus sign is due to the orientation conventions, since ∂C passes once through each of the three marked points on the vanishing cycles. Hence

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}}=\frac{\psi_{xy}\psi_{yz}\psi_{zx}\,\zeta_+^3}{\psi_{yx}\psi_{zy}\psi_{xz}\,\zeta_-^3}=-q_C\left(\frac{\zeta_+}{\zeta_-}\right)^3.$$

□

Remark 4.9. If $[\omega+iB]\cdot[\bar{C}]=0$, then $q_C=1$ and the ratio between $\alpha_{xy}\alpha_{yz}\alpha_{zx}$ and $\alpha_{yx}\alpha_{zy}\alpha_{xz}$ becomes equal to -1 irrespective of the value of q_F; this corresponds to a classical (commutative) Del Pezzo surface.

Moreover, in the limit where $[\omega]\cdot[\Sigma_0]\to\infty$, we have $q_F=0$ and the ratio becomes $-q_C$, which corresponds to the toric case studied in [3].

Proposition 4.10. *There exist constants $\alpha_{xx},\alpha_{yy},\alpha_{zz}\in\mathbb{C}$ such that*

$$m_2(x_0,x_1)=\alpha_{xx}\bar{x},\quad m_2(y_0,y_1)=\alpha_{yy}\bar{y},\quad m_2(z_0,z_1)=\alpha_{zz}\bar{z},$$

and these constants satisfy the relation

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}}=-\frac{q_F}{q_C}\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n\,q_C^n\,q_F^{3n(n-1)/2}}{\sum_{n\in\mathbb{Z}}(-1)^n\,q_C^n\,q_F^{n(3n-1)/2}}\right)^3=-\frac{q_F}{q_C}\left(\frac{\zeta_0}{\zeta_-}\right)^3.$$

Proof. The argument is similar to the proof of Proposition 4.7. The immersed triangular regions which contribute to $m_2(x_0,x_1)$ all have vertices $\bar{x}$ as their third vertex, and can be indexed by integers $n\in\mathbb{Z}$ in a manner such that $\partial T_{xx,n}-\partial T_{xx,n'}=(n-n')\,\partial C$ for all $n,n'\in\mathbb{Z}$. We can choose the integer labels in such a way that, after deforming to the standard configuration, $T_{xx,0}$ and $T_{xx,1}=T_{xx,0}+C$ are the two embedded triangles with vertices x_0, x_1, $\bar{x}$ visible on Figure 8. It is then easy to check that $T_{xx,n}=T_{xy,0}+nC+\frac{3}{2}n(n-1)\Sigma_0$. Hence, denoting by ψ_{xx} the coefficient associated to $T_{xx,0}$, we have

$$\alpha_{xx}=\zeta_0\psi_{xx},$$

by the same argument as in previous calculations. Similarly, with the obvious notations, we have $\alpha_{yy}=\zeta_0\psi_{yy}$ and $\alpha_{zz}=\zeta_0\psi_{zz}$. Moreover, $T_{xx,0}+T_{yy,0}+T_{zz,0}-T_{yx,0}-T_{zy,0}-T_{xz,0}=\Sigma_0-C$, which implies (by the same argument as above) that

$$\frac{\psi_{xx}\psi_{yy}\psi_{zz}}{\psi_{yx}\psi_{zy}\psi_{xz}}=-\frac{q_F}{q_C}.$$

Therefore

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}}=\frac{\psi_{xx}\psi_{yy}\psi_{zz}\,\zeta_0^3}{\psi_{yx}\psi_{zy}\psi_{xz}\,\zeta_-^3}=-\frac{q_F}{q_C}\left(\frac{\zeta_0}{\zeta_-}\right)^3.$$

□

When $q_F = 0$ (in particular in the toric case) we have $\alpha_{xx}\alpha_{yy}\alpha_{zz} = 0$, as in [3]. The same conclusion also holds when $q_C = 1$ (the commutative case). In fact, when $q_C = 1$ each of the constants α_{xx}, α_{yy}, α_{zz} is zero, since in that case we have $\zeta_0 = 0$ (because the terms corresponding to n and $1-n$ in the series defining ζ_0 exactly cancel each other).

Definition 4.11. *Let $q_i = \exp(2\pi i\,[B+i\omega]\cdot[\bar{C}_i])$, and define*

$$\zeta_{i,+} = \sum_{n\in\mathbb{Z}} (-1)^n\, q_i^n\, q_F^{n(3n+1)/2},\quad \zeta_{i,-} = \sum_{n\in\mathbb{Z}} (-1)^n\, q_i^n\, q_F^{n(3n-1)/2},\quad \zeta_{i,0} = \sum_{n\in\mathbb{Z}} (-1)^n\, q_i^n\, q_F^{3n(n-1)/2}.$$

Proposition 4.12. *There exist constants $\beta_{\bar{x},i}, \beta_{\bar{y},i}, \beta_{\bar{z},i} \in \mathbb{C}$ such that*

$$m_2(\bar{x}, c_i) = \beta_{\bar{x},i} a_i, \quad m_2(\bar{y}, c_i) = \beta_{\bar{y},i} a_i, \quad m_2(\bar{z}, c_i) = \beta_{\bar{z},i} a_i,$$

and these constants satisfy the relations

$$\frac{\beta_{\bar{z},i}^2\, \alpha_{xy}\alpha_{zz}}{\beta_{\bar{x},i}\beta_{\bar{y},i}\, \alpha_{zy}\alpha_{xz}} = \left(\frac{\zeta_{i,-}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,+}\,\zeta_{i,0}},$$

$$\frac{\beta_{\bar{x},i}^2\, \alpha_{yz}\alpha_{xx}}{\beta_{\bar{y},i}\beta_{\bar{z},i}\, \alpha_{xz}\alpha_{yx}} = -q_i \left(\frac{\zeta_{i,+}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,0}\,\zeta_{i,-}}, \quad \text{and} \quad \frac{\beta_{\bar{y},i}^2\, \alpha_{zx}\alpha_{yy}}{\beta_{\bar{z},i}\beta_{\bar{x},i}\, \alpha_{yx}\alpha_{zy}} = -\frac{q_F}{q_i} \left(\frac{\zeta_{i,0}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,-}\,\zeta_{i,+}},$$

where ζ_+, ζ_-, ζ_0, $\zeta_{i,+}$, $\zeta_{i,-}$, $\zeta_{i,0}$, q_i and q_F are as in Definitions 4.6 and 4.11.

Proof. As before, the constants $\beta_{\bar{x},i}, \beta_{\bar{y},i}, \beta_{\bar{z},i}$ are the sums of infinite series corresponding to all immersed triangular regions with vertices at a_i, c_i, and one of $\bar{x}, \bar{y}, \bar{z}$. For example the coefficient $\beta_{\bar{z},i}$ associated to composition $m_2(\bar{x}, c_i)$ is the sum of an infinite series of contributions associated to triangular regions $T_{\bar{z},i,n}$, $n \in \mathbb{Z}$. The integer labels can be chosen so that $\partial T_{\bar{z},i,n} - \partial T_{\bar{z},i,n'} = (n-n')\partial C_i$ and, after deforming to the configuration in Figure 9, $T_{\bar{z},i,0}$ becomes the smallest triangle with vertices $\bar{z}$, a_i, c_i (i.e., the triangle which appears with coefficient -2 in the 2-chain C_i). Then one easily checks that $T_{\bar{z},i,n} = T_{\bar{z},i,0} + nC_i + \frac{1}{2}n(3n-1)\Sigma_0$. Therefore, denoting by $\psi_{\bar{z},i}$ the coefficient associated to $T_{\bar{z},i,0}$, the same argument as in the previous calculations yields the formula

$$\beta_{\bar{z},i} = \zeta_{i,-}\psi_{\bar{z},i}.$$

Similarly, denote by $T_{\bar{x},i,n}$, $n \in \mathbb{Z}$, the immersed triangles contributing to $m_2(\bar{x}, c_i)$, in such a way that $\partial T_{\bar{x},i,n} - \partial T_{\bar{x},i,n'} = (n-n')\partial C_i$, and $T_{\bar{x},i,0}$ corresponds to the smallest triangle with vertices $\bar{x}$, a_i, c_i in Figure 9 (i.e. the triangle which appears with coefficient $+2$ in the 2-chain C_i). Then $T_{\bar{x},i,n} = T_{\bar{x},i,0} + nC_i + \frac{1}{2}n(3n+1)\Sigma_0$. Therefore, denoting by $\psi_{\bar{x},i}$ the contribution of $T_{\bar{x},i,0}$, we have $\beta_{\bar{x},i} = \zeta_{i,+}\psi_{\bar{x},i}$.

Finally, labelling the triangles with vertices $\bar{y}$, a_i, c_i by integers in such a way that $T_{\bar{y},i,0}$ and $T_{\bar{y},i,1} = T_{\bar{y},i,0} + C_i$ correspond to the negative and positive parts of C_i respectively, it is easy to check that $T_{\bar{y},i,n} = T_{\bar{y},i,0} + nC_i + \frac{3}{2}n(n-1)\Sigma_0$, so denoting by $\psi_{\bar{y},i}$ the contribution of $T_{\bar{y},i,0}$ we have $\beta_{\bar{y},i} = \zeta_{i,0}\psi_{\bar{y},i}$. It follows that

$$\frac{\beta_{\bar{z},i}^2\, \alpha_{xy}\alpha_{zz}}{\beta_{\bar{x},i}\beta_{\bar{y},i}\, \alpha_{zy}\alpha_{xz}} = \frac{\psi_{\bar{z},i}^2\, \psi_{xy}\psi_{zz}}{\psi_{\bar{x},i}\psi_{\bar{y},i}\, \psi_{zy}\psi_{xz}}\, \frac{\zeta_{i,-}^2\, \zeta_+\,\zeta_0}{\zeta_{i,+}\,\zeta_{i,0}\,\zeta_-^2}.$$

Moreover, the 2-chains $2\,T_{\bar{z},i,0}+T_{xy,0}+T_{zz,0}$ and $T_{\bar{x},i,0}+T_{\bar{y},i,0}+T_{zy,0}+T_{xz,0}$ are equal, which implies that $\psi_{\bar{z},i}^2\,\psi_{xy}\psi_{zz}=\psi_{\bar{x},i}\psi_{\bar{y},i}\,\psi_{zy}\psi_{xz}$ and completes the proof of the first identity.

The arguments are the same for

$$\frac{\beta_{\bar{x},i}^2\,\alpha_{yz}\alpha_{xx}}{\beta_{\bar{y},i}\beta_{\bar{z},i}\,\alpha_{xz}\alpha_{yx}}=\frac{\psi_{\bar{x},i}^2\,\psi_{yz}\psi_{xx}}{\psi_{\bar{y},i}\psi_{\bar{z},i}\,\psi_{xz}\psi_{yx}}\,\frac{\zeta_{i,+}^2\,\zeta_+\,\zeta_0}{\zeta_{i,0}\,\zeta_{i,-}\,\zeta_-^2},$$

observing that $2\,T_{\bar{x},i,0}+T_{yz,0}+T_{xx,0}-T_{\bar{y},i,0}-T_{\bar{z},i,0}-T_{xz,0}-T_{yx,0}=C_i$ (for which the corresponding weight is $-q_i$), and for

$$\frac{\beta_{\bar{y},i}^2\,\alpha_{zx}\alpha_{yy}}{\beta_{\bar{z},i}\beta_{\bar{x},i}\,\alpha_{yx}\alpha_{zy}}=\frac{\psi_{\bar{y},i}^2\,\psi_{zx}\psi_{yy}}{\psi_{\bar{z},i}\psi_{\bar{x},i}\,\psi_{yx}\psi_{zy}}\,\frac{\zeta_{i,0}^2\,\zeta_+\,\zeta_0}{\zeta_{i,-}\,\zeta_{i,+}\,\zeta_-^2},$$

observing that $2\,T_{\bar{y},i,0}+T_{zx,0}+T_{yy,0}-T_{\bar{z},i,0}-T_{\bar{x},i,0}-T_{yx,0}-T_{zy,0}=\Sigma_0-C_i$ (for which the corresponding weight is $-q_F/q_i$). $\square$

Corollary 4.13. *The constants $\beta_{\bar{x},i}$, $\beta_{\bar{y},i}$, $\beta_{\bar{z},i}$ satisfy the relations:* $\dfrac{\beta_{\bar{z},i}^3}{\beta_{\bar{x},i}^3}\,\dfrac{\alpha_{xy}\alpha_{yx}\alpha_{zz}}{\alpha_{yz}\alpha_{zy}\alpha_{xx}}=-\dfrac{1}{q_i}\Big(\dfrac{\zeta_{i,-}}{\zeta_{i,+}}\Big)^3$, $\dfrac{\beta_{\bar{x},i}^3}{\beta_{\bar{y},i}^3}\,\dfrac{\alpha_{yz}\alpha_{zy}\alpha_{xx}}{\alpha_{zx}\alpha_{xz}\alpha_{yy}}=\dfrac{q_i^2}{q_F}\Big(\dfrac{\zeta_{i,+}}{\zeta_{i,0}}\Big)^3$, *and* $\dfrac{\beta_{\bar{y},i}^3}{\beta_{\bar{z},i}^3}\,\dfrac{\alpha_{zx}\alpha_{xz}\alpha_{yy}}{\alpha_{xy}\alpha_{yx}\alpha_{zz}}=-\dfrac{q_F}{q_i}\Big(\dfrac{\zeta_{i,0}}{\zeta_{i,-}}\Big)^3$.

Proposition 4.14. *For all $0\le i,j<k$ we have the identities*

$$\frac{\beta_{\bar{y},i}\,\beta_{\bar{z},j}}{\beta_{\bar{y},j}\,\beta_{\bar{z},i}}=\tilde{q}_{i,j}\,\frac{\zeta_{i,0}\,\zeta_{j,-}}{\zeta_{j,0}\,\zeta_{i,-}},\quad \frac{\beta_{\bar{z},i}\,\beta_{\bar{x},j}}{\beta_{\bar{z},j}\,\beta_{\bar{x},i}}=\tilde{q}_{i,j}\,\frac{\zeta_{i,-}\,\zeta_{j,+}}{\zeta_{j,-}\,\zeta_{i,+}},\quad \text{and}\quad \frac{\beta_{\bar{x},i}\,\beta_{\bar{y},j}}{\beta_{\bar{x},j}\,\beta_{\bar{y},i}}=\tilde{q}_{i,j}^{-2}\,\frac{\zeta_{i,+}\,\zeta_{j,0}}{\zeta_{j,+}\,\zeta_{i,0}},$$

where $\tilde{q}_{i,j}=\exp(2\pi i\,[B+i\omega]\cdot[\bar{\Delta}_{i,j}])$, and $\zeta_{i,+}$, $\zeta_{i,-}$, $\zeta_{i,0}$ are as in Definition 4.11.

Proof. We claim that $T_{\bar{y},i,0}+T_{\bar{z},j,0}-T_{\bar{y},j,0}-T_{\bar{z},i,0}=\Delta_{i,j}$. Indeed, consider first a situation in which L_{3+i} lies in the position represented in Figure 9, and L_{3+j} lies close to it, but is slightly shifted towards the lower-right direction. Then the intersection points a_j and c_j lie close to a_i and c_i, and following the triangular regions through the small deformation which takes L_{3+i} to L_{3+j}, we easily see that $T_{\bar{z},j,0}$ is obtained by slightly truncating $T_{\bar{z},i,0}$ on its L_{3+i} side. Similarly, $T_{\bar{y},j,0}$ is obtained by slightly truncating $T_{\bar{y},i,0}$, and since $\Delta_{i,j}$ is simply the thin strip in between L_{3+i} and L_{3+j} the claim follows.

The same property remains true if L_{3+i} and L_{3+j} are further apart from each other. This can be checked explicitly for example in the configuration of Figure 8, where $\Delta_{i,j}$ is as pictured on Figure 9 (right). (In this configuration the deformation from L_{3+i} to L_{3+j} passes through $\bar{y}$ and $\bar{z}$, so the triangles $T_{\bar{z},i,0}$ and $T_{\bar{z},j,0}$ lie on opposite sides of $\bar{z}$, and similarly for $T_{\bar{y},i,0}$ and $T_{\bar{y},j,0}$; this latter triangle is now the small region to the lower-right of $\bar{y}$ on Figure 8).

As a consequence, we have the identity

$$\frac{\psi_{\bar{y},i}\psi_{\bar{z},j}}{\psi_{\bar{y},j}\psi_{\bar{z},i}}=\tilde{q}_{i,j},$$

which implies the first formula in the proposition. The two other formulas are proved similarly, using the equalities $T_{\bar{z},i,0}+T_{\bar{x},j,0}-T_{\bar{z},j,0}-T_{\bar{x},i,0}=\Delta_{i,j}$ and $T_{\bar{x},i,0}+T_{\bar{y},j,0}-T_{\bar{x},j,0}-T_{\bar{y},i,0}=-2\,\Delta_{i,j}$. $\square$

Remark 4.15. The various ratios computed in Propositions 4.7–4.14 are *intrinsic* quantities attached to the symplectic geometry of W_k, i.e. they are invariant under Hamiltonian deformations, irrespective of whether the vanishing cycles are represented by geodesics or not. Equivalently, they are invariant under *rescalings* of the chosen generators of the morphism spaces in $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$. On the other hand, if we allow ourselves to use the fact that the vanishing cycles are geodesics in a flat torus, we can also compute some interesting non-intrinsic quantities (i.e., quantities which depend on a particular choice of scaling of the generators).

For example, the invariance of L_0, L_1, L_2 under the translation of the torus which maps x_0 to y_0 (and y_0 to z_0, z_0 to x_0) implies that, for suitable choices of the marked points associated to the spin structures and of the isomorphisms between lines used to calculate boundary holonomies, $\alpha_{xy} = \alpha_{yz} = \alpha_{zx}$, $\alpha_{yx} = \alpha_{xy} = \alpha_{xz}$, and $\alpha_{xx} = \alpha_{yy} = \alpha_{zz}$. In fact, going over the calculations in the proofs of Propositions 4.7 and 4.10, and observing that, in terms of areas and boundary holonomies, the contributions of $T_{xy,0} - T_{yx,0}$ and $T_{xx,0} - T_{yx,0}$ are equivalent to those of $\frac{1}{3}C$ and $\frac{1}{3}(\Sigma_0 - C)$ respectively, one easily checks that there exists a constant $s \neq 0$ such that

$$\begin{aligned} &\alpha_{xy} = \alpha_{yz} = \alpha_{zx} = s\, q_C^{1/3}\, \zeta_+, \\ &\alpha_{xx} = \alpha_{yy} = \alpha_{zz} = s\, q_F^{1/3} q_C^{-1/3}\, \zeta_0, \\ &\alpha_{yx} = \alpha_{zy} = \alpha_{xz} = -s\, \zeta_-, \end{aligned} \tag{4.2}$$

where by definition $q_C^{1/3} = \exp(\frac{2\pi i}{3}[B + i\omega] \cdot [\bar{C}])$ and $q_F^{1/3} = \exp(\frac{2\pi i}{3}\,[B + i\omega] \cdot [\Sigma_0])$. Similarly, for suitable choices we have

$$\beta_{\bar{x},i} = s_i\, q_i^{1/3}\, \zeta_{i,+}, \quad \beta_{\bar{y},i} = s_i\, q_F^{1/3}\, q_i^{-1/3}\, \zeta_{i,0}, \quad \text{and } \beta_{\bar{z},i} = -s_i\, \zeta_{i,-}, \tag{4.3}$$

where s_i is a non-zero constant and $q_i^{1/3} = \exp(\frac{2\pi i}{3}[B + i\omega] \cdot [\bar{C}_i])$.

The formulas (4.2) and (4.3) are only valid in the flat case, when the complexified symplectic form on Σ_0 is translation-invariant and the vanishing cycles are geodesics; however, in the general case we can always modify our choices of generators of the various morphism spaces by suitable scaling factors (or equivalently, modify the vanishing cycles by certain Hamiltonian isotopies) in order to make these formulas hold. It is therefore these simpler formulas that we will use in order to determine the mirror map in §5 below.

4.4. **Simple degenerations.** In this section we consider the situation where the symplectic area of one of the 2-cycles $\bar{\Delta}_{i,j}$ becomes a multiple of that of the fiber Σ_0. The vanishing cycles L_{3+i} and L_{3+j} are then Hamiltonian isotopic to each other in Σ_0, and hence cannot be represented by disjoint geodesics anymore. However we can still represent L_{3+i} by a closed geodesic, and L_{3+j} by a small generic Hamiltonian perturbation of L_{3+i}, intersecting it transversely in two points. These two intersection points have Maslov indices 0 and 1 respectively (if we choose the same graded lifts as previously), and for this configuration we have:

Lemma 4.16. *If there exist integers $n \in \mathbb{Z}$ and $i < j$ such that $[\omega] \cdot [\bar{\Delta}_{i,j}] = n\,[\omega] \cdot [\Sigma_0]$, then* $\mathrm{Hom}(L_{3+i}, L_{3+j})$ *is graded isomorphic to $H^*(S^1) \otimes \mathbb{C}$. Moreover, the differential*

$$m_1 : \mathrm{Hom}^0(L_{3+i}, L_{3+j}) \to \mathrm{Hom}^1(L_{3+i}, L_{3+j})$$

is zero if $[B] \cdot [\bar{\Delta}_{i,j}] \in \mathbb{Z} + n\,[B] \cdot [\Sigma_0]$, and an isomorphism otherwise.

Proof. The only contributions to m_1 come from the two disks D' and D'' bounded by L_{3+i} and L_{3+j}. The 2-chain $D' - D''$ in Σ_0 has symplectic area zero, and is in fact given by $D' - D'' = \Delta_{i,j} - n\Sigma_0$. Hence we can compare the coefficients ψ' and ψ'' associated to these two disks by the same argument as in §4.3. Namely, ψ' and ψ'' differ by a sign factor, a holonomy factor, and an area factor.

In this case the sign factor is -1 (the sign rule for odd degree morphisms is slightly more subtle than that for even degree morphisms [18]; here we can see directly that the signs for D' and D'' have to be different since the untwisted Floer homology of L_{3+i} and L_{3+j} is non-trivial); the holonomy factor is the total holonomy along $\partial(D' - D'') = L_{3+j} - L_{3+i}$, i.e. $\exp(2\pi i \int_{D_{3+i} - D_{3+j}} B)$; and the area factor is $\exp(2\pi i \int_{D' - D''} B + i\omega)$. It follows that

$$\psi' = -\exp\bigl(2\pi i\,[B + i\omega] \cdot ([\bar{\Delta}_{i,j}] - n[\Sigma_0])\bigr)\,\psi'',$$

since $D' - D'' + D_{3+i} - D_{3+j} = \bar{\Delta}_{i,j} - n\Sigma_0$. Since m_1 is determined by the sum $\psi' + \psi''$, we conclude that $m_1 = 0$ if and only if $[B + i\omega] \cdot ([\bar{\Delta}_{i,j}] - n[\Sigma_0])$ is an integer. □

In other words, if $[B + i\omega] \cdot [\bar{\Delta}_{i,j}] \in \mathbb{Z} \oplus ([B + i\omega] \cdot [\Sigma_0])\,\mathbb{Z}$, then (L_{3+i}, ∇_{3+i}) and (L_{3+j}, ∇_{3+j}) are essentially identical, and we have a non-cancelling pair of extra morphisms of degrees 0 and 1 from L_{3+i} to L_{3+j}; this mirrors the situation in which $\mathbb{CP}^2$ is blown up twice at infinitely close points, in which case there is a rational -2-curve and the derived category of coherent sheaves is richer than in the generic case. In all other situations the intersection points between L_{3+i} and L_{3+j}, if any, are killed by the twisted Floer differential (even when L_{3+i} and L_{3+j} are Hamiltonian isotopic).

Remark 4.17. It is important to note that, due to the presence of immersed convex polygonal regions with two edges on $L_0 \cup L_1 \cup L_2$ and two edges on $L_{3+i} \cup L_{3+j}$ (with a corner at the intersection point of Maslov index 1), we have to consider not only the Floer differential m_1, but also the higher-order composition m_3. For example, when L_{3+i} and L_{3+j} are Hamiltonian isotopic the composition

$$m_3 : \mathrm{Hom}(L_0, L_2) \otimes \mathrm{Hom}(L_2, L_{3+i}) \otimes \mathrm{Hom}^1(L_{3+i}, L_{3+j}) \longrightarrow \mathrm{Hom}(L_0, L_{3+j})$$

is in general non-zero (and similarly with L_1 instead of L_0 or L_2).

As in §2.2, it is possible to describe things in a simpler and more unified manner by considering a suitable mutation of the exceptional collection $(L_0, \dots, L_{k+2})$. Assume for simplicity that the two vanishing cycles which may coincide are L_3 and L_4, while the others are represented by distinct geodesics. Then we can modify the system of arcs $\{\gamma_i\}$ considered so far to a new ordered system of arcs $\{\gamma'_i\}$ such that $\gamma'_i = \gamma_i$ for $i \notin \{2, 3\}$, $\gamma'_3 = \gamma_2$, and γ'_2 connects the origin to $\lambda_3 \approx \infty$

along the negative real axis. This gives rise to a new category $\mathrm{Lag_{vc}}(W_k,\{\gamma'_i\})$, in which all objects but one can be identified with the objects L_i, $i\neq 3$ of $\mathrm{Lag_{vc}}(W_k,\{\gamma_i\})$; thus, we denote by $L_0,L_1,L',L_2,L_4,\dots,L_{k+2}$ the objects of $\mathrm{Lag_{vc}}(W_k,\{\gamma'_i\})$. The morphisms and compositions not involving L' are as in $\mathrm{Lag_{vc}}(W_k,\{\gamma_i\})$.

The new vanishing cycle L' is Hamiltonian isotopic to the image of L_3 under the positive Dehn twist along L_2. In particular, with the notations of Lemma 3.1, and for a suitable choice of orientation, its homology class is $[L']=[L_2]-[L_3]=b$. Choosing a geodesic representative, we have $|L_0\cap L'|=2$, $|L_1\cap L'|=1$, $|L'\cap L_2|=1$, and $|L'\cap L_{3+i}|=1$ for $i\geq 1$, and all morphisms in $\mathrm{Lag_{vc}}(W_k,\{\gamma'_i\})$ have degree 0.

Because L' is Hamiltonian isotopic to the image of L_3 under the Dehn twist along L_2, the fiber Σ_0 contains a 2-chain Δ' with $\partial\Delta'=L'+L_4-L_2$ and such that $\int_{\Delta'}\omega=\int_{\Delta_{3,4}}\omega$. Capping off Δ' with the appropriate Lefschetz thimbles, we obtain a 2-cycle $\bar\Delta'$ in M_k, with $[\bar\Delta']=[\bar\Delta_{3,4}]$ in $H_2(M_k,\mathbb{Z})$. The composition

$$\mathrm{Hom}(L',L_2)\otimes\mathrm{Hom}(L_2,L_4)\longrightarrow\mathrm{Hom}(L',L_4)$$

corresponds to an infinite series of triangular immersed regions in Σ_0, of which in general two are embedded. The case where the symplectic area of Δ' is a multiple of that of the fiber corresponds precisely to the situation where the two embedded triangular regions have equal symplectic areas. In general, the immersed triangles contributing to the composition can be labelled T'_n, $n\in\mathbb{Z}$, in such a way that $T'_n=T'_0+n\Delta'+\frac{1}{2}n(n-1)\Sigma_0$. Arguing as before, one easily shows that the composition is given by the contribution of T'_0 multiplied by the factor

$$\sum_{n\in\mathbb{Z}}(-1)^n\,{q'}^{\,n}\,q_F^{n(n-1)/2},\quad\text{where } q'=\exp(2\pi i[B+i\omega]\cdot[\bar\Delta'])=\tilde q_{3,4}.$$

This multiplicative factor vanishes if and only if $q'=q_F^k$ for some $k\in\mathbb{Z}$ (an easy way to see this is to view this factor as a theta function, see below), i.e. iff $[B+i\omega]\cdot[\bar\Delta']\in\mathbb{Z}\oplus([B+i\omega]\cdot[\Sigma_0])\mathbb{Z}$. Hence, as in §2.2 the mutation makes it possible to avoid dealing with a non-trivial differential, and provides an alternative description in which the simple degeneration corresponds to one of the composition maps becoming identically zero.

4.5. **Modular invariance and theta functions.** In this section we study the modularity properties of the category $\mathrm{Lag_{vc}}(W_k,\{\gamma_i\})$ with respect to some of the parameters governing deformations of the complexified symplectic structure, and the relation with theta functions.

Proposition 4.18. *Consider two complexified symplectic forms $\kappa=B+i\omega$ and $\kappa'=B'+i\omega'$ on M_k, such that $[\kappa']\cdot[\Sigma_0]=[\kappa]\cdot[\Sigma_0]$ and $[\kappa']-[\kappa]\in H^2(M_k,\mathbb{Z})\oplus(\kappa\cdot[\Sigma_0])\,H^2(M_k,\mathbb{Z})$. Then the categories $\mathrm{Lag_{vc}}(W_k,\kappa,\{\gamma_i\})$ and $\mathrm{Lag_{vc}}(W_k,\kappa',\{\gamma_i\})$ are equivalent.*

Proof. First consider the situation where $\omega'=\omega$, and $B'=B+d\chi$ for some 1-form χ. Then the vanishing cycles L_i remain the same, but the associated flat connections differ, and we can e.g. take

$\nabla'_i = \nabla_i - 2\pi i\chi$. Then the contribution of a pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_0, \cup L_i)$ is actually the same in both cases, since the holonomy term changes by $\exp(-2\pi i \int_{u(\partial D^2)} \chi)$, while the weight factor changes by $\exp(2\pi i \int_{D^2} u^* d\chi) = \exp(2\pi i \int_{u(\partial D^2)} \chi)$. So, in the more general situation where $[B' - B] \in H^2(M_k, \mathbb{Z})$ and $[B' - B] \cdot [\Sigma_0] = 0$ (still assuming $\omega' = \omega$), after modifying B by an exact term we can assume that B and B' coincide over Σ_0, and that the integral of $B' - B$ over each thimble D_i is a multiple of 2π. In this situation the vanishing cycles L_i are the same, and the associated flat connections are gauge equivalent (since their holonomies differ by multiples of 2π), so the corresponding twisted Floer theories are identical.

Next, consider the situation where $[B + i\omega]$ changes by an integer multiple of $[B + i\omega] \cdot [\Sigma_0]$. After adding an exact term to $\kappa = B + i\omega$ (which does not affect the category of vanishing cycles by Lemma 3.2 and by the above remark), we can assume that κ and κ' coincide over Σ_0, and that the relative cohomology class of $\kappa' - \kappa$ is an element of $(\kappa \cdot [\Sigma_0])H^2(M_k, \Sigma_0; \mathbb{Z})$.

Let D_i and D'_i be the thimbles associated to the arc γ_i and to the symplectic forms ω and ω' respectively. The integrality assumption on $\kappa' - \kappa$ implies that there exists an integer $n_i \in \mathbb{Z}$ such that $\int_{D_i} \kappa' = n_i[\kappa] \cdot [\Sigma_0] + \int_{D_i} \kappa$. Since D_i and D'_i can be deformed continuously into each other (by deforming the horizontal distribution), there exists a 2-chain K_i in Σ_0 such that $[D_i + K_i - D'_i] = 0$ in $H_2(M_k)$. Then $\int_{K_i} \omega = \int_{K_i} \omega' = -\int_{D_i} \omega' = -n_i[\omega] \cdot [\Sigma_0]$. Since the symplectic area of the 2-chain $K_i \subset \Sigma_0$ is an integer multiple of that of the fiber, the two vanishing cycles $L'_i = \partial D'_i$ and $L_i = \partial D_i$ are mutually Hamiltonian isotopic in Σ_0, and hence we can assume that $L'_i = L_i$. Moreover, in $H_2(M_k, L_i)$ we have $[D'_i] = [D_i] - n_i[\Sigma_0]$. Therefore, $\int_{D'_i} B' = \int_{D_i} B' - n_i \int_{\Sigma_0} B' = (\int_{D_i} B + n_i[B] \cdot [\Sigma_0]) - n_i[B] \cdot [\Sigma_0] = \int_{D_i} B$. So the flat connections ∇_i and ∇'_i have the same holonomy, which implies that (L_i, ∇_i) and (L'_i, ∇'_i) behave identically for twisted Floer theory. □

This property explains the invariance of the structure coefficients (α_{xy}, etc.) under certain changes of variables. More precisely, one easily checks that $\zeta_+(q_C q_F^3, q_F) = -q_C^{-1} q_F^{-2} \zeta_+(q_C, q_F)$, $\zeta_-(q_C q_F^3, q_F) = -q_C^{-1} q_F^{-1} \zeta_-(q_C, q_F)$, and $\zeta_0(q_C q_F^3, q_F) = -q_C^{-1} \zeta_0(q_C, q_F)$. This implies that the quantities considered in Propositions 4.7 and 4.10 are invariant under the change of variables $(q_C, q_F) \mapsto (q_C q_F^3, q_F)$; a closer examination shows that the individual constants α_{xy}, etc. are also invariant under this change of variables.

On the other hand, one easily checks that $\zeta_+(q_C q_F, q_F) = -q_C^{-1} \zeta_0(q_C, q_F)$, $\zeta_0(q_C q_F, q_F) = \zeta_-(q_C, q_F)$, and $\zeta_-(q_C q_F, q_F) = \zeta_+(q_C, q_F)$, which may seem surprising at first. The reason is that this change of variables corresponds to a non-Hamiltonian deformation of e.g. L_1 which sweeps exactly once through the entire fiber Σ_0. This deformation preserves the intersection points, but induces a non-trivial permutation of their labels: namely, x_0, y_0, z_0 become y_0, z_0, x_0 respectively, and x_1, y_1, z_1 become z_1, x_1, y_1 respectively. Thus, for example, $\alpha_{xy}(q_C, q_F) = \alpha_{yx}(q_C q_F, q_F) = \alpha_{zz}(q_C q_F^2, q_F)$ (and similarly for the other coefficients).

Another way to understand these invariance properties is to relate the functions ζ_+, ζ_-, and ζ_0 to theta functions. Recall that the ordinary theta function is an analytic function defined by

$$\theta(z,\tau)=\sum_{n\in\mathbb{Z}}\exp(\pi i n^2\tau+2\pi i n z),$$

where $z\in\mathbb{C}$ and $\tau\in\mathcal{H}$ (here $\mathcal{H}$ is the upper half-plane $\{\operatorname{Im}\tau>0\}$). This function is quasiperiodic with respect to the lattice $\Lambda_\tau\subset\mathbb{C}$ generated by 1 and τ, and its behavior under translation by an element of the lattice is given by the formula

$$\theta(z+u\tau+v,\tau)=\exp(-\pi i u^2\tau-2\pi i u z)\theta(z,\tau).$$

The zeros of the theta function are the infinite set $\left\{z=(n+\frac{1}{2})+(m+\frac{1}{2})\tau \mid n,m\in\mathbb{Z}\right\}$.

Here we consider theta functions with rational characteristics $a,b\in\mathbb{Q}$, defined by

$$\theta_{a,b}(z,\tau)=\sum_{n\in\mathbb{Z}}\exp(\pi i(n+a)^2\tau+2\pi i(n+a)(z+b)).$$

Let us introduce new variables $q=\exp(\pi i\tau)$ and $w=\exp(\pi i z)$. Now the following three θ-functions play a very important role in our considerations:

$$\begin{aligned}
\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau) &= \exp(\tfrac{i\pi}{2})\,q^{3/4}\sum_{n\in\mathbb{Z}}(-1)^n w^{6n+3}q^{3n^2+3n},\\
\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau) &= \exp(\tfrac{i\pi}{6})\,q^{1/12}\sum_{n\in\mathbb{Z}}(-1)^n w^{6n+1}q^{3n^2+n},\\
\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau) &= \exp(-\tfrac{i\pi}{6})\,q^{1/12}\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-1}q^{3n^2-n}.
\end{aligned}$$

The zero set of the function $\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau)$ is $\left\{\frac{n}{3}+m\tau \mid n,m\in\mathbb{Z}\right\}$, while the zero sets of the functions $\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau)$ and $\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)$ are

$$\left\{\tfrac{n}{3}+(m+\tfrac{1}{3})\tau \mid n,m\in\mathbb{Z}\right\}\quad\text{and}\quad\left\{\tfrac{n}{3}+(m-\tfrac{1}{3})\tau \mid n,m\in\mathbb{Z}\right\}$$

respectively. These three theta functions can be viewed as holomorphic sections of a line bundle of degree 3 on the elliptic curve $E=\mathbb{C}/\Lambda_\tau$; considering the zero sets, we see that this line bundle is $\mathbb{L}=\mathcal{O}_E(3\cdot(0))$. These three sections of $\mathbb{L}$ determine an embedding of the elliptic curve $E=\mathbb{C}/\Lambda_\tau$ into the projective plane, given by

$$z\mapsto(\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau):\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau):\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)).$$

Observe that the two functions

$$\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau)\,\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau)\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)\quad\text{and}\quad\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau)^3+\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau)^3+\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)^3$$

coincide up to a constant multiplicative factor, since they both correspond to holomorphic sections of the line bundle $\mathbb{L}^{\otimes 3}$ over E, and an easy calculation shows that they have the same zero set $\{\frac{n}{3}+\frac{m}{3}\tau\,|\,n,m\in\mathbb{Z}\}$. Therefore, the image of the above embedding of E into $\mathbb{P}^2$ is the cubic given by the equation

$$(A^3+B^3+C^3)XYZ-ABC(X^3+Y^3+Z^3)=0,$$

where (A,B,C) are the values of the three theta functions at any given point of $\mathbb{C}/\Lambda_\tau$ (not in $\frac{1}{3}\Lambda_\tau$).

Consider the function

$$\left(\frac{\theta_{\frac{1}{6},\frac{1}{2}}(3z,3\tau)}{\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)}\right)^3 = -\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n\, w^{6n+1}\, q^{n(3n+1)}}{\sum_{n\in\mathbb{Z}}(-1)^n\, w^{6n-1}\, q^{n(3n-1)}}\right)^3.$$

Substituting $q^2=q_F$ and $w^6=q_C$, one easily checks that this coincides with the expression which appears in Proposition 4.7,

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -q_C\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n\, q_C^n\, q_F^{n(3n+1)/2}}{\sum_{n\in\mathbb{Z}}(-1)^n\, q_C^n\, q_F^{n(3n-1)/2}}\right)^3.$$

Similarly,

$$\left(\frac{\theta_{\frac{1}{2},\frac{1}{2}}(3z,3\tau)}{\theta_{\frac{5}{6},\frac{1}{2}}(3z,3\tau)}\right)^3 = q^2\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n+3}q^{3n^2+3n}}{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-1}q^{3n^2-n}}\right)^3 = -q^2\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-3}q^{3n^2-3n}}{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-1}q^{3n^2-n}}\right)^3.$$

After the same substitution $q^2=q_F$ and $w^6=q_C$, this coincides with the expression given in Proposition 4.10,

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -\frac{q_F}{q_C}\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n\, q_C^n\, q_F^{3n(n-1)/2}}{\sum_{n\in\mathbb{Z}}(-1)^n\, q_C^n\, q_F^{n(3n-1)/2}}\right)^3.$$

Similarly, in the case where (4.2) holds, one easily checks that

$$\begin{aligned} \alpha_{xy}=\alpha_{yz}=\alpha_{zx} &= \tilde{s}\, e^{-2i\pi/3}\,\theta_{\frac{1}{6},\frac{1}{2}}(3z_0,3\tau),\\ \alpha_{xx}=\alpha_{yy}=\alpha_{zz} &= \tilde{s}\,\theta_{\frac{1}{2},\frac{1}{2}}(3z_0,3\tau),\\ \alpha_{yx}=\alpha_{zy}=\alpha_{xz} &= \tilde{s}\, e^{2i\pi/3}\,\theta_{\frac{5}{6},\frac{1}{2}}(3z_0,3\tau), \end{aligned} \tag{4.4}$$

where $\tau=[B+i\omega]\cdot[\Sigma_0]$, $z_0=\frac{1}{3}[B+i\omega]\cdot[\bar{C}]$, and $\tilde{s}=e^{i\pi/2}\,q_F^{-1/24}\,q_C^{1/6}\,s\neq 0$. Similar interpretations can be made for the quantities considered in Propositions 4.12–4.14 and in (4.3).

5. Proof of the main theorems

The derived categories considered in §2 depend on an elliptic curve E, two degree 3 line bundles $\mathcal{L}_1,\mathcal{L}_2$ over E, and k points $p_1,\dots,p_k$ on E. Meanwhile, the categories considered in §4 depend on a cohomology class $[B+i\omega]\in H^2(M_k,\mathbb{C})$. We now show how to relate these two sets of parameters.

Fix the cohomology class $[B+i\omega]\in H^2(M_k,\mathbb{C})$, and consider the category $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ studied in §4. With the notations of §3.4, assume that $[\omega]\cdot[\bar{\Delta}_{i,j}]$ is not an integer multiple of $[\omega]\cdot[\Sigma_0]$ for any $i,j\in\{0,\dots,k-1\}$. Then $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ admits a full strong exceptional collection $(L_0,\dots,L_{k+2})$, whose properties have been studied in §4. In particular, the objects and morphisms in this exceptional collection are the same as for the exceptional collection $\sigma=(\mathcal{O}_{X_K},\pi^*\mathcal{T}_{\mathbb{P}^2}(-1),\pi^*\mathcal{O}_{\mathbb{P}^2}(1),\mathcal{O}_{l_1},\dots,\mathcal{O}_{l_k})$ considered in §2 for the derived category of coherent sheaves on a (possibly noncommutative) Del Pezzo surface. Hence, our goal is now to compare the composition laws and show that, for a suitable choice of the parameters $(E,\mathcal{L}_1,\mathcal{L}_2,K)$, the algebra of homomorphisms of the exceptional collection $(L_0,\dots,L_{k+2})$ is isomorphic to the algebra $B_{K,\mu}$ considered in §2. More precisely, we claim:

Proposition 5.1. *Let E be the elliptic curve $\mathbb{C}/\Lambda_\tau$, where $\tau = [B+i\omega]\cdot[\Sigma_0]$, realized as a plane cubic via the embedding $j : E \to \mathbb{P}^2$ given by $z \mapsto (\vartheta_+(z) : \vartheta_0(z) : \vartheta_-(z))$, where*

$$\vartheta_+(z) = e^{-2i\pi/3}\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau), \quad \vartheta_0(z) = \theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau), \quad \textit{and } \vartheta_-(z) = e^{2i\pi/3}\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau).$$

Let $z_0 = \frac{1}{3}[B+i\omega]\cdot[\bar{C}]$, and for $i \in \{0,\dots,k-1\}$ let $p_i = \frac{1}{3}[B+i\omega]\cdot[\bar{C}_i]$. Finally, let $\mathcal{L}_1 = \mathcal{O}_E(3\cdot(-z_0))$ and $\mathcal{L}_2 = \mathcal{O}_E(3\cdot(0))$. Then the algebra of homomorphisms of the exceptional collection $(L_0,\dots,L_{k+2})$ is isomorphic to $B_{K,\mu}$, where μ is determined by $(E,\mathcal{L}_1,\mathcal{L}_2)$ via Construction 2.9 and $K = \{j(z_0+p_0),\dots,j(z_0+p_{k-1})\}$.

Proof. After a suitable rescaling of the chosen bases of the morphism spaces (or just by deforming to the situation where the fiber is flat and the vanishing cycles are geodesics), we can assume that the compositions of morphisms between the objects $L_0,\dots,L_{k+2}$ are given by the formulas (4.2) and (4.3). We identify the vector spaces $U = \mathrm{Hom}(L_0,L_1)$, $V = \mathrm{Hom}(L_1,L_2)$, and $W = \mathrm{Hom}(L_0,L_2)$ with $\mathbb{C}^3$ by considering the bases (x_0,y_0,z_0), (x_1,y_1,z_1), and $(\bar{x},\bar{y},\bar{z})$. The composition tensor $\mu : V\otimes U \to W$ is determined by the three constants $a = \alpha_{xy} = \alpha_{yz} = \alpha_{zx}$, $b = \alpha_{xx} = \alpha_{yy} = \alpha_{zz}$, and $c = \alpha_{yx} = \alpha_{zy} = \alpha_{xz}$. In particular, given an element $v = (X,Y,Z) \in V$, the composition map $\mu_v = \mu(v,\cdot) : U \to W$ is given by the matrix

$$\begin{pmatrix} \alpha_{xx}X & \alpha_{yz}Z & \alpha_{zy}Y \\ \alpha_{xz}Z & \alpha_{yy}Y & \alpha_{zx}X \\ \alpha_{xy}Y & \alpha_{yx}X & \alpha_{zz}Z \end{pmatrix} = \begin{pmatrix} bX & aZ & cY \\ cZ & bY & aX \\ aY & cX & bZ \end{pmatrix} \tag{5.1}$$

which has rank 2 precisely when

$$\det(\mu_v) = (a^3+b^3+c^3)\,XYZ - abc(X^3+Y^3+Z^3) = 0. \tag{5.2}$$

By (4.4), the constants a,b,c are (up to a non-zero constant factor) the values of the theta functions $\vartheta_+, \vartheta_0, \vartheta_-$ at the point z_0. Therefore, by the discussion in §4.5, there are two possibilities:

(1) if $z_0 \in \frac{1}{3}\Lambda_\tau$, then $abc = 0$ and μ_v always has rank 2; as explained in §2.3 this corresponds to a commutative situation;

(2) if $z_0 \notin \frac{1}{3}\Lambda_\tau$, then (5.2) defines a cubic $\Gamma_V \subset \mathbb{P}(V) = \mathbb{P}^2$, and this cubic is precisely the image of the embedding j.

The same situation holds for μ_u; interestingly, under the chosen identifications of $\mathbb{P}(U)$ and $\mathbb{P}(V)$ with $\mathbb{P}^2$, the two subschemes $\Gamma_U \subset \mathbb{P}(U)$ and $\Gamma_V \subset \mathbb{P}(V)$ determined by the equations $\det(\mu_u) = 0$ and $\det(\mu_v) = 0$ coincide exactly. However, with this description, the isomorphism $\sigma : \Gamma_V \to \Gamma_U$ which takes v to the point of Γ_U corresponding to $\mathrm{Ker}\,\mu_v$ is *not* the identity map. Here the reader is referred to the discussion on pp. 37–38 of [2], which we follow loosely.

Given a point $v = (X:Y:Z) \in \Gamma_V$, the kernel of μ_v can be obtained as the cross-product of any two of the rows of the matrix (5.1). Taking e.g. the first two rows, we obtain that the corresponding point of Γ_U is

$$\sigma(X:Y:Z) = (a^2XZ - bcY^2 : c^2YZ - abX^2 : b^2XY - acZ^2). \tag{5.3}$$

Observe that j maps the origin to $(1:0:-1) \in \Gamma_V$, and that the corresponding point in Γ_U is $\sigma(1:0:-1) = (a:b:c) = j(z_0)$. Hence, considering only the situation where $\Gamma_U \simeq \Gamma_V \simeq E$, and identifying E with Γ_V by means of the embedding j, the identification of E with Γ_U is given by the embedding $\sigma \circ j$, which is the composition of j with the translation by z_0. Therefore, the line bundles on E induced by the two inclusions of E into $\mathbb{P}(U)$ and $\mathbb{P}(V)$ are respectively $(\sigma \circ j)^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_E(3 \cdot (-z_0)) = \mathcal{L}_1$ and $j^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_E(3 \cdot (0)) = \mathcal{L}_2$. It then follows from the discussion in §2.3 that the composition tensor μ corresponds to the data $(E, \mathcal{L}_1, \mathcal{L}_2)$. This remains true even when $z_0 \in \frac{1}{3}\Lambda_\tau$, since in that case we have $\mathcal{L}_1 \simeq \mathcal{L}_2$ and the composition tensor associated to the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ is that of the usual projective plane (see Remark 2.10).

Next we consider the composition $\mathrm{Hom}(L_2, L_{3+i}) \otimes W \longrightarrow \mathrm{Hom}(L_0, L_{3+i})$. Choosing generators of the lines $\mathrm{Hom}(L_2, L_{3+i})$ and $\mathrm{Hom}(L_0, L_{3+i})$ we can view this map as a linear form on W. In the given basis of W, this linear form is given by $\big(\beta_{\bar{x},i}\,,\, \beta_{\bar{y},i}\,,\, \beta_{\bar{z},i}\big)$, which by (4.3) coincides up to a non-zero constant factor with

$$\big(\vartheta_+(p_i)\,,\, \vartheta_0(p_i)\,,\, \vartheta_-(p_i)\big).$$

On the other hand we know from §2.3 that the kernel of this linear form should be exactly $\mathrm{Im}\,\mu_{v_i}$, where $v_i \in \Gamma_V$ is the point being blown up.

For any $v = (X:Y:Z) \in \Gamma_V$, the projection $W \to W/\mathrm{Im}\,\mu_v$ is a linear form given up to a scaling factor by the dot product of any two columns of the matrix (5.1). Taking e.g. the first two columns, we obtain that the expression of this linear form relatively to our chosen basis of W is

$$\big(c^2XZ - abY^2\,,\, a^2YZ - bcX^2\,,\, b^2XY - acZ^2\big).$$

Interestingly, if we assume that $(X:Y:Z) = \sigma(\tilde{X}:\tilde{Y}:\tilde{Z})$, where σ is the transformation given by (5.3), then this expression simplifies to a scalar multiple of $(\tilde{X}\,,\,\tilde{Y}\,,\,\tilde{Z})$. Hence, we conclude that $v_i = \sigma(j(p_i)) = j(z_0 + p_i)$. □

Remark 5.2. At this point the reader may legitimately be concerned that, since the homology classes $[\bar{C}]$ and $[\bar{C}_i]$ are canonically defined only up to a multiple of $[\Sigma_0]$, and since $[B]$ is only defined up to an element of $H^2(M_k, \mathbb{Z})$, the points z_0 and p_i of E are canonically determined only up to translations by elements of $\frac{1}{3}\Lambda_\tau$. However, the line bundle $\mathcal{L}_1 = \mathcal{O}_E(3 \cdot (-z_0))$ is not affected by this ambiguity in the determination of z_0, and neither are the relative positions of the points p_i, since the quantity $p_j - p_i = [B + i\omega] \cdot [\bar{\Delta}_{i,j}]$ is well-defined up to an element of Λ_τ. Moreover, a simultaneous translation of all the blown up points by an element of $\frac{1}{3}\Lambda_\tau$ amounts to an automorphism of the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$, which does not actually affect the category. (From the point of view of the embedding j, this automorphism simply permutes the homogeneous coordinates X, Y, Z and multiplies them by cubic roots of unity; this is consistent with the observation made after the proof of Proposition 4.18).

Theorems 1.4 and 1.6 now follow directly from the discussion. Namely, in the case of a blowup of $\mathbb{CP}^2$ at a set $K = \{p_0, \ldots, p_{k-1}\}$ of k distinct points (Theorem 1.4), we consider a cubic curve

$E \subset \mathbb{CP}^2$ which contains all the points of K, and view it as an elliptic curve $\mathbb{C}/\Lambda_\tau$ for some $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. This allows us to view the points p_i as elements of $\mathbb{C}/\Lambda_\tau$ (well-defined up to a simultaneous translation of all p_i by an element of $\frac{1}{3}\Lambda_\tau$, since the origin can be chosen at any of the flexes of E; however by Remark 5.2 this does not matter for our construction). Then we equip M_k with a complexified symplectic structure such that $[B+i\omega]\cdot[\Sigma_0] = \tau$, $[B+i\omega]\cdot[\bar{C}] = 0$, and $[B+i\omega]\cdot[\bar{C}_i] = 3p_i$. The existence of such a $B+i\omega$ follows from a standard result about symplectic structures on Lefschetz fibrations:

Proposition 5.3 (Gompf)**.** *Given any cohomology class $[\zeta] \in H^2(M_k,\mathbb{R})$ such that $[\zeta]\cdot[\Sigma_0] > 0$, the manifold M_k admits a symplectic structure in the cohomology class $[\zeta]$, for which the fibers of W_k are symplectic submanifolds.*

Proof. The map $W_k : M_k \to \mathbb{C}$ is a Lefschetz fibration, and the argument given in the proof of [7, Theorem 10.2.18] can be adapted in a straightforward manner to this situation, even though the base of the fibration is not compact. (Alternatively, one can also work with the compactified fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$). The symplectic form ω constructed by this argument lies in the cohomology class $t[\zeta] + W_k^*([\mathrm{vol}_\mathbb{C}])$ for some constant $t > 0$; since the area form on $\mathbb{C}$ is exact, we have $[\omega] = t[\zeta]$, and scaling ω by a constant factor we obtain the desired result. □

By Proposition 5.1 the algebra of homomorphisms of the exceptional collection $(L_0,\dots,L_{k+2})$ is then isomorphic to B_K, which implies that $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k)) \cong \mathbf{D}^b(\text{mod–}B_K) \cong \mathbf{D}^b(\mathrm{coh}(X_K))$.

In the case of a noncommutative blowup of $\mathbb{P}^2$ (Theorem 1.6), consider the triple $(E,\mathcal{L}_1,\mathcal{L}_2)$ associated to the underlying noncommutative $\mathbb{P}^2$, and view again E as a quotient $\mathbb{C}/\Lambda_\tau$. Choose z_0 (well-defined up to an element of $\frac{1}{3}\Lambda_\tau$) such that $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \simeq \mathcal{O}_E(3\cdot(z_0) - 3\cdot(0)) \in \mathrm{Pic}^0(E)$. As explained in §2.3, the blown up points must all lie in $\Gamma_V \subset \mathbb{P}(V)$, and under the identification $\Gamma_V \simeq E$ they can be viewed as elements $p_i \in \mathbb{C}/\Lambda_\tau$. Equip M_k with a complexified symplectic structure such that $[B+i\omega]\cdot[\Sigma_0] = \tau$, $[B+i\omega]\cdot[\bar{C}] = 3z_0$, and $[B+i\omega]\cdot[\bar{C}_i] = 3(p_i - z_0)$. By Proposition 5.1 the algebra of homomorphisms of the exceptional collection $(L_0,\dots,L_{k+2})$ is then isomorphic to $B_{K,\mu}$, which yields the desired equivalence of categories.

Theorem 1.5 is proved similarly, working with the mutated exceptional collections τ' (introduced in §2.2) and $(L_0, L_1, L', L_2, L_4, \dots, L_{k+2})$ (introduced in §4.4). The details are left to the reader.

Remark 5.4. The construction carried out for Theorem 1.4 also applies to some limit situations in which X_K is actually not a Del Pezzo surface. For example, the argument applies equally well to the situation where $\mathbb{CP}^2$ is blown up at nine points which lie at the intersection of two elliptic curves. In this case the mirror is an elliptic fibration over $\mathbb{C}$ for which the compactification has a smooth fiber at infinity. Compared to that of $\mathbb{CP}^2$ ($k=0$), this extreme case where $k=9$ lies at the opposite end of the spectrum that we consider.

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