EQUIVALENCES OF DERIVED CATEGORIES AND K3 SURFACES

DMITRI ORLOV

Abstract. We consider derived categories of coherent sheaves on smooth projective varieties. We prove that any equivalence between them can be represented by an object on the product. Using this, we give a necessary and sufficient condition for equivalence of derived categories of two K3 surfaces.

§0. Introduction

Let $D^b(X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X . The category $D^b(X)$ has the structure of a triangulated category (see [V], [GM]). We shall consider $D^b(X)$ as a triangulated category.

In this paper we are concerned with the problem of description for varieties, which have equivalent derived categories of coherent sheaves.

In the paper [Mu1], Mukai showed that for an abelian variety A and its dual $\hat{A}$ the derived categories $D^b(A)$ and $D^b(\hat{A})$ are equivalent . Equivalences of another type appeared in [BO1]. They are induced by certain birational transformations which are called flops.

Further, it was proved in the paper [BO2] that if X is a smooth projective variety with either ample canonical or ample anticanonical sheaf, then any other algebraic variety X' such that $D^b(X') \simeq D^b(X)$ is biregularly isomorphic to X .

The aim of this paper is to give some description for equivalences between derived categories of coherent sheaves. The main result is Theorem 2.2. of §2 . It says that any full and faithful exact functor $F : D^b(M) \longrightarrow D^b(X)$ having left (or right) adjoint functor can be represented by an object $E \in D^b(M \times X)$, i.e. $F(\bullet) \cong R^{\bullet}\pi_*(E \overset{L}{\otimes} p^*(\bullet))$, where π and p are the projections on M and X respectively.

In §3 , basing on the Mukai's results [Mu2], we show that two K3 surfaces S_1 and S_2 over field $\mathbb{C}$ have equivalent derived categories of coherent sheaves iff the lattices of transcendental cycles T_{S_1} and T_{S_2} are Hodge isometric.

I would like to thank A. Polishchuk for useful notices.

§1. Preliminaries

1.1. We collect here some facts relating to triangulated categories. Recall that a triangulated category is an additive category with additional structures:

a) *an additive autoequivalence* $T : \mathcal{D} \longrightarrow \mathcal{D}$ *, which is called a translation functor* (we usually write $X[n]$ instead of $T^n(X)$ and $f[n]$ instead of $T^n(f)$),

b) *a class of distinguished triangles:*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

And these structures must satisfy the usual set of axioms (see [V]).

If X , Y are objects of a triangulated category $\mathcal{D}$, then $\mathrm{Hom}^i_{\mathcal{D}}(X\,,\,Y)$ means $\mathrm{Hom}_{\mathcal{D}}(X\,,\,Y[i])$.

An additive functor $F:\mathcal{D}\longrightarrow\mathcal{D}'$ between two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is called exact if

a) *it commutes with the translation functor, i.e there is fixed an isomorphism of functors:*

$$t_F : F\circ T \xrightarrow{\sim} T'\circ F,$$

b) *it takes every distinguished triangle to a distinguished triangle* (using the isomorphism t_F , we replace $F(X[1])$ by $F(X)[1]$).

The following lemma will be needed for the sequel.

1.2. Lemma [BK] *If a functor* $G:\mathcal{D}'\longrightarrow\mathcal{D}$ *is a left (or right) adjoint to an exact functor* $F:\mathcal{D}\longrightarrow\mathcal{D}'$ *then functor* G *is also exact .*

Proof. Since G is the left adjoint functor to F , there exist canonical morphisms of functors $id_{\mathcal{D}'}\to F\circ G,\; G\circ F\longrightarrow id_{\mathcal{D}}$. Let us consider the following sequence of natural morphisms:

$$G\circ T'\longrightarrow G\circ T'\circ F\circ G\xrightarrow{\sim} G\circ F\circ T\circ G\longrightarrow T\circ G$$

We obtain the natural morphism $G\circ T'\longrightarrow T\circ G$. This morphism is an isomorphism. Indeed, for any two objects $A\in\mathcal{D}$ and $B\in\mathcal{D}'$ we have isomorphisms :

$$\mathrm{Hom}(G(B[1])\,,\,A)\cong\mathrm{Hom}(B[1]\,,\,F(A))\cong\mathrm{Hom}(B\,,\,F(A)[-1])\cong$$

$$\mathrm{Hom}(B\,,\,F(A[-1]))\cong\mathrm{Hom}(G(B)\,,\,A[-1])\cong\mathrm{Hom}(G(B)[1]\,,\,A)$$

This implies that the natural morphism $G\circ T'\longrightarrow T\circ G$ is an isomorphism.

Let now $A\xrightarrow{\alpha}B\longrightarrow C\longrightarrow A[1]$ be a distinguished triangle in $\mathcal{D}'$. We have to show that G takes this triangle to a distinguished one.

Let us include the morphism $G(\alpha):G(A)\to G(B)$ into a distinguished triangle:

$$G(A)\longrightarrow G(B)\longrightarrow Z\longrightarrow G(A)[1].$$

Applying functor F to it, we obtain a distinguished triangle:

$$FG(A)\longrightarrow FG(B)\longrightarrow F(Z)\longrightarrow FG(A)[1]$$

(we use the commutation isomorphisms like $T'\circ F\xrightarrow{\sim}F\circ T$ with no mention).

Using morphism $id\to F\circ G$, we get a commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ FG(A) & \xrightarrow{FG(\alpha)} & FG(B) & \longrightarrow & F(Z) & \longrightarrow & FG(A)[1] \end{array}$$

By axioms of triangulated categories there exists a morphism $\mu : C \to F(Z)$ that completes this commutative diagram. Since G is left adjoint to F , the morphism μ defines $\nu : G(C) \to Z$. It is clear that ν makes the following diagram commutative:

$$\begin{array}{ccccccc} G(A) & \longrightarrow & G(B) & \longrightarrow & G(C) & \longrightarrow & G(A)[1] \\ \wr\downarrow & & \wr\downarrow & & \downarrow\nu & & \wr\downarrow \\ G(A) & \longrightarrow & G(B) & \longrightarrow & Z & \longrightarrow & G(A)[1] \end{array}$$

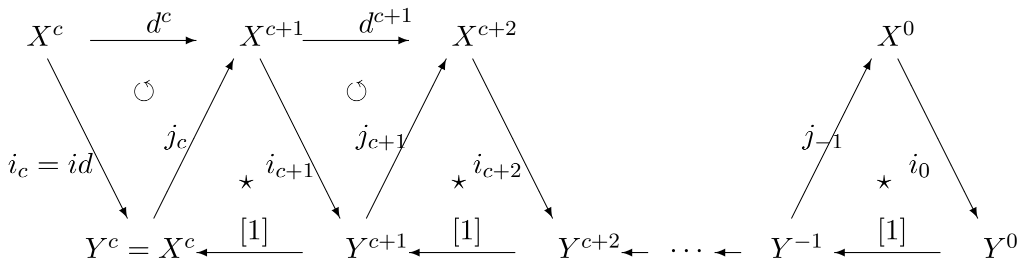
To prove the lemma, it suffices to show that ν is an isomorphism. For any object $Y \in \mathcal{D}$ let us consider the diagram for Hom :

$$\begin{array}{ccccc} \to \mathrm{Hom}(G(A)[1]\,,\,Y) & \to & \mathrm{Hom}(Z\,,\,Y) & \to & \mathrm{Hom}(G(B)\,,\,Y) \to \\ \downarrow\wr & & \downarrow \mathrm{H_Y}(\nu) & & \downarrow\wr \\ \to \mathrm{Hom}(G(A)[1]\,,\,Y) & \to & \mathrm{Hom}(G(C)\,,\,Y) & \to & \mathrm{Hom}(G(B)\,,\,Y) \to \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \to \mathrm{Hom}(A[1]\,,\,F(Y)) & \to & \mathrm{Hom}(C\,,\,F(Y)) & \to & \mathrm{Hom}(B\,,\,F(Y)) \to \end{array}$$

Since the lower sequence is exact, the middle sequence is exact also. By the lemma about five homomorphisms, for any Y the morphism $\mathrm{H}(\nu)$ is an isomorphism . Thus $\nu : G(C) \to Z$ is an isomorphism too. This concludes the proof. $\square$

1.3. Let $X^\bullet = \{X^c \stackrel{d^c}{\to} X^{c+1} \stackrel{d^{c+1}}{\to} \cdots \to X^0\}$ be a bounded complex over a triangulated category $\mathcal{D}$, i.e. all compositions $d^{i+1} \circ d^i$ are equal to 0 ($c < 0$).

A left Postnikov system, attached to $X^\bullet$, is, by definition, a diagram



in which all triangles marked with $\star$ are distinguished and triangles marked with $\circlearrowleft$ are commutative (i.e. $j_k \circ i_k = d^k$). An object $E \in \mathrm{Ob}\mathcal{D}$ is called a left convolution of $X^\bullet$, if there exists a left Postnikov system, attached to $X^\bullet$ such that $E = Y^0$. The class of all convolutions of $X^\bullet$ will be denoted by $\mathrm{Tot}(X^\bullet)$. Clearly the Postnikov systems and convolutions are stable under exact functors between triangulated categories.

The class $\mathrm{Tot}(X^\bullet)$ may contain many non-isomorphic elements and may be empty. Further we shall give a sufficient condition for $\mathrm{Tot}(X^\bullet)$ to be non-empty and for its objects to be isomorphic. The following lemma is needed for the sequel(see [BBD]).

1.4. Lemma *Let* g *be a morphism between two objects* Y *and* Y' *, which are included into two distinguished triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

If $v'gu = 0$ *, then there exist morphisms* $f : X \to X'$ *and* $h : Z \to Z'$ *such that the triple* (f, g, h) *is a morphism of triangles.*

If, in addition, $\mathrm{Hom}(X[1]\,,\,Z') = 0$ *then this triple is uniquely determined by* g *.*

Now we prove two lemmas which are generalizations of the previous one for Postnikov diagrams.

1.5. Lemma *Let* $X^\bullet = \{X^c \xrightarrow{d^c} X^{c+1} \xrightarrow{d^{c+1}} \cdots \to X^0\}$ *be a bounded complex over a triangulated category* $\mathcal{D}$ *. Suppose it satisfies the following condition:*

$$\mathrm{Hom}^i(X^a\,,\,X^b) = 0 \;\; \text{for} \;\; i < 0 \;\; \text{and} \;\; a < b. \tag{1}$$

Then there exists a convolution for this complex and all convolutions are isomorphic (non-canonically).

If, in addition,

$$\mathrm{Hom}^i(X^a\,,\,Y^0) = 0 \;\; \text{for} \;\; i < 0 \;\; \text{and for all} \;\; a \tag{2}$$

for some convolution Y^0 *(and, consequently, for any one), then all convolutions are canonically isomorphic.*

1.6. Lemma *Let* $X_1^\bullet$ *and* $X_2^\bullet$ *be bounded complexes that satisfy (1), and let* $(f_c, ..., f_0)$ *be a morphism of these complexes:*

$$\begin{array}{ccccccc} X_1^c & \xrightarrow{d_1^c} & X_1^{c+1} & \longrightarrow & \cdots & \longrightarrow & X_1^0 \\ \downarrow f_c & & \downarrow f_{c+1} & & & & \downarrow f_0 \\ X_2^c & \xrightarrow{d_2^c} & X_2^{c+1} & \longrightarrow & \cdots & \longrightarrow & X_2^0 \end{array}$$

Suppose that

$$\mathrm{Hom}^i(X_1^a\,,\,X_2^b) = 0 \;\; \text{for} \;\; i < 0 \;\; \text{and} \;\; a < b. \tag{3}$$

Then for any convolution Y_1^0 *of* $X_1^\bullet$ *and for any convolution* Y_2^0 *of* $X_2^\bullet$ *there exists a morphism* $f : Y_1^0 \to Y_2^0$ *that commutes with the morphism* f_0 *. If, in addition,*

$$\mathrm{Hom}^i(X_1^a\,,\,Y_2^0) = 0 \;\; \text{for} \;\; i < 0 \;\; \text{and for all} \;\; a \tag{4}$$

then this morphism is unique.

Proof. We shall prove both lemmas together. Let Y^{c+1} be a cone of the morphism d^c :

$$X^c \xrightarrow{d^c} X^{c+1} \xrightarrow{\alpha} Y^{c+1} \longrightarrow X^c[1]$$

By assumption $d^{c+1} \circ d^c = 0$ and $\mathrm{Hom}(X^c[1]\,,\,X^{c+2}) = 0$, hence there exists a unique morphism $\bar{d}^{c+1} : Y^{c+1} \to X^{c+2}$ such that $\bar{d}^{c+1} \circ \alpha = d^{c+1}$.

Let us consider a composition $d^{c+2}\circ\bar{d}^{c+1}: Y^{c+1}\to X^{c+3}$. We know that $d^{c+2}\circ\bar{d}^{c+1}\circ\alpha = d^{c+2}\circ d^{c+1}=0$, and at the same time we have $\mathrm{Hom}(X^c[1]\,,\,X^{c+3})=0$. This implies that the composition $d^{c+2}\circ\bar{d}^{c+1}$ is equal to 0 .

Moreover, consider the distinguished triangle for Y^{c+1} . It can easily be checked that $\mathrm{Hom}^i(Y^{c+1}\,,\,X^b)=0$ for $i<0$ and $b>c+1$. Hence the complex $Y^{c+1}\longrightarrow X^{c+2}\longrightarrow\cdots\longrightarrow X^0$ satisfies the condition (1). By induction, we can suppose that it has a convolution. This implies that the complex $X^\bullet$ has a convolution too. Thus, the class $\mathrm{Tot}(X^\bullet)$ is non-empty.

Now we shall show that under the conditions (3) any morphism of complexes can be extended to a morphism of Postnikov systems.

Let us consider cones Y_1^{c+1} and Y_2^{c+1} of the morphisms d_1^c and d_2^c . There exists a morphism $g_{c+1}: Y_1^{c+1}\to Y_2^{c+1}$ such that one has the morphism of distinguished triangles:

$$\begin{array}{ccccccc} X_1^c & \xrightarrow{d_1^c} & X_1^{c+1} & \xrightarrow{\alpha} & Y_1^{c+1} & \longrightarrow & X_1^c[1] \\ \downarrow{\scriptstyle f_c} & & \downarrow{\scriptstyle f_{c+1}} & & \downarrow{\scriptstyle g_{c+1}} & & \downarrow{\scriptstyle f_c[1]} \\ X_2^c & \xrightarrow{d_2^c} & X_2^{c+1} & \xrightarrow{\beta} & Y_2^{c+1} & \longrightarrow & X_2^c[1] \end{array}$$

As above, there exist uniquely determined morphisms $\bar{d}_i^{c+1}: Y_i^{c+1}\to X_i^{c+2}$ for $i=1,2$. Consider the following diagram:

$$\begin{array}{ccc} Y_1^{c+1} & \xrightarrow{\bar{d}_1^{c+1}} & X_1^{c+2} \\ \downarrow{\scriptstyle g_{c+1}} & & \downarrow{\scriptstyle f_{c+2}} \\ Y_2^{c+1} & \xrightarrow{\bar{d}_2^{c+1}} & X_2^{c+2} \end{array}$$

Let us show that this square is commutative. Denote by h the difference $f_{c+2}\circ\bar{d}_1^{c+1}-\bar{d}_2^{c+1}\circ g_{c+1}$. We have $h\circ\alpha = f_{c+2}\circ d_1^{c+1}-d_2^{c+1}\circ f_{c+1}=0$ and, by assumption, $\mathrm{Hom}(X_1^c[1]\,,\,X_2^{c+2})=0$. It follows that $h=0$. Therefore, we obtain the morphism of new complexes:

$$\begin{array}{ccccccc} Y_1^{c+1} & \xrightarrow{\bar{d}_1^{c+1}} & X_1^{c+2} & \longrightarrow & \cdots & \longrightarrow & X_1^0 \\ \downarrow{\scriptstyle g_{c+1}} & & \downarrow{\scriptstyle f_{c+2}} & & & & \downarrow{\scriptstyle f_0} \\ Y_2^{c+1} & \xrightarrow{\bar{d}_2^{c+1}} & X_2^{c+2} & \longrightarrow & \cdots & \longrightarrow & X_2^0 \end{array}$$

It can easily be checked that these complexes satisfy the conditions (1) and (3) of the lemmas. By the induction hypothesis, this morphism can be extended to a morphism of Postnikov systems, attached to these complexes. Hence there exists a morphism of Postnikov systems, attached to $X_1^\bullet$ and $X_2^\bullet$.

Moreover, we see that if all morphisms f_i are isomorphisms, then a morphism of Postnikov systems is an isomorphism too. Therefore, under the condition (1) all objects from the class $\mathrm{Tot}(X^\bullet)$ are isomorphic.

Now let us consider a morphism of the rightmost distinguished triangles of Postnikov systems:

$$\begin{array}{ccccccc} Y_1^{-1} & \stackrel{j_{1,-1}}{\longrightarrow} & X_1^0 & \stackrel{i_{1,0}}{\longrightarrow} & Y_1^0 & \longrightarrow & Y_1^{-1}[1] \\ \downarrow g_{-1} & & \downarrow f_0 & & \downarrow g_0 & & \downarrow g_{-1}[1] \\ Y_2^{-1} & \stackrel{j_{2,-1}}{\longrightarrow} & X_2^0 & \stackrel{i_{2,0}}{\longrightarrow} & Y_2^0 & \longrightarrow & Y_2^{-1}[1] \end{array}$$

If the complexes $X_i^{\bullet}$ satisfy the condition (4) (i.e. $\mathrm{Hom}^i(X_1^a\ ,\ Y_2^0) = 0$ for $i < 0$ and all a), then we get $\mathrm{Hom}(Y_1^{-1}[1]\ ,\ Y_2^0) = 0$. It follows from Lemma 1.4. that g_0 is uniquely determined. This concludes the proof of both lemmas. □

§2. Equivalences of derived categories

2.1. Let X and M be smooth projective varieties over field k . Denote by $D^b(X)$ and $D^b(M)$ the bounded derived categories of coherent sheaves on X and M respectively. Recall that a derived category has the structure of a triangulated category.

For every object $E \in D^b(M \times X)$ we can define an exact functor Φ_E from $D^b(M)$ to $D^b(X)$. Denote by p and π the projections of $M \times X$ onto M and X respectively:

$$\begin{array}{ccc} M \times X & \stackrel{\pi}{\longrightarrow} & X \\ p\downarrow & & \\ M & & \end{array}$$

Then Φ_E is defined by the following formula:

$$\Phi_E(\bullet) := \pi_*(E \otimes p^*(\bullet)) \tag{5}$$

(we always shall write shortly $f_*, f^*, \otimes$ and etc. instead of $R^{\bullet}f_*, L^{\bullet}f^*, \overset{L}{\otimes}$, because we consider only derived functors).

The functor Φ_E has the left and the right adjoint functors Φ_E^* and $\Phi_E^!$ respectively, defined by the following formulas:

$$\Phi_E^*(\bullet) = p_*(E^{\vee} \otimes \pi^*(\omega_X[dimX] \otimes (\bullet))),$$

$$\Phi_E^!(\bullet) = \omega_M[dimM] \otimes p_*(E^{\vee} \otimes (\bullet)),$$

where ω_X and ω_M are the canonical sheaves on X and M , and $E^{\vee} := \boldsymbol{R}^{\bullet}\mathcal{H}om(E, \mathcal{O}_{M\times X})$.

Let F be an exact functor from the derived category $D^b(M)$ to the derived category $D^b(X)$. Denote by F^* and $F^!$ the left and the right adjoint functors for F respectively, when they exist. Note that if there exists the left adjoint functor F^* , then the right adjoint functor $F^!$ also exists and

$$F^! = S_M \circ F^* \circ S_X^{-1},$$

where S_X and S_M are Serre functors on $D^b(X)$ and $D^b(M)$. They are equal to $(\bullet) \otimes \omega_X[dimX]$ and $(\bullet) \otimes \omega_M[dimM]$ (see [BK]).

What can we say about the category of all exact functors between $D^b(M)$ and $D^b(X)$? It seems to be true that any functor can be represented by an object on the product $M \times X$ for smooth projective varieties M and X . But we are unable prove it. However, when F is full and faithfull, it can be represented. The main result of this chapter is the following theorem.

2.2. Theorem *Let F be an exact functor from $D^b(M)$ to $D^b(X)$, where M and X are smooth projective varieties. Suppose F is full and faithful and has the right (and,consequently, the left) adjoint functor.*

Then there exists an object $E \in D^b(M \times X)$ such that F is isomorphic to the functor Φ_E defined by the rule (5), and this object is unique up to isomorphism.

2.3. Let F be an exact functor from a derived category $D^b(\mathcal{A})$ to a derived category $D^b(\mathcal{B})$. We say that F is bounded if there exist $z \in \mathbb{Z}, n \in \mathbb{N}$ such that for any $A \in \mathcal{A}$ the cohomology objects $H^i(F(A))$ are equal to 0 for $i \notin [z, z+n]$.

2.4. Lemma *Let M and X be smooth projective varieties. If an exact functor $F : D^b(M) \longrightarrow D^b(X)$ has a left adjoint functor then it is bounded.*

Proof. Let $G : D^b(X) \longrightarrow D^b(M)$ be a left adjoint functor to F . Take a very ample invertible sheaf $\mathcal{L}$ on X . It gives the embedding $i : X \hookrightarrow \mathbb{P}^N$. For any $i < 0$ we have right resolution of the sheaf $\mathcal{O}(i)$ on $\mathbb{P}^N$ in terms of the sheaves $\mathcal{O}(j)$, where $j = 0, 1, .., N$ (see [Be]). It is easily seen that this resolution is of the form

$$\mathcal{O}(i) \stackrel{\sim}{\longrightarrow} \Big\{ V_0 \otimes \mathcal{O} \longrightarrow V_1 \otimes \mathcal{O}(1) \longrightarrow \cdots \longrightarrow V_N \otimes \mathcal{O}(N) \longrightarrow 0 \Big\}$$

where all V_k are vector spaces. The restriction of this resolution to X gives us the resolution of the sheaf $\mathcal{L}^i$ in terms of the sheaves $\mathcal{L}^j$, where $j = 0, 1, ..., N$. Since the functor G is exact that there exist z' and n' such that $H^k(G(\mathcal{L}^i))$ are equal 0 for $k \notin [z', z'+n']$. This follows from the existence of the spectral sequence

$$E_1^{p,q} = V_p \otimes H^q(G(\mathcal{L}^p)) \Rightarrow H^{p+q}(G(\mathcal{L}^i)).$$

As all nonzero terms of this spectral sequence are concentrated in some rectangle, so it follows that for all i cohomologies $H^{\bullet}(G(\mathcal{L}^i))$ are concentrated in some segment.

Now, notice that if $\mathrm{Hom}^j(\mathcal{L}^i , F(A)) = 0$ for all $i \ll 0$, then $H^j(F(A))$ is equal to 0 . Further, by assumption, the functor G is left adjoint to F , hence

$$\mathrm{Hom}^j(\mathcal{L}^i , F(A)) \cong \mathrm{Hom}^j(G(\mathcal{L}^i) , A).$$

If now A is a sheaf on M , then $\mathrm{Hom}^j(G(\mathcal{L}^i) , A) = 0$ for all $i < 0$ and $j \notin [-z' - n', -z' + dimM]$, and thus $H^j(F(A)) = 0$ for the same j . □

2.5. Remark *We shall henceforth assume that for any sheaf $\mathcal{F}$ on M the cohomology objects $H^i(F(\mathcal{F}))$ are nonzero only if $i \in [-a, 0]$.*

2.6. Now we begin constructing an object $E \in D^b(M \times X)$. Firstly, we shall consider a closed embedding $j : M \hookrightarrow \mathbb{P}^N$ and shall construct an object $E' \in D^b(\mathbb{P}^N \times X)$.

Secondly, we shall show that there exists an object $E \in D^b(M \times X)$ such that $E' = (j \times id)_* E$. After that we shall prove that functors F and Φ_E are isomorphic.

Let $\mathcal{L}$ be a very ample invertible sheaf on M such that $\mathrm{H}^i(\mathcal{L}^k) = 0$ for any $k > 0$, when $i \neq 0$. By j denote the closed embedding $j : M \hookrightarrow \mathbb{P}^N$ with respect to $\mathcal{L}$.

Recall that there exists a resolution of the diagonal on the product $\mathbb{P}^N \times \mathbb{P}^N$ (see[Be]). Let us consider the following complex of sheaves on the product:

$$0 \to \mathcal{O}(-N) \boxtimes \Omega^N(N) \xrightarrow{d_{-N}} \mathcal{O}(-N+1) \boxtimes \Omega^{N-1}(N-1) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \xrightarrow{d_{-1}} \mathcal{O} \boxtimes \mathcal{O} \tag{6}$$

This complex is a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal Δ.

Now by F' denote the functor from $D^b(\mathbb{P}^N)$ to $D^b(X)$, which is the composition $F \circ j^*$. Consider the product

$$\begin{array}{ccc} \mathbb{P}^N \times X & \xrightarrow{\pi'} & X \\ {\scriptstyle q}\downarrow & & \\ \mathbb{P}^N & & \end{array}$$

Denote by

$$d'_{-i} \in \mathrm{Hom}_{\mathbb{P}^N \times X}(\mathcal{O}(-i) \boxtimes F'(\Omega^i(i))\,,\ \mathcal{O}(-i+1) \boxtimes F'(\Omega^{i-1}(i-1)))$$

the image d_{-i} under the following through map.

$$\mathrm{Hom}(\mathcal{O}(-i) \boxtimes \Omega^i(i)\,,\ \mathcal{O}(-i+1) \boxtimes \Omega^{i-1}(i-1)) \xrightarrow{\sim}$$

$$\mathrm{Hom}(\mathcal{O} \boxtimes \Omega^i(i)\,,\ \mathcal{O}(1) \boxtimes \Omega^{i-1}(i-1)) \xrightarrow{\sim}$$

$$\mathrm{Hom}(\Omega^i(i)\,,\ \mathrm{H}^0(\mathcal{O}(1)) \otimes \Omega^{i-1}(i-1)) \longrightarrow$$

$$\mathrm{Hom}(F'(\Omega^i(i))\,,\ \mathrm{H}^0(\mathcal{O}(1)) \otimes F'(\Omega^{i-1}(i-1))) \xrightarrow{\sim}$$

$$\mathrm{Hom}(\mathcal{O} \boxtimes F'(\Omega^i(i))\,,\ \mathcal{O}(1) \boxtimes F'(\Omega^{i-1}(i-1))) \xrightarrow{\sim}$$

$$\mathrm{Hom}(\mathcal{O}(-i) \boxtimes F'(\Omega^i(i))\,,\ \mathcal{O}(-i+1) \boxtimes F'(\Omega^{i-1}(i-1)))$$

It can easily be checked that the composition $d_{-i+1} \circ d_{-i}$ is equal to 0. We omit the check.

Consider the following complex $C^\bullet$

$$C^\bullet := \{\mathcal{O}(-N) \boxtimes F'(\Omega^N(N)) \xrightarrow{d'_{-N}} \cdots \longrightarrow \mathcal{O}(-1) \boxtimes F'(\Omega^1(1)) \xrightarrow{d'_{-1}} \mathcal{O} \boxtimes F'(\mathcal{O})\}$$

over the derived category $D^b(\mathbb{P}^N \times X)$. For $l < 0$ we have

$$\mathrm{Hom}^l(\mathcal{O}(-i) \boxtimes F'(\Omega^i(i))\,,\ \mathcal{O}(-k) \boxtimes F'(\Omega^k(k))) \cong$$

$$\mathrm{Hom}^l(\mathcal{O} \boxtimes F'(\Omega^i(i))\,,\ \mathrm{H}^0(\mathcal{O}(i-k)) \otimes F'(\Omega^k(k))) \cong$$

$$\mathrm{Hom}^l(j^*(\Omega^i(i))\,,\ \mathrm{H}^0(\mathcal{O}(i-k)) \otimes j^*(\Omega^k(k))) = 0$$

Hence, by Lemma 1.5. , there exists a convolution of the complex $C^\bullet$, and all convolutions are isomorphic. By E' denote some convolution of $C^\bullet$ and by γ_0 denote the morphism $\mathcal{O}\boxtimes F'(\mathcal{O}) \xrightarrow{\gamma_0} E'$. (Further we shall see that all convolutions of $C^\bullet$ are canonically isomorphic). Now let $\Phi_{E'}$ be the functor from $D^b(\mathbb{P}^N)$ to $D^b(X)$, defined by (5).

2.7. Lemma *There exist canonically defined isomorphisms* $f_k : F'(\mathcal{O}(k)) \xrightarrow{\sim} \Phi_{E'}(\mathcal{O}(k))$ *for all* $k\in\mathbb{Z}$ *, and these isomorphisms are functorial, i.e. for any* $\alpha : \mathcal{O}(k)\to\mathcal{O}(l)$ *the following diagram commutes*

$$\begin{array}{ccc} F'(\mathcal{O}(k)) & \xrightarrow{F'(\alpha)} & F'(\mathcal{O}(l)) \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_l} \\ \Phi_{E'}(\mathcal{O}(k)) & \xrightarrow{\Phi_{E'}(\alpha)} & \Phi_{E'}(\mathcal{O}(l)) \end{array}$$

Proof. At first, assume that $k\ge 0$.

Consider the resolution (6) of the diagonal $\Delta\subset\mathbb{P}^N\times\mathbb{P}^N$ and, after tensoring it with $\mathcal{O}(k)\boxtimes\mathcal{O}$, push forward onto the second component. We get the following resolution of $\mathcal{O}(k)$ on $\mathbb{P}^N$

$$\{\mathrm{H}^0(\mathcal{O}(k-N))\otimes\Omega^N(N)\longrightarrow\cdots\longrightarrow\mathrm{H}^0(\mathcal{O}(k-1))\otimes\Omega^1(1)\longrightarrow\mathrm{H}^0(\mathcal{O}(k))\otimes\mathcal{O}\}\xrightarrow{\delta_k}\mathcal{O}(k)$$

Consequently $F'(\mathcal{O}(k))$ is a convolution of the complex $D_k^\bullet$:

$$\mathrm{H}^0(\mathcal{O}(k-N))\otimes F'(\Omega^N(N))\longrightarrow\cdots\longrightarrow\mathrm{H}^0(\mathcal{O}(k-1))\otimes F'(\Omega^1(1))\longrightarrow\mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O})$$

over $D^b(X)$.

On the other hand, let us consider the complex $C_k^\bullet := q^*\mathcal{O}(k)\otimes C^\bullet$ on $\mathbb{P}^N\times X$ with the morphism $\gamma_k : \mathcal{O}(k)\boxtimes F'(\mathcal{O}) \longrightarrow q^*\mathcal{O}(k)\otimes E'$, and push it forward onto the second component. It follows from the construction of the complex $C^\bullet$ that $\pi'_*(C_k^\bullet) = D_k^\bullet$. So we see that $F'(\mathcal{O}(k))$ and $\Phi_{E'}(\mathcal{O}(k))$ both are convolutions of the same complex $D_k^\bullet$.

By assumption the functor F is full and faithful, hence, if $\mathcal{G}$ and $\mathcal{H}$ are locally free sheaves on $\mathbb{P}^N$ then we have

$$\mathrm{Hom}^i(F'(\mathcal{G})\,,\,F'(\mathcal{H})) = \mathrm{Hom}^i(j^*(\mathcal{G})\,,\,j^*(\mathcal{H})) = 0$$

for $i<0$. Therefore the complex $D_k^\bullet$ satisfies the conditions (1) and (2) of Lemma 1.5. . Hence there exists a uniquely defined isomorphism $f_k : F'(\mathcal{O}(k)) \xrightarrow{\sim} \Phi_{E'}(\mathcal{O}(k))$, completing the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_k)} & F'(\mathcal{O}(k)) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle f_k} \\ \mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{\pi'_*(\gamma_k)} & \Phi_{E'}(\mathcal{O}(k)) \end{array}$$

Now we have to show that these morphisms are functorial. For any $\alpha : \mathcal{O}(k)\to\mathcal{O}(l)$ we have the commutative squares

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}(k))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_k)} & F'(\mathcal{O}(k)) \\ {\scriptstyle \mathrm{H}^0(\alpha)\otimes id}\downarrow & & \downarrow{\scriptstyle F'(\alpha)} \\ \mathrm{H}^0(\mathcal{O}(l))\otimes F'(\mathcal{O}) & \xrightarrow{F'(\delta_l)} & F'(\mathcal{O}(l)) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}(k)) \otimes F'(\mathcal{O}) & \stackrel{\pi'_*(\gamma_k)}{\longrightarrow} & \Phi_{E'}(\mathcal{O}(k)) \\ {\scriptstyle \mathrm{H}^0(\alpha)\otimes id}\downarrow & & \downarrow{\scriptstyle \Phi_{E'}(\alpha)} \\ \mathrm{H}^0(\mathcal{O}(l)) \otimes F'(\mathcal{O}) & \stackrel{\pi'_*(\gamma_l)}{\longrightarrow} & \Phi_{E'}(\mathcal{O}(l)) \end{array}$$

Therefore we have the equalities:

$f_l \circ F'(\alpha) \circ F'(\delta_k) = f_l \circ F'(\delta_l) \circ (\mathrm{H}^0(\alpha) \otimes id) = \pi'_*(\gamma_l) \circ (\mathrm{H}^0(\alpha) \otimes id) = \Phi_{E'}(\alpha) \circ \pi'_*(\gamma_k) = \Phi_{E'}(\alpha) \circ f_k \circ F'(\delta_k)$

Since the complexes $D_k^\bullet$ and $D_l^\bullet$ satisfy the conditions of Lemma 1.6. there exists only one morphism $h : F'(\mathcal{O}(k)) \to \Phi_{E'}(\mathcal{O}(l))$ such that

$$h \circ F'(\delta_k) = \pi'_*(\gamma_l) \circ (\mathrm{H}^0(\alpha) \otimes id)$$

Hence $f_l \circ F'(\alpha)$ coincides with $\Phi_{E'}(\alpha) \circ f_k$.

Now, consider the case $k < 0$.

Let us take the following right resolution for $\mathcal{O}(k)$ on $\mathbb{P}^N$.

$$\mathcal{O}(k) \stackrel{\sim}{\longrightarrow} \{V_0^k \otimes \mathcal{O} \longrightarrow \cdots \longrightarrow V_N^k \otimes \mathcal{O}(N)\}$$

By Lemma 1.6. , the morphism of the complexes over $D^b(X)$

$$\begin{array}{ccccc} V_0^k \otimes F'(\mathcal{O}) & \longrightarrow & \cdots & \longrightarrow & V_N^k \otimes F'(\mathcal{O}(N)) \\ {\scriptstyle id\otimes f_0}\downarrow\wr & & & & {\scriptstyle id\otimes f_N}\downarrow\wr \\ V_0^k \otimes \Phi_{E'}(\mathcal{O}) & \longrightarrow & \cdots & \longrightarrow & V_N^k \otimes \Phi_{E'}(\mathcal{O}(N)) \end{array}$$

gives us the uniquely determined morphism $f_k : F'(\mathcal{O}(k)) \longrightarrow \Phi_{E'}(\mathcal{O}(k))$.

It is not hard to prove that these morphisms are functorial. The proof is left to the reader.
□

2.8. Now we must prove that there exists an object $E \in D^b(M \times X)$ such that $j_*E \cong E'$.

Let $\mathcal{L}$ be a very ample invertible sheaf on M and let $j : M \hookrightarrow \mathbb{P}^N$ be an embedding with respect to $\mathcal{L}$. By A denote the graded algebra $\bigoplus_{i=0}^{\infty} \mathrm{H}^0(M, \mathcal{L}^i)$.

Let $B_0 = k$, and $B_1 = A_1$. For $m \geq 2$, we define B_m as

$$B_m = Ker(B_{m-1} \otimes A_1 \longrightarrow B_{m-2} \otimes A_2) \tag{7}$$

2.9. Definition *A is said to be n -Koszul if the following sequence is exact*

$$B_n \otimes_k A \longrightarrow B_{n-1} \otimes_k A \longrightarrow \cdots \longrightarrow B_1 \otimes_k A \longrightarrow A \longrightarrow k \longrightarrow 0$$

Assume that A is n-Koszul. Let $R_0 = \mathcal{O}_M$. For $m \geq 1$, denote by R_m the kernel of the morphism $B_m \otimes \mathcal{O}_M \longrightarrow B_{m-1} \otimes \mathcal{L}$. Using (7), we obtain the canonical morphism $R_m \longrightarrow A_1 \otimes R_{m-1}$. (actually, $\mathrm{Hom}(R_m , R_{m-1}) \cong A_1^*$).

Since A is n -Koszul, we have the exact sequences

$$0 \longrightarrow R_m \longrightarrow B_m \otimes \mathcal{O}_M \longrightarrow B_{m-1} \otimes \mathcal{L} \longrightarrow \cdots \longrightarrow B_1 \otimes \mathcal{L}^{m-1} \longrightarrow \mathcal{L}^m \longrightarrow 0$$

for $m \le n$.

We have the canonical morphisms $f_m : j^*\Omega^m(m) \longrightarrow R_m$, because $\Lambda^i A_1 \subset B_i$ and there exist the exact sequences on $\mathbb{P}^N$

$$0 \longrightarrow \Omega^m(m) \longrightarrow \Lambda^m A_1 \otimes \mathcal{O} \longrightarrow \Lambda^{m-1} A_1 \otimes \mathcal{O}(1) \longrightarrow \cdots \longrightarrow \mathcal{O}(m) \longrightarrow 0$$

It is known that for any n there exists l such that the Veronese algebra $A^l = \bigoplus_{i=0}^{\infty} \mathrm{H}^0(M, \mathcal{L}^{il})$ is n -Koszul.(Moreover, it was proved in [Ba] that A^l is Koszul for $l \gg 0$).

Using the technique of [IM] and substituting $\mathcal{L}$ with $\mathcal{L}^j$, when j is sufficiently large , we can choose for any n a very ample $\mathcal{L}$ such that

1) algebra A is n -Koszul,

2) the complex

$$\mathcal{L}^{-n} \boxtimes R_n \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes R_1 \longrightarrow \mathcal{O}_M \boxtimes R_0 \longrightarrow \mathcal{O}_\Delta$$

on $M \times M$ is exact,

3) the following sequences on M .

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_{k-1} \otimes R_1 \longrightarrow A_k \otimes R_0 \longrightarrow \mathcal{L}^k \longrightarrow 0$$

are exact for any $k \ge 0$. Here, by definition, if $k - i < 0$, then $A_{k-i} = 0$. (see Appendix for proof).

Let us denote by T_k the kernel of the morphism $A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1}$.

Consider the following complex over $D^b(M \times X)$

$$\mathcal{L}^{-n} \boxtimes F(R_n) \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes F(R_1) \longrightarrow \mathcal{O}_M \boxtimes F(R_0) \qquad (8)$$

Here the morphism $\mathcal{L}^{-k} \boxtimes F(R_k) \longrightarrow \mathcal{L}^{-k+1} \boxtimes F(R_{k-1})$ is induced by the canonical morphism $R_k \longrightarrow A_1 \otimes R_{k-1}$ with respect to the following sequence of isomorphisms

$$\mathrm{Hom}(\mathcal{L}^{-k} \boxtimes F(R_k)\, ,\ \mathcal{L}^{-k+1} \boxtimes F(R_{k-1})) \cong \mathrm{Hom}(F(R_k)\, ,\ \mathrm{H}^0(\mathcal{L}) \otimes F(R_{k-1})) \cong$$

$$\cong \mathrm{Hom}(R_k\, ,\ A_1 \otimes R_{k-1})$$

By Lemma 1.5. , there is a convolution of the complex (8) and all convolutions are isomorphic. Let $G \in D^b(M \times X)$ be a convolution of this complex.

For any $k \ge 0$, object $\pi_*(G \otimes p^*(\mathcal{L}^k))$ is a convolution of the complex

$$A_{k-n} \otimes F(R_n) \longrightarrow A_{k-n+1} \otimes F(R_{n-1}) \longrightarrow \cdots \longrightarrow A_k \otimes F(R_0).$$

On the other side, we know that $T_k[n] \oplus \mathcal{L}^k$ is a convolution of the complex

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_k \otimes R_0,$$

because $\mathrm{Ext}^{n+1}(\mathcal{L}^k\, ,\ T_k) = 0$ for $n \gg 0$. Therefore, by Lemma 1.5. , we have $\pi_*(G \otimes p^*(\mathcal{L}^k)) \cong F(T_k[n] \oplus \mathcal{L}^k)$.

It follows immediately from Remark 2.5. that the cohomology sheaves $H^i(\pi_*(G \otimes p^*(\mathcal{L}^k))) = H^i(F(T_k)[n]) \oplus H^i(F(\mathcal{L}^k))$ concentrate on the union $[-n-a, -n] \cup [-a, 0]$ for any $k > 0$ (a was defined in 2.5.). Therefore the cohomology sheaves $H^i(G)$ also concentrate on $[-n-a, -n] \cup [-a, 0]$. We can assume that $n > dimM + dimX + a$.

This implies that $G \cong C \oplus E$, where E, C are objects of $D^b(M \times X)$ such that $H^i(E) = 0$ for $i \notin [-a, 0]$ and $H^i(C) = 0$ for $i \notin [-n-a, -n]$. Moreover, we have $\pi_*(E \otimes p^*(\mathcal{L}^k)) \cong F(\mathcal{L}^k)$.

Now we show that $j_*(E) \cong E'$. Let us consider the morphism of the complexes over $D^b(\mathbb{P}^N \times X)$.

$$\begin{array}{ccccc} \mathcal{O}(-n) \boxtimes F'(\Omega^n(n)) & \longrightarrow & \cdots & \longrightarrow & \mathcal{O} \boxtimes F'(\mathcal{O}) \\ \big\downarrow {\scriptstyle can \boxtimes F(f_n)} & & & & \big\downarrow {\scriptstyle can \boxtimes F(f_0)} \\ j_*(\mathcal{L}^{-n}) \boxtimes F(R_n) & \longrightarrow & \cdots & \longrightarrow & j_*(\mathcal{O}_M) \boxtimes F(R_0) \end{array}$$

By Lemma 1.6. , there exists a morphism of convolutions $\phi : K \longrightarrow j_*(G)$. If $N > n$, then K is not isomorphic to E' , but there is a distinguished triangle

$$S \longrightarrow K \longrightarrow E' \longrightarrow S[1]$$

and the cohomology sheaves $H^i(S) \neq 0$ only if $i \in [-n-a, -n]$. Now, since $\mathrm{Hom}(S\, ,\, j_*(E)) = 0$ and $\mathrm{Hom}(S[1]\, ,\, j_*(E)) = 0$, we have a uniquely determined morphism $\psi : E' \longrightarrow j_*(E)$ such that the following diagram commutes

$$\begin{array}{ccc} K & \stackrel{\phi}{\longrightarrow} & j_*(G) \\ \downarrow & & \downarrow \\ E' & \stackrel{\psi}{\longrightarrow} & j_*(E) \end{array}$$

We know that $\pi'_*(E' \otimes q^*(\mathcal{O}(k))) \cong F(\mathcal{L}^k) \cong \pi_*(E \otimes p^*(\mathcal{L}^k))$. Let ψ_k be the morphism $\pi'_*(E' \otimes q^*(\mathcal{O}(k))) \longrightarrow \pi_*(E \otimes p^*(\mathcal{L}^k))$ induced by ψ . The morphism ψ_k can be included in the following commutative diagram:

$$\begin{array}{ccccc} S^k A_1 \otimes F(\mathcal{O}) & \stackrel{can}{\longrightarrow} & F(\mathcal{L}^k) & \stackrel{\sim}{\longrightarrow} & \pi'_*(E' \otimes q^*(\mathcal{O}(k))) \\ {\scriptstyle can} \big\downarrow & & & & \big\downarrow {\scriptstyle \psi_k} \\ A_k \otimes F(\mathcal{O}) & \stackrel{can}{\longrightarrow} & F(\mathcal{L}^k) & \stackrel{\sim}{\longrightarrow} & \pi_*(E \otimes p^*(\mathcal{L}^k)) \end{array}$$

Thus we see that ψ_k is an isomorphism for any $k \geq 0$. Hence ψ is an isomorphism too. This proves the following:

2.10. Lemma *There exists an object* $E \in D^b(M \times X)$ *such that* $j_*(E) \cong E'$ *, where* E' *is the object from* $D^b(\mathbb{P}^N \times X)$ *, constructed in 2.6. .*

2.11. Now, we prove some statements relating to abelian categories. they are needed for the sequel.

Let $\mathcal{A}$ be a k -linear abelian category (henceforth we shall consider only k -linear abelian categories). Let $\{P_i\}_{i \in \mathbb{Z}}$ be a sequence of objects from $\mathcal{A}$.

2.12. Definition *We say that this sequence is* ample *if for every object* $X \in \mathcal{A}$ *there exists* N *such that for all* $i < N$ *the following conditions hold:*

a) the canonical morphism $\mathrm{Hom}(P_i\, ,\, X) \otimes P_i \longrightarrow X$ *is surjective,*

b) $\mathrm{Ext}^j(P_i\, ,\, X) = 0$ *for any* $j \neq 0$ *,*

c) $\mathrm{Hom}(X\, ,\, P_i) = 0$.

It is clear that if $\mathcal{L}$ is an ample invertible sheaf on a projective variety in usual sense, then the sequence $\{\mathcal{L}^i\}_{i\in\mathbb{Z}}$ in the abelian category of coherent sheaves is ample.

2.13. Lemma *Let $\{P_i\}$ be an ample sequence in an abelian category $\mathcal{A}$. If X is an object in $D^b(\mathcal{A})$ such that $\mathrm{Hom}^{\bullet}(P_i\, ,\, X)=0$ for all $i\ll 0$, then X is the zero object.*

Proof. If $i\ll 0$ then

$$\mathrm{Hom}(P_i\, ,\, H^k(X))\cong \mathrm{Hom}^k(P_i\, ,\, X)=0$$

The morphism $\mathrm{Hom}(P_i\, ,\, H^k(X))\otimes P_i\longrightarrow H^k(X)$ must be surjective for $i\ll 0$, hence $H^k(X)=0$ for all k . Thus X is the zero object. □

2.14. Lemma *Let $\{P_i\}$ be an ample sequence in an abelian category $\mathcal{A}$ of finite homological dimension. If X is an object in $D^b(\mathcal{A})$ such that $\mathrm{Hom}^{\bullet}(X\, ,\, P_i)=0$ for all $i\ll 0$. Then X is the zero object.*

Proof. Assume that the cohomology objects of X are concentrated in a segment $[a,0]$. There exists the canonical morphism $X\longrightarrow H^0(X)$. Consider a surjective morphism $P_{i_1}^{\oplus k_1}\longrightarrow H^0(X)$. By Y_1 denote the kernel of this morphism. Since $\mathrm{Hom}^{\bullet}(X\, ,\, P_{i_1})=0$ we have $\mathrm{Hom}^1(X\, ,\, Y_1)\neq 0$. Further take a surjective morphism $P_{i_2}^{\oplus k_2}\longrightarrow Y_1$. By Y_2 denote the kernel of this morphism. Again, since $\mathrm{Hom}^{\bullet}(X\, ,\, P_{i_2})=0$, we obtain $\mathrm{Hom}^2(X\, ,\, Y_2)\neq 0$. Iterating this procedure as needed, we get contradiction with the assumption that $\mathcal{A}$ is of finite homological dimension. □

2.15. Lemma *Let $\mathcal{B}$ be an abelian category, $\mathcal{A}$ an abelian category of finite homological dimension, and $\{P_i\}$ an ample sequence in $\mathcal{A}$. Suppose F is an exact functor from $D^b(\mathcal{A})$ to $D^b(\mathcal{B})$ such that it has right and left adjoint functors $F^!$ and F^* respectively. If the maps*

$$\mathrm{Hom}^k(P_i\, ,\, P_j)\xrightarrow{\sim}\mathrm{Hom}^k(F(P_i)\, ,\, F(P_j))$$

are isomorphisms for $i<j$ and all k . Then F is full and faithful.

Proof. Let us take the canonical morphism $f_i: P_i\longrightarrow F^!F(P_i)$ and consider a distinguished triangle

$$P_i\xrightarrow{f_i}F^!F(P_i)\longrightarrow C_i\longrightarrow P_i[1].$$

Since for $j\ll 0$ we have isomorphisms:

$$\mathrm{Hom}^k(P_j\, ,\, P_i)\xrightarrow{\sim}\mathrm{Hom}^k(F(P_j)\, ,\, F(P_i))\cong \mathrm{Hom}^k(P_j\, ,\, F^!F(P_i)).$$

We see that $\mathrm{Hom}^{\bullet}(P_j\, ,\, C_i)=0$ for $j\ll 0$. It follows from Lemma 2.13. that $C_i=0$. Hence f_i is an isomorphism.

Now, take the canonical morphism $g_X: F^*F(X)\longrightarrow X$ and consider a distinguished triangle

$$F^*F(X)\xrightarrow{g_X}X\longrightarrow C_X\longrightarrow F^*F(X)[1]$$

We have the following sequence of isomorphisms

$$\mathrm{Hom}^k(X\,,\,P_i) \xrightarrow{\sim} \mathrm{Hom}^k(X\,,\,F^!F(P_i)) \cong \mathrm{Hom}^k(F^*F(X)\,,\,P_i)$$

This implies that $\mathrm{Hom}^\bullet(C_X\,,\,P_i)=0$ for all i . By Lemma 2.14. , we obtain $C_X=0$. Hence g_X is an isomorphism. It follows that F is full and faithful. □

Let $\mathcal{A}$ be an abelian category possessing an ample sequence $\{P_i\}$. Denote by $D^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. Let us consider the full subcategory $j:\mathcal{C}\hookrightarrow D^b(\mathcal{A})$ such that $\mathrm{Ob}\mathcal{C}:=\{P_i \mid i\in\mathbb{Z}\}$. Now we would like to show that if there exists an isomorphism of a functor $F: D^b(\mathcal{A})\longrightarrow D^b(\mathcal{A})$ to identity functor on the subcategory $\mathcal{C}$, then it can be extended to the whole $D^b(\mathcal{A})$.

2.16. Proposition *Let* $F: D^b(\mathcal{A})\longrightarrow D^b(\mathcal{A})$ *be an autoequivalence. Suppose there exists an isomorphism* $f: j\xrightarrow{\sim} F\,|_{\mathcal{C}}$ *(where* $j:\mathcal{C}\hookrightarrow D^b(\mathcal{A})$ *is a natural embedding). Then it can be extended to an isomorphism* $id\xrightarrow{\sim}F$ *on the whole* $D^b(\mathcal{A})$ *.*

Proof. First, we can extend the transformation f to all direct sums of objects $\mathcal{C}$ componentwise , because F takes direct sums to direct sums.

Note that $X\in D^b(\mathcal{A})$ is isomorphic to an object in $\mathcal{A}$ iff $\mathrm{Hom}^j(P_i\,,\,X)=0$ for $j\neq 0$ and $i\ll 0$. It follows that $F(X)$ is isomorphic to an object in $\mathcal{A}$, because

$$\mathrm{Hom}^j(P_i\,,\,F(X))\cong \mathrm{Hom}^j(F(P_i)\,,\,F(X))\cong \mathrm{Hom}^j(P_i\,,\,X)=0$$

for $j\neq 0$ and $i\ll 0$.

2.16.1 At first, let X be an object from $\mathcal{A}$. Take a surjective morphism $v: P_i^{\oplus k}\longrightarrow X$. We have the morphism $f_i: P_i^{\oplus k}\longrightarrow F(P_i^{\oplus k})$ and two distinguished triangles:

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & P_i^{\oplus k} & \xrightarrow{v} & X & \longrightarrow & Y[1] \\ & & \downarrow f_i & & & & \\ F(Y) & \xrightarrow{F(u)} & F(P_i^{\oplus k}) & \xrightarrow{F(v)} & F(X) & \longrightarrow & F(Y)[1] \end{array}$$

Now we show that $F(v)\circ f_i\circ u=0$. Consider any surjective morphism $w: P_j^{\oplus l}\longrightarrow Y$. It is sufficient to check that $F(v)\circ f_i\circ u\circ w=0$. Let $f_j: P_j^{\oplus l}\longrightarrow F(P_j^{\oplus l})$ be the canonical morphism. Using the commutation relations for f_i and f_j , we obtain

$$F(v)\circ f_i\circ u\circ w = F(v)\circ F(u\circ w)\circ f_j = F(v\circ u\circ w)\circ f_j = 0$$

because $v\circ u=0$.

Since $\mathrm{Hom}(Y[1]\,,\,F(X))=0$, by Lemma 1.4. , there exists a unique morphism $f_X: X\longrightarrow F(X)$ that commutes with f_i .

2.16.2 Let us show that f_X does not depend from morphism $v: P_i^{\oplus k}\longrightarrow X$. Consider two surjective morphisms $v_1: P_{i_1}^{\oplus k_1}\longrightarrow X$ and $v_2: P_{i_2}^{\oplus k_2}\longrightarrow X$. We can take two surjective morphisms $w_1: P_j^{\oplus l}\longrightarrow P_{i_1}^{\oplus k_1}$ and $w_2: P_j^{\oplus l}\longrightarrow P_{i_2}^{\oplus k_2}$ such that the following diagram is commutative:

$$\begin{array}{ccc} P_j^{\oplus l} & \xrightarrow{w_2} & P_{i_2}^{\oplus k_2} \\ \downarrow w_1 & & \downarrow v_2 \\ P_{i_1}^{\oplus k_1} & \xrightarrow{v_1} & X \end{array}$$

Clearly, it is sufficient to check the coincidence of the morphisms, constructed by v_1 and $v_1 \circ w_1$. Now, let us consider the following commutative diagram:

$$\begin{array}{ccccc} P_j^{\oplus l} & \xrightarrow{w_1} & P_{i_1}^{\oplus k_1} & \xrightarrow{v_1} & X \\ \downarrow f_j & & \downarrow v_2 & & \downarrow f_X \\ F(P_j^{\oplus l}) & \xrightarrow{F(w_1)} & F(P_{i_1}^{\oplus k_1}) & \xrightarrow{F(v_1)} & F(X) \end{array}$$

Here the morphism f_X is constructed by v_1 . Both squares of this diagram are commutative. Since there exists only one morphism from X to $F(X)$ that commutes with f_j , we see that the f_X , constructed by v_1 , coincides with the morphism, constructed by $v_1 \circ w_1$.

2.16.3 Now we must show that for any morphism $X \xrightarrow{\phi} Y$ we have equality:

$$f_Y \circ \phi = F(\phi) \circ f_X$$

Take a surjective morphism $P_j^{\oplus l} \xrightarrow{v} Y$. Choose a surjective morphism $P_i^{\oplus k} \xrightarrow{u} X$ such that the composition $\phi \circ u$ lifts to a morphism $\psi : P_i^{\oplus k} \longrightarrow P_j^{\oplus l}$. We can do it, because for $i \ll 0$ the map $\mathrm{Hom}(P_i^{\oplus k}, P_j^{\oplus l}) \to \mathrm{Hom}(P_i^{\oplus k}, Y)$ is surjective. We get the commutative square:

$$\begin{array}{ccc} P_i^{\oplus k} & \xrightarrow{u} & X \\ \downarrow \psi & & \downarrow \phi \\ P_j^{\oplus l} & \xrightarrow{v} & Y \end{array}$$

By h_1 and h_2 denote $f_Y \circ \phi$ and $F(\phi) \circ f_X$ respectively. We have the following sequence of equalities:

$$h_1 \circ u = f_Y \circ \phi \circ u = f_Y \circ v \circ \psi = F(v) \circ f_j \circ \psi = F(v) \circ F(\psi) \circ f_i$$

and

$$h_2 \circ u = F(\phi) \circ f_X \circ u = F(\phi) \circ F(u) \circ f_i = F(\phi \circ u) \circ f_i = F(v \circ \psi) \circ f_i = F(v) \circ F(\psi) \circ f_i$$

Consequently, the following square is commutative for $t = 1, 2$.

$$\begin{array}{ccccccc} Z & \longrightarrow & P_i^{\oplus k} & \xrightarrow{u} & X & \longrightarrow & Z[1] \\ & & F(\psi) \circ f_i \downarrow & & \downarrow h_t & & \\ F(W) & \longrightarrow & F(P_j^{\oplus l}) & \xrightarrow{F(v)} & F(Y) & \longrightarrow & F(W)[1] \end{array}$$

By Lemma 1.4. , as $\mathrm{Hom}(Z[1], F(Y)) = 0$, we obtain $h_1 = h_2$. Thus, $f_Y \circ \phi = F(\phi) \circ f_X$.

Now take a cone C_X of the morphism f_X . Using the following isomorphisms

$$\mathrm{Hom}(P_i, X) \cong \mathrm{Hom}(F(P_i), F(X)) \cong \mathrm{Hom}(P_i, F(X)),$$

we obtain $\mathrm{Hom}^j(P_i, C_X) = 0$ for all j , when $i \ll 0$. Hence, by Lemma 2.13. , $C_X = 0$ and f_X is an isomorphism.

2.16.4 Let us define $f_{X[n]} : X[n] \longrightarrow F(X[n]) \cong F(X)[n]$ for any $X \in \mathcal{A}$ by

$$f_{X[n]} = f_X[n].$$

It is easily shown that these transformations commute with any $u \in \mathrm{Ext}^k(X\ ,\ Y)$. Indeed, since any element $u \in \mathrm{Ext}^k(X\ ,\ Y)$ can be represented as a composition $u = u_0 u_1 \cdots u_k$ of some elements $u_i \in \mathrm{Ext}^1(Z_i\ ,\ Z_{i+1})$ and $Z_0 = X, Z_k = Y$, we have only to check it for $u \in \mathrm{Ext}^1(X\ ,\ Y)$. Consider the following diagram:

$$\begin{array}{ccccccc} Y & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{u} & Y[1] \\ f_Y\downarrow & & \downarrow f_Z & & & & \downarrow f_Y[1] \\ F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(X) & \xrightarrow{F(u)} & F(Y)[1] \end{array}$$

By an axiom of triangulated categories there exists a morphism $h : X \to F(X)$ such that (f_Y, f_Z, h) is a morphism of triangles. On the other hand, since $\mathrm{Hom}(Y[1]\ ,\ F(X)) = 0$, by Lemma 1.4. , h is a unique morphism that commutes with f_Z . But f_X also commutes with f_Z . Hence we have $h = f_X$. This implies that

$$f_Y[1] \circ u = F(u) \circ f_X$$

2.16.5 The rest of the proof is by induction over the length of a segment, in which the cohomology objects of X are concentrated. Let X be an object from $D^b(\mathcal{A})$ and suppose that its cohomology objects $H^p(X)$ are concentrated in a segment $[a, 0]$. Take $v : P_i^{\oplus k} \longrightarrow X$ such that

$$\begin{array}{lll} a) & \mathrm{Hom}^j(P_i\ ,\ H^p(X)) = 0 & \text{for all } p \text{ and for } j \neq 0, \\ b) & u : P_i^{\oplus k} \longrightarrow H^0(X) & \text{is the surjective morphism,} \\ c) & \mathrm{Hom}(H^0(X)\ ,\ P_i) = 0. & \end{array} \tag{9}$$

Here u is the composition v with the canonical morphism $X \longrightarrow H^0(X)$. Consider a distinguished triangle:

$$Y[-1] \longrightarrow P_i^{\oplus k} \xrightarrow{v} X \longrightarrow Y$$

By the induction hypothesis, there exists the isomorphism f_Y and it commutes with f_i . So we have the commutative diagram:

$$\begin{array}{ccccccc} Y[-1] & \longrightarrow & P_i^{\oplus k} & \xrightarrow{v} & X & \longrightarrow & Y \\ f_Y[-1]\downarrow & & \downarrow f_i & & & & \downarrow f_Y \\ F(Y)[-1] & \longrightarrow & F(P_i^{\oplus k}) & \xrightarrow{F(v)} & F(X) & \longrightarrow & F(Y) \end{array}$$

Moreover we have the following sequence of equalities

$$\mathrm{Hom}(X\ ,\ F(P_i^{\oplus k})) \cong \mathrm{Hom}(X\ ,\ P_i^{\oplus k}) \cong \mathrm{Hom}(H^0(X)\ ,\ P_i^{\oplus k}) = 0$$

Hence, by Lemma 1.4. , there exists a unique morphism $f_X : X \longrightarrow F(X)$ that commutes with f_Y .

2.16.6 We must first show that f_X is correctly defined. Suppose we have two morphisms $v_1 : P_{i_1}^{\oplus k_1} \longrightarrow X$ and $v_2 : P_{i_2}^{\oplus k_2} \longrightarrow X$. As above, we can find P_j and surjective morphisms w_1, w_2 such that the following diagram is commutative

$$\begin{array}{ccc} P_j^{\oplus l} & \xrightarrow{w_2} & P_{i_2}^{\oplus k_2} \\ \downarrow{w_1} & & \downarrow{u_2} \\ P_{i_1}^{\oplus k_1} & \xrightarrow{u_1} & H^0(X) \end{array}$$

We can find a morphism $\phi : Y_j \longrightarrow Y_{i_1}$ such that the triple (w_1, id, ϕ) is a morphism of distinguished triangles.

$$\begin{array}{ccccccc} P_j^{\oplus l} & \xrightarrow{v_1 \circ w_1} & X & \longrightarrow & Y_j & \longrightarrow & P_j^{\oplus l}[1] \\ w_1 \downarrow & & \downarrow{id} & & \downarrow{\phi} & & \downarrow{w_1[1]} \\ P_{i_1}^{\oplus k_1} & \xrightarrow{v_1} & X & \longrightarrow & Y_{i_1} & \longrightarrow & P_{i_1}^{\oplus k_1}[1] \end{array}$$

By the induction hypothesis, the following square is commutative.

$$\begin{array}{ccc} Y_j & \xrightarrow{\phi} & Y_{i_1} \\ f_{Y_j} \downarrow & & \downarrow f_{Y_i} \\ F(Y_j) & \xrightarrow{F(\phi)} & F(Y_{i_1}) \end{array}$$

Hence, we see that the f_X , constructed by $v_1 \circ w_1$, commutes with $f_{Y_{i_1}}$ and, consequently, coincides with the f_X , constructed by v_1 ; because such morphism is unique by Lemma 1.4. . Therefore morphism f_X does not depend on a choice of morphism $v : P_i^{\oplus k} \longrightarrow X$.

2.16.7 Finally, let us prove that for any morphism $\phi : X \longrightarrow Y$ the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ f_X \downarrow & & \downarrow f_Y \\ F(X) & \xrightarrow{F(\phi)} & F(Y) \end{array} \tag{10}$$

Suppose the cohomology objects of X are concentrated on a segment $[a, 0]$ and the cohomology objects of Y are concentrated on $[b, c]$.

Case 1. If $c < 0$, we take a morphism $v : P_i^{\oplus k} \longrightarrow X$ that satisfies conditions (9) and $\mathrm{Hom}(P_i^{\oplus k}, Y) = 0$. We have a distinguished triangle:

$$P_i^{\oplus k} \xrightarrow{v_1} X \xrightarrow{\alpha} Z \longrightarrow P_i^{\oplus k}[1]$$

Applying the functor $\mathrm{Hom}(- , Y)$ to this triangle we found that there exists a morphism $\psi : Z \longrightarrow Y$ such that $\phi = \psi \circ \alpha$. We know that f_X , defined above, satisfy

$$F(\alpha) \circ f_X = f_Z \circ \alpha$$

If we assume that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & Y \\ f_Z \downarrow & & \downarrow f_Y \\ F(Z) & \xrightarrow{F(\psi)} & F(Y) \end{array}$$

commutes, then diagram (10) commutes too.

This means that for verifying the commutativity of (10) we can substitute X by an object Z . And the cohomology objects of Z are concentrated on the segment $[a, -1]$.

Case 2. If $c \geq 0$, we take a surjective morphism $v : P_i^{\oplus k} \longrightarrow Y[c]$ that satisfies conditions (9) and $\mathrm{Hom}(H^c(X)\,,\, P_i^{\oplus k}) = 0$. Consider a distinguished triangle

$$P_i^{\oplus k}[-c] \stackrel{v[-c]}{\longrightarrow} Y \stackrel{\beta}{\longrightarrow} W \longrightarrow P_i^{\oplus k}[-c+1]$$

Note that the cohomology objects of W are concentrated on $[b, c-1]$.

By ψ denote the composition $\beta \circ \phi$. If we assume that the following square

$$\begin{array}{ccc} X & \stackrel{\psi}{\longrightarrow} & W \\ f_X \downarrow & & \downarrow f_W \\ F(X) & \stackrel{F(\psi)}{\longrightarrow} & F(W) \end{array}$$

commutes, then, since $F(\beta) \circ f_Y = f_W \circ \beta$,

$$F(\beta) \circ (f_Y \circ \phi - F(\phi) \circ f_X) = f_W \circ \psi - F(\psi) \circ f_X = 0.$$

We chose P_i such that $\mathrm{Hom}(X\,,\, P_i^{\oplus k}[-c]) = 0$. As $F(P_i^{\oplus k})$ is isomorphic to $P_i^{\oplus k}$, then $\mathrm{Hom}(X\,,\, F(P_i^{\oplus k}[-c])) = 0$. Applying the functor $\mathrm{Hom}(X\,,\, F(-))$ to the above triangle we found that the composition with $F(\beta)$ gives an inclusion of $\mathrm{Hom}(X\,,\, F(Y))$ into $\mathrm{Hom}(X\,,\, F(W))$. This follows that $f_Y \circ \phi = F(\phi) \circ f_X$, i.e. our diagram (10) commutes.

Combining case 1 and case 2, we can reduce the checking of commutativity of diagram (10) to the case when X and Y are objects from the abelian category $\mathcal{A}$. But for those it has already been done. Thus the proposition is proved. $\Box$

2.17. **Proof of theorem**. 1) EXISTENCE. Using Lemma 2.10. and Lemma 2.7., we can construct an object $E \in D^b(M \times X)$ such that there exists an isomorphism of the functors $\bar{f} : F\big|_{\mathcal{C}} \stackrel{\sim}{\longrightarrow} \Phi_E\big|_{\mathcal{C}}$ on full subcategory $\mathcal{C} \subset D^b(M)$, where $\mathrm{Ob}\mathcal{C} = \{\mathcal{L}^i \mid i \in \mathbb{Z}\}$ and $\mathcal{L}$ is a very ample invertible sheaf on M such that for any $k > 0$ $\mathrm{H}^i(M, \mathcal{L}^k) = 0$, when $i \neq 0$.

By Lemma 2.15. the functor Φ_E is full and faithfull. Moreover, the functors $F^! \circ \Phi_E$ and $\Phi_E^* \circ F$ are full and faithful too, because we have the isomorphisms:

$$F^!(\bar{f}) : F^! \circ F\big|_{\mathcal{C}} \cong id_{\mathcal{C}} \stackrel{\sim}{\longrightarrow} F^! \circ \Phi_E\big|_{\mathcal{C}}$$

$$\Phi_E^*(\bar{f}) : \Phi_E^* \circ F\big|_{\mathcal{C}} \stackrel{\sim}{\longrightarrow} \Phi_E^* \circ \Phi_E\big|_{\mathcal{C}} \cong id_{\mathcal{C}}$$

and conditions of Lemma 2.15. is fulfilled.

Further, the functors $F^! \circ \Phi_E$ and $\Phi_E^* \circ F$ are equivalences, because they are adjoint each other.

Consider the isomorphism $F^!(\bar{f}) : F^! \circ F\big|_{\mathcal{C}} \cong id_{\mathcal{C}} \stackrel{\sim}{\longrightarrow} F^! \circ \Phi_E\big|_{\mathcal{C}}$ on the subcategory $\mathcal{C}$. By Proposition 2.16. we can extend it onto the whole $D^b(M)$, so $id \stackrel{\sim}{\longrightarrow} F^! \circ \Phi_E$.

Since $F^!$ is the right adjoint to F, we get the morphism of the functors $f : F \longrightarrow \Phi_E$ such that $f|_{\mathcal{C}} = \bar{f}$. It can easily be checked that f is an isomorphism. Indeed, let C_Z be a cone of the morphism $f_Z : F(Z) \longrightarrow \Phi_E(Z)$. Since $F^!(f_Z)$ is an isomorphism, we obtain $F^!(Z) = 0$. Therefore, this implies that $\mathrm{Hom}(F(Y), C_Z) = 0$ for any object Y . Further, there are isomorphisms $F(\mathcal{L}^k) \cong \Phi_E(\mathcal{L}^k)$ for any k . Hence, we have

$$\mathrm{Hom}^i(\mathcal{L}^k, \Phi_E^!(C_Z)) = \mathrm{Hom}^i(\Phi_E(\mathcal{L}^k), C_Z)) = \mathrm{Hom}^i(F(\mathcal{L}^k), C_Z)) = 0$$

for all k and i .

Thus, we obtain $\Phi_E^!(C_Z) = 0$. This implies that $\mathrm{Hom}(\Phi_E(Z), C_Z) = 0$. Finally, we get $F(Z) = C_Z[-1] \oplus \Phi_E(Z)$. But we know that $\mathrm{Hom}(F(Z)[1], C_Z) = 0$. Thus, $C_Z = 0$ and f is an isomorphism.

2) Uniqueness. Suppose there exist two objects E and E_1 of $D^b(M \times X)$ such that $\Phi_{E_1} \cong F \cong \Phi_{E_2}$. Let us consider the complex (8) over $D^b(M \times X)$ (see the proof Lemma 2.10.).

$$\mathcal{L}^{-n} \boxtimes F(R_n) \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes F(R_1) \longrightarrow \mathcal{O}_M \boxtimes F(R_0)$$

By Lemma 1.5. , there exists a convolution of this complex and all convolutions are isomorphic. Let $G \in D^b(M \times X)$ be a convolution of the complex (8). Now consider the following complexes

$$\mathcal{L}^{-n} \boxtimes F(R_n) \longrightarrow \cdots \longrightarrow \mathcal{L}^{-1} \boxtimes F(R_1) \longrightarrow \mathcal{O}_M \boxtimes F(R_0) \longrightarrow E_k$$

Again by Lemma 1.5. , there exists a unique up to isomorphism convolutions of these complexes.

Hence we have the canonical morphisms $G \longrightarrow E_1$ and $G \longrightarrow E_2$. Moreover, it has been proved above (see the proof of Lemma 2.10.) that $C_1 \oplus E_1 \cong G \cong C_2 \oplus E_2$ for large n , where E_k, C_k are objects of $D^b(M \times X)$ such that $H^i(E_k) = 0$ for $i \notin [-a, 0]$ and $H^i(C_k) = 0$ for $i \notin [-n-a, -n]$ (a was defined in 2.5.). Thus E_1 and E_2 are isomorphic.

This completes the proof of Theorem 2.2. □

2.18. Theorem *Let M and X be smooth projective varieties. Suppose $F : D^b(M) \longrightarrow D^b(X)$ is an equivalence. Then there exists a unique up to isomorphism object $E \in D^b(M \times X)$ such that the functors F and Φ_E are isomorphic.*

It follows immediately from Theorem 2.2.

§3. Derived categories of K3 surfaces

3.1. In this chapter we are trying to answer the following question: When are derived categories of coherent sheaves on two different K3 surfaces over field $\mathbb{C}$ equivalent?

This question is interesting, because there exists a procedure for recovering a variety from its derived category of coherent sheaves if the canonical (or anticanonical) sheaf is ample. Besides, if $D^b(X) \simeq D^b(Y)$ and X is a smooth projective K3 surface, then Y is also a smooth projective K3 surface. This is true, because the dimension of a variety and Serre functor are invariants of a derived category.

The following theorem is proved in [BO2].

3.2. Theorem *(see* [BO2]*) Let* X *be smooth irreducible projective variety with either ample canonical or ample anticanonical sheaf. If* $D = D^b(X)$ *is equivalent to* $D^b(X')$ *for some other smooth algebraic variety, then* X *is isomorphic to* X' *.*

However, there exist examples of varieties that have equivalent derived categories, if the canonical sheaf is not ample. Here we give an explicit description for K3 surfaces with equivalent derived categories.

3.3. Theorem *Let* S_1 *and* S_2 *be smooth projective K3 surfaces over field* $\mathbb{C}$ *. Then the derived categories* $D^b(S_1)$ *and* $D^b(S_2)$ *are equivalent as triangulated categories iff there exists a Hodge isometry* $f_T : T_{S_1} \xrightarrow{\sim} T_{S_2}$ *between the lattices of transcendental cycles of* S_1 *and* S_2 *.*

Recall that the lattice of transcendental cycles T_S is the orthogonal complement to Neron-Severi lattice N_S in $H^2(S, \mathbb{Z})$. Hodge isometry means that the one-dimensional subspace $H^{2,0}(S_1) \subset T_{S_1} \otimes \mathbb{C}$ goes to $H^{2,0}(S_2) \subset T_{S_2} \otimes \mathbb{C}$.

Now we need some basic facts about K3 surfaces (see [Mu2]). If S is a K3 surface, then the Todd class td_S of S is equal to $1 + 2w$, where $1 \in H^0(S, \mathbb{Z})$ is the unit element of the cohomology ring $H^*(S, \mathbb{Z})$ and $w \in H^4(S, \mathbb{Z})$ is the fundamental cocycle of S . The positive square root $\sqrt{td_S} = 1 + w$ lies in $H^*(S, \mathbb{Z})$ too.

Let E be an object of $D^b(S)$ then the Chern character

$$ch(E) = r(E) + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2)$$

belongs to integral cohomology $H^*(S, \mathbb{Z})$.

For an object E , we put $v(E) = ch(E)\sqrt{td_S} \in H^*(S, \mathbb{Z})$ and call it the vector associated to E (or Mukai vector).

We can define a symmetric integral bilinear form $(\,,)$ on $H^*(S, \mathbb{Z})$ by the rule

$$(u, u') = rs' + sr' - \alpha\alpha' \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}$$

for every pair $u = (r, \alpha, s), u' = (r', \alpha', s') \in H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. By $\widetilde{H}(S, \mathbb{Z})$ denote $H^*(S, \mathbb{Z})$ with this inner product $(\,,)$ and call it Mukai lattice.

For any objects E and F , inner product $(v(E), v(F))$ is equal to the H^4 component of $ch(E)^\vee \cdot ch(F) \cdot td_S$. Hence, by Riemann-Roch- Grothendieck theorem, we have

$$(v(E), v(F)) = \chi(E, F) := \sum_i (-1)^i dim \mathrm{Ext}^i(E\,,\ F)$$

Let us suppose that $D^b(S_1)$ and $D^b(S_2)$ are equivalent. By Theorem 2.2. there exists an object $E \in D^b(S_1 \times S_2)$ such that the functor Φ_E gives this equivalence.

Now consider the algebraic cycle $Z := p^*\sqrt{td_{S_1}} \cdot ch(E) \cdot \pi^*\sqrt{td_{S_2}}$ on the product $S_1 \times S_2$, where p and π are the projections

$$\begin{array}{ccc} S_1 \times S_2 & \xrightarrow{\pi} & S_2 \\ {\scriptstyle p}\downarrow & & \\ S_1 & & \end{array}$$

It follows from the following lemma that the cycle Z belongs to integral cohomology $H^*(S_1 \times S_2, \mathbb{Z})$.

3.4. Lemma [Mu2] *For any object* $E \in D^b(S_1 \times S_2)$ *the Chern character* $ch(E)$ *is integral, which means that it belongs to* $H^*(S_1 \times S_2, \mathbb{Z})$

The cycle Z defines a homomorphism from integral cohomology of S_1 to integral cohomology of S_2 :

$$\begin{array}{cccc} f: & H^*(S_1,\mathbb{Z}) & \longrightarrow & H^*(S_2,\mathbb{Z}) \\ & \cup & & \cup \\ & \alpha & \mapsto & \pi_*(Z\cdot p^*(\alpha)) \end{array}$$

The following proposition is similar to Theorem 4.9 from [Mu2].

3.5. Proposition *If* Φ_E *is full and faithful functor from* $D^b(S_1)$ *to* $D^b(S_2)$ *then:*
1) f *is an isometry between* $\widetilde{H}(S_1,\mathbb{Z})$ *and* $\widetilde{H}(S_2,\mathbb{Z})$,
2) the inverse of f *is equal to the homomorphism*

$$\begin{array}{cccc} f': & H^*(S_2,\mathbb{Z}) & \longrightarrow & H^*(S_1,\mathbb{Z}) \\ & \cup & & \cup \\ & \beta & \mapsto & p_*(Z^\vee\cdot \pi^*(\beta)) \end{array}$$

defined by $Z^\vee = p^*\sqrt{td_{S_1}}\cdot ch(E^\vee)\cdot \pi^*\sqrt{td_{S_2}}$, *where* $E^\vee := \boldsymbol{R}^{\boldsymbol{\cdot}}\mathcal{H}om(E, \mathcal{O}_{S_1\times S_2})$.

Proof. The left and right adjoint functors to Φ_E are:

$$\Phi_E^* = \Phi_E^! = p_*(E^\vee \otimes \pi^*(\bullet))[2]$$

Since Φ_E is full and faithful, the composition $\Phi_E^*\circ\Phi_E$ is isomorphic to $id_{D^b(S_1)}$.

Functor $id_{D^b(S_1)}$ is given by the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta\subset S_1\times S_1$.

Using the projection formula and Grothendieck-Riemann-Roch theorem, it can easily be shown that the composition $f'\circ f$ is given by the cycle $p_1^*\sqrt{td_{S_1}}\cdot ch(\mathcal{O}_\Delta)\cdot p_2^*\sqrt{td_{S_1}}$, where p_1, p_2 are the projections of $S_1\times S_1$ to the summands. But this cycle is equal to Δ .

Therefore, $f'\circ f$ is the identity, and, hence, f is an isomorphism of the lattices, because these lattices are free abelian groups of the same rank.

Let $\nu_S : S \longrightarrow Spec\mathbb{C}$ be the structure morphism of S . Then our inner product (α,α') on $\widetilde{H}(S,\mathbb{Z})$ is equal to $\nu_*(\alpha^\vee\cdot\alpha')$. Hence, by the projection formula, we have

$$\begin{aligned} (\alpha, f(\beta)) &= \nu_{S_2,*}(\alpha^\vee\cdot \pi_*(\pi^*\sqrt{td_{S_2}}\cdot ch(E)\cdot p^*\sqrt{td_{S_1}}\cdot p^*(\beta))) = \\ &= \nu_{S_2,*}\pi_*(\pi^*(\alpha^\vee)\cdot p^*(\beta)\cdot ch(E)\cdot\sqrt{td_{S_1\times S_2}}) = \\ &= \nu_{S_1\times S_2,*}(\pi^*(\alpha^\vee)\cdot p^*(\beta)\cdot ch(E)\cdot\sqrt{td_{S_1\times S_2}}) \end{aligned}$$

for every $\alpha\in H^*(S_2,\mathbb{Z}), \beta\in H^*(S_1,\mathbb{Z})$. In a similar way, we have

$$(\beta, f'(\alpha)) = \nu_{S_1\times S_2,*}(p^*(\beta^\vee)\cdot\pi^*(\alpha)\cdot ch(E)^\vee\cdot\sqrt{td_{S_1\times S_2}})$$

Therefore, $(\alpha, f(\beta)) = (f'(\alpha),\beta)$. Since $f'\circ f$ is the identity, we obtain

$$(f(\alpha), f(\alpha')) = (f'f(\alpha),\alpha') = (\alpha,\alpha')$$

Thus, f is an isometry. $\square$

3.6. Consider the isometry f . Since the cycle Z is algebraic, we obtain two isometries $f_{alg} : -N_{S_1} \perp U \xrightarrow{\sim} -N_{S_2} \perp U$ and $f_{\mathcal{T}} : T_{S_1} \xrightarrow{\sim} T_{S_2}$, where N_{S_1}, N_{S_2} are Neron-Severi lattices, and T_{S_1}, T_{S_2} are the lattices of transcendental cycles. It is clear $f_{\mathcal{T}}$ is a Hodge isometry.

Thus we have proved that if the derived categories of two K3 surfaces are equivalent, then there exists a Hodge isometry between the lattices of transcendental cycles.

3.7. Let us begin to prove the converse. Suppose we have a Hodge isometry

$$f_{\mathcal{T}} : T_{S_2} \xrightarrow{\sim} T_{S_1}$$

It implies from the following proposition that we can extend this isometry to Mukai lattices.

3.8. Proposition [Ni] *Let $\phi_1, \phi_2 : T \longrightarrow H$ be two primitive embedding of a lattice T in an even unimodular lattice H . Assume that the orthogonal complement $N := \phi_1(T)^{\perp}$ in H contains the hyperbolic lattice $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a sublattice.*

Then ϕ_1 and ϕ_2 are equivalent, that means there exists an isometry γ of H such that $\phi_1 = \gamma\phi_2$.

We know that the orthogonal complement of T_S in Mukai lattice $\widetilde{H}(S,\mathbb{Z})$ is isomorphic to $N_S \perp U$. By Proposition 3.8. , there exists an isometry

$$f : \widetilde{H}(S_2,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S_1,\mathbb{Z})$$

such that $f\big|_{T_{S_2}} = f_{\mathcal{T}}$.

Put $v = f(0,0,1) = (r,l,s)$ and $u = f(1,0,0) = (p,k,q)$.

We may assume that $r > 1$. One may do this, because there are two types of isometries on Mukai lattice that are identity on the lattice of transcendental cycles. First type is multiplication by Chern character e^m of a line bundle:

$$\phi_m(r,l,s) = (r, l + rm, s + (m,l) + \frac{r}{2}m^2)$$

Second type is the change r and s (see [Mu2]). So we can change f in such a way that $r > 1$ and $f\big|_{T_{S_2}} = f_{\mathcal{T}}$.

First, note that vector $v \in U \perp -N_{S_1}$ is isotropic, i.e $(v,v) = 0$. It was proved by Mukai in his brilliant paper [Mu2] that there exists a polarization A on S_1 such that the moduli space $\mathcal{M}_A(v)$ of stable bundles with respect to A , for which vector Mukai is equal to v , is projective smooth K3 surface. Moreover, this moduli space is fine, because there exists the vector $u \in U \perp -N_{S_1}$ such that $(v,u) = 1$. Therefore we have a universal vector bundle $\mathcal{E}$ on the product $S_1 \times \mathcal{M}_A(v)$.

The universal bundle $\mathcal{E}$ gives the functor $\Phi_{\mathcal{E}} : D^b(\mathcal{M}_A(v)) \longrightarrow D^b(S_1)$.

Let us assume that $\Phi_{\mathcal{E}}$ is an equivalence of derived categories. In this case, the cycle $Z = \pi_{S_1}^* \sqrt{td_{S_1}} \cdot ch(\mathcal{E}) \cdot p^* \sqrt{td_{\mathcal{M}}}$ induces the Hodge isometry

$$g : \widetilde{H}(\mathcal{M}_A(v),\mathbb{Z}) \longrightarrow \widetilde{H}(S_1,\mathbb{Z}),$$

such that $g(0,0,1) = v = (r,l,s)$. Hence, $f^{-1} \circ g$ is an isometry too, and it sends $(0,0,1)$ to $(0,0,1)$. Therefore $f^{-1} \cdot g$ gives the Hodge isometry between the second cohomologies, because for a K3 surface S

$$H^2(S,\mathbb{Z}) = (0,0,1)^{\perp} \Big/ \mathbb{Z}(0,0,1).$$

Consequently, by the strong Torelli theorem (see [LP]), the surfaces S_2 and $\mathcal{M}_A(v)$ are isomorphic. Hence the derived categories of S_1 and S_2 are equivalent.

3.9. Thus, to conclude the proof of Theorem 3.3. , it remains to show that the functor $\Phi_{\mathcal{E}}$ is an equivalence.

First, we show that the functor $\Phi_{\mathcal{E}}$ is full and faithful. This is a special case of the following more general statement, proved in [BO1].

3.10. Theorem [BO1] *Let* M *and* X *be smooth algebraic varieties and* $E \in D^b(M \times X)$ *. Then* Φ_E *is fully faithful functor, iff the following orthogonality conditions are verified:*

i) $\mathrm{Hom}^i_X(\Phi_E(\mathcal{O}_{t_1}),\ \Phi_E(\mathcal{O}_{t_2})) = 0$ *for every* i *and* $t_1 \neq t_2$.

ii) $\mathrm{Hom}^0_X(\Phi_E(\mathcal{O}_t),\ \Phi_E(\mathcal{O}_t)) = k,$

$\mathrm{Hom}^i_X(\Phi_E(\mathcal{O}_t),\ \Phi_E(\mathcal{O}_t)) = 0,$ *for* $i \notin [0, dimM]$.

Here t , t_1 , t_2 *are points of* M , $\mathcal{O}_{t_i}$ *are corresponding skyscraper sheaves.*

In our case, $\Phi_{\mathcal{E}}(\mathcal{O}_t) = E_t$, where E_t is stable sheaf with respect to the polarization A on S_1 for which $v(E_t) = v$. All these sheaves are simple and $\mathrm{Ext}^i(E_t\ ,\ E_t) = 0$ for $i \notin [0,2]$. This implies that condition 2) of Theorem 3.10. is fulfilled.

All E_t are stable sheaves, hence $\mathrm{Hom}(E_{t_1}\ ,\ E_{t_2}) = 0$. Further, by Serre duality $\mathrm{Ext}^2(E_{t_1}\ ,\ E_{t_2}) = 0$. Finally, since the vector v is isotropic, we obtain $\mathrm{Ext}^1(E_{t_1}\ ,\ E_{t_2}) = 0$.

This yields that $\Phi_{\mathcal{E}}$ is full and faithful. As our situation is not symmetric (a priori), it is not clear whether the adjoint functor to $\Phi_{\mathcal{E}}$ is also full and faithful. Some additional reasoning is needed.

3.11. Theorem *In the above notations, the functor* $\Phi_{\mathcal{E}} : D^b(\mathcal{M}_A(v)) \longrightarrow D^b(S_1)$ *is an equivalence.*

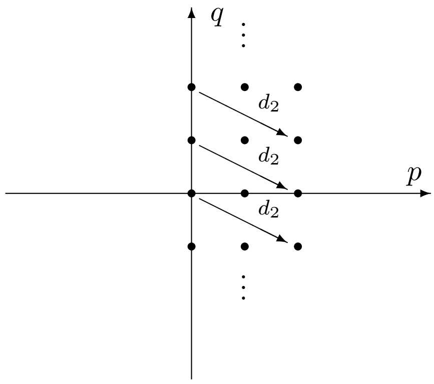
Proof. Assume the converse, i.e. $\Phi_{\mathcal{E}}$ is not an equivalence, then, since the functor $\Phi_{\mathcal{E}}$ is full and faithful, there exists an object $C \in D^b(S_1)$ such that $\Phi^*_{\mathcal{E}}(C) = 0$. By Proposition 3.5. , the functor $\Phi_{\mathcal{E}}$ induces the isometry f on the Mukai lattices, hence the Mukai vector $v(C)$ is equal to 0 .

Object C satisfies the conditions $\mathrm{Hom}^i(C\ ,\ E_t) = 0$ for every i and all $t \in \mathcal{M}_A(v)$, where E_t are stable bundles on S_1 with the Mukai vector v .

Denote by $H^i(C)$ the cohomology sheaves of the object C . There is a spectral sequence which converges to $\mathrm{Hom}^i(C\ ,\ E_t)$

$$E_2^{p,q} = \mathrm{Ext}^p(H^{-q}(C)\ ,\ E_t) \Longrightarrow \mathrm{Hom}^{p+q}(C\ ,\ E_t) \tag{11}$$

It is depicted in the following diagram



We can see that $\mathrm{Ext}^1(H^q(C)\ ,\ E_t) = 0$ for every q and all t , and every morphism d_2 is an isomorphism.

To prove the theorem, we need the following lemma.

3.12. Lemma *Let* G *be a sheaf on K3 surface* S_1 *such that* $\mathrm{Ext}^1(G\ ,\ E_t) = 0$ *for all* t *. Then there exists an exact sequence*

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 0$$

that satisfies the following conditions:

1) $\mathrm{Ext}^i(G_1\ ,\ E_t) = 0$ *for every* $i \neq 2,$ *and* $\mathrm{Ext}^2(G_1\ ,\ E_t) \cong \mathrm{Ext}^2(G\ ,\ E_t)$
2) $\mathrm{Ext}^i(G_2\ ,\ E_t) = 0$ *for every* $i \neq 0,$ *and* $\mathrm{Hom}(G_2\ ,\ E_t) \cong \mathrm{Hom}(G\ ,\ E_t)$

and $\mu_A(G_2) < \mu_A(G) < \mu_A(G_1)$.

Proof. Firstly, there is a short exact sequence

$$0 \longrightarrow T \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow 0,$$

where T is a torsion sheaf, and $\widetilde{G}$ is torsion free.

Secondly, there is a Harder-Narasimhan filtration $0 = I_0 \subset ... \subset I_n = \widetilde{G}$ for $\widetilde{G}$ such that the successive quotients I_i/I_{i-1} are A -semistable, and $\mu_A(I_i/I_{i-1}) > \mu_A(I_j/I_{j-1})$ for $i < j$.

Now, combining T and the members of the filtration for which $\mu_A(I_i/I_{i-1}) > \mu_A(E_t)$ (resp. $=$, $<$) to one, we obtain the 3-member filtration on G

$$0 = J_0 \subset J_1 \subset J_2 \subset J_3 = G.$$

Let K_i be the quotients sheaves J_i/J_{i-1} . We have

$$\mu_A(K_1) > \mu_A(K_2) = \mu_A(E_t) > \mu_A(K_3)$$

(we suppose, if needed, $\mu_A(T) = +\infty$).

Moreover, it follows from stability of E_t that

$$\mathrm{Hom}(K_1\,,\ E_t)=0 \qquad \text{and} \qquad \mathrm{Ext}^2(K_3\,,\ E_t)=0$$

Combining this with the assumption that $\mathrm{Ext}^1(G\,,\ E_t)=0$, we get $\mathrm{Ext}^1(K_2\,,\ E_t)=0$.

To prove the lemma it remains to show that $K_2=0$.

Note that K_2 is A -semistable. Hence there is a Jordan-Hölder filtration for K_2 such that the successive quotients are A -stable. The number of the quotients is finite. Therefore we can take t_0 such that

$$\mathrm{Hom}(K_2\,,\ E_{t_0})=0 \qquad \text{and} \qquad \mathrm{Ext}^2(K_2\,,\ E_{t_0})=0$$

Consequently, $\chi(v(K_2),v(E_t))=0$. Thus, as $\mathrm{Ext}^1(K_2\,,\ E_t)=0$ for all t , we obtain $\mathrm{Ext}^i(K_2\,,\ E_t)=0$ for every i and all t .

Further, let us consider $\Phi^*_{\mathcal{E}}(K_2)$. We have

$$\mathrm{Hom}^\bullet(\Phi^*_{\mathcal{E}}(K_2)\,,\ \mathcal{O}_t)\cong \mathrm{Hom}^\bullet(K_2\,,\ E_t)=0,$$

This implies $\Phi^*_{\mathcal{E}}(K_2)=0$. Hence $v(K_2)=0$, because f is an isometry. And, finally, $K_2=0$. The lemma is proved. □

Let us return to the theorem. The object C possesses at least two non-zero consequent cohomology sheaves $H^p(C)$ and $H^{p+1}(C)$. They satisfy the condition of Lemma 3.12. Hence there exist decompositions with conditions 1),2):

$$0\longrightarrow H_1^p\longrightarrow H^p(C)\longrightarrow H_2^p\longrightarrow 0 \quad \text{and} \quad 0\longrightarrow H_1^{p+1}\longrightarrow H^{p+1}(C)\longrightarrow H_2^{p+1}\longrightarrow 0$$

Now consider the canonical morphism $H^{p+1}(C)\longrightarrow H^p(C)[2]$. It induces the morphism $s: H_1^{p+1}\longrightarrow H_2^p[2]$. By Z denote a cone of s .

Since d_2 of the spectral sequence (11) is an isomorphism, we obtain

$$\mathrm{Hom}^\bullet(Z\,,\ E_t)=0 \qquad \text{for all } t.$$

Consequently, we have $\Phi^*_{\mathcal{E}}(Z)=0$. On the other hand, we know that $\mu_A(H_1^{p+1})>\mu_A(E_t)>\mu_A(H_2^p)$. Therefore $v(Z)\neq 0$. This contradiction proves the theorem. □

There exists the another version of Theorem 3.3.

3.13. Theorem *Let S_1 and S_2 be smooth projective K3 surfaces over field $\mathbb{C}$. Then the derived categories $D^b(S_1)$ and $D^b(S_2)$ are equivalent as triangulated categories iff there exists a Hodge isometry $f:\widetilde{H}(S_1,\mathbb{Z})\xrightarrow{\sim}\widetilde{H}(S_2,\mathbb{Z})$ between the Mukai lattices of S_1 and S_2 .*

Here the 'Hodge isometry' means that the one-dimensional subspace $H^{2,0}(S_1)\subset\widetilde{H}(S_1,\mathbb{Z})\otimes\mathbb{C}$ goes to $H^{2,0}(S_2)\subset\widetilde{H}(S_2,\mathbb{Z})\otimes\mathbb{C}$.

Appendix.

The facts, collected in this appendix, are not new; they are known. However, not having a good reference, we regard it necessary to give a proof for the statement, which is used in the main text. We exploit the technique from [IM].

Let X be a smooth projective variety and L be a very ample invertible sheaf on X such that $\mathrm{H}^i(X, L^k) = 0$ for any $k > 0$, when $i \neq 0$. Denote by A the coordinate algebra for X with respect to L, i.e. $A = \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X, L^k)$.

Now consider the variety X^n. First, we introduce some notations. Define subvarieties $\Delta^{(n)}_{(i_1,\dots,i_k)(i_{k+1},\dots,i_m)} \subset X^n$ by the following rule:

$$\Delta^{(n)}_{(i_1,\dots,i_k)(i_{k+1},\dots,i_m)} := \{(x_1,\dots,x_n) | x_{i_1} = \cdots = x_{i_k}; x_{i_{k+1}} = \cdots = x_m\}$$

By $S^{(n)}_i$ denote $\Delta^{(n)}_{(n,\dots,i)}$. It is clear that $S^{(n)}_i \cong X^i$.

Further, let $T^{(n)}_i := \bigcup_{k=1}^{i-1} \Delta^{(n)}_{(n,\dots,i)(k,k-1)}$ (note that $T^{(n)}_1$ and $T^{(n)}_2$ are empty) and let $\Sigma^{(n)} := \bigcup_{k=1}^{n} \Delta^{(n)}_{(k,k-1)}$. We see that $T^{(n)}_i \subset S^{(n)}_i$. Denote by $\mathcal{I}^{(n)}_i$ the kernel of the restriction map $\mathcal{O}_{S^{(n)}_i} \longrightarrow \mathcal{O}_{T^{(n)}_i} \longrightarrow 0$.

Using induction by n, it can easily be checked that the following complex on X^n

$$P^\bullet_n : 0 \longrightarrow J_{\Sigma^{(n)}} \longrightarrow \mathcal{I}^{(n)}_n \longrightarrow \mathcal{I}^{(n)}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{I}^{(n)}_2 \longrightarrow \mathcal{I}^{(n)}_1 \longrightarrow 0$$

is exact. (Note that $\mathcal{I}^{(n)}_1 = \mathcal{O}_{\Delta^{(n)}_{n,\dots,1}}$ and $\mathcal{I}^{(n)}_2 = \mathcal{O}_{\Delta^{(n)}_{n,\dots,2}}$). For example, for $n = 2$ this complex is a short exact sequence on $X \times X$:

$$P^\bullet_2 : 0 \longrightarrow J_\Delta \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

Denote by $\pi^{(n)}_i$ the projection of X^n onto i^{th} component, and by $\pi^{(n)}_{ij}$ denote the projection of X^n onto the product of i^{th} and j^{th} components.

Let $B_n := \mathrm{H}^0(X^n, J_{\Sigma^{(n)}} \otimes (L \boxtimes \cdots \boxtimes L))$ and let $R_{n-1} := R^0 \pi^{(n)}_{1*}(J_{\Sigma^{(n)}} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L))$.

Proposition A.1 *Let L be a very ample invertible sheaf on X as above. Suppose that for any m such that $1 < m \leq n + dim X + 2$ the following conditions hold:*

a) $\mathrm{H}^i(X^m, J_{\Sigma^{(m)}} \otimes (L \boxtimes \cdots \boxtimes L)) = 0$ *for* $i \neq 0$
b) $R^i \pi^{(m)}_{1*}(J_{\Sigma^{(m)}} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L)) = 0$ *for* $i \neq 0$
c) $R^i \pi^{(m)}_{1m*}(J_{\Sigma^{(m)}} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L \boxtimes \mathcal{O})) = 0$ *for* $i \neq 0$

Then we have:

1) algebra A is n-Koszul, i.e the sequence

$$B_n \otimes_k A \longrightarrow B_{n-1} \otimes_k A \longrightarrow \cdots \longrightarrow B_1 \otimes_k A \longrightarrow A \longrightarrow k \longrightarrow 0$$

is exact;

2) the following complexes on X:

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_{k-1} \otimes R_1 \longrightarrow A_k \otimes R_0 \longrightarrow \mathcal{L}^k \longrightarrow 0$$

are exact for any $k \geq 0$ (if $k - i < 0$, then $A_{k-i} = 0$ by definition);

3) the complex

$$L^{-n} \boxtimes R_n \longrightarrow \cdots \longrightarrow L^{-1} \boxtimes R_1 \longrightarrow \mathcal{O}_M \boxtimes R_0 \longrightarrow \mathcal{O}_\Delta$$

gives n-resolution of the diagonal on $X \times X$, i.e. it is exact.

Proof.

1) First, note that

$$\mathrm{H}^i(X^m, \mathcal{I}_k^{(m)} \otimes (L \boxtimes \cdots \boxtimes L)) = \mathrm{H}^i(X^{k-1}, J_{\Sigma(k-1)} \otimes (L \boxtimes \cdots \boxtimes L)) \otimes A_{m-k+1}$$

By condition a), they are trivial for $i \neq 0$.

Consider the complexes $P_m^\bullet \otimes (L \boxtimes \cdots \boxtimes L)$ for $m \leq n + dimX + 1$. Applying the functor H^0 to these complexes and using condition a), we get the exact sequences:

$$0 \longrightarrow B_m \longrightarrow B_{m-1} \otimes_k A_1 \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m-1} \longrightarrow A_m \longrightarrow 0$$

for $m \leq n + dimX + 1$.

Now put $m = n + dimX + 1$. Denote by $W_m^\bullet$ the complex

$$\mathcal{I}_m^{(m)} \longrightarrow \mathcal{I}_{m-1}^{(m)} \longrightarrow \cdots \longrightarrow \mathcal{I}_2^{(m)} \longrightarrow \mathcal{I}_1^{(m)} \longrightarrow 0$$

Take the complex $W_m^\bullet \otimes (L \boxtimes \cdots \boxtimes L \boxtimes L^i)$ and apply functor H^0 to it. We obtain the following sequence:

$$B_{m-1} \otimes_k A_i \longrightarrow B_{m-2} \otimes_k A_{i+1} \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m-1} \longrightarrow A_m \longrightarrow 0$$

The cohomologies of this sequence are $\mathrm{H}^j(X^m, J_{\Sigma(m)} \otimes (L \boxtimes \cdots \boxtimes L \boxtimes L^i))$. It follows from condition b) that

$$\mathrm{H}^j(X^m, J_{\Sigma(m)} \otimes (L \boxtimes \cdots \boxtimes L \boxtimes L^i)) = \mathrm{H}^j(X, R^0\pi_{m*}^{(m)}(J_{\Sigma(m)} \otimes (L \boxtimes \cdots \boxtimes L \boxtimes \mathcal{O})) \otimes L^i)$$

Hence they are trivial for $j > dimX$. Consequently, we have the exact sequences:

$$B_n \otimes_k A_{m-n+i-1} \longrightarrow B_{n-1} \otimes_k A_{m-n+i} \longrightarrow \cdots \longrightarrow B_1 \otimes_k A_{m+i-2} \longrightarrow A_{m+i-1}$$

for $i \geq 1$. And for $i \leq 1$ the exactness was proved above.

Thus, algebra A is n-Koszul.

2) The proof is the same as for 1). We have isomorphisms

$$R^i\pi_{1*}^{(m)}(\mathcal{I}_k^{(m)} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L)) \cong R^i\pi_{1*}^{(k-1)}(J_{\Sigma(k-1)} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L)) \otimes A_{m-k+1}$$

Applying functor $R^0\pi_{1*}^{(m)}$ to the complexes $P_m^\bullet \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L))$ for $m \leq n + dimX + 2$, we obtain the exact complexes on X

$$0 \longrightarrow R_{m-1} \longrightarrow A_1 \otimes R_{m-2} \longrightarrow \cdots \longrightarrow A_{m-2} \otimes R_1 \longrightarrow A_{m-1} \otimes R_0 \longrightarrow \mathcal{L}^{m-1} \longrightarrow 0$$

for $m \leq n + dimX + 2$.

Put $m = n + dimX + 2$. Applying functor $R^0\pi_{1*}^{(m)}$ to the complex $W_m^\bullet \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L \boxtimes L^i))$, we get the complex

$$A_i \otimes R_{m-2} \longrightarrow \cdots \longrightarrow A_{m+i-3} \otimes R_1 \longrightarrow A_{m+i-2} \otimes R_0 \longrightarrow \mathcal{L}^{m+i-2} \longrightarrow 0$$

The cohomologies of this complex are

$$R^j\pi_{1*}^{(m)}(J_{\Sigma(m)} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L \boxtimes L^i)) \cong R^j p_{1*}(R^0\pi_{1m*}^{(m)}(J_{\Sigma(m)} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L \boxtimes \mathcal{O})) \otimes (\mathcal{O} \boxtimes L^i))$$

They are trivial for $j > dimX$. Thus, the sequences

$$A_{k-n} \otimes R_n \longrightarrow A_{k-n+1} \otimes R_{n-1} \longrightarrow \cdots \longrightarrow A_{k-1} \otimes R_1 \longrightarrow A_k \otimes R_0 \longrightarrow \mathcal{L}^k \longrightarrow 0$$

are exact for all $k \geq 0$.

3) Consider the complex $W^{\bullet}_{n+2} \otimes (\mathcal{O} \boxtimes L \boxtimes \cdots \boxtimes L \boxtimes L^{-i})$. Applying the functor $R^0 \pi^{(n+2)}_{1(n+2)*}$ to it, we obtain the following complex on $X \times X$:

$$L^{-n} \boxtimes R_n \longrightarrow \cdots \longrightarrow L^{-1} \boxtimes R_1 \longrightarrow \mathcal{O}_M \boxtimes R_0 \longrightarrow \mathcal{O}_{\Delta}$$

By condition c), it is exact.

This finishes the proof.

Note that for any ample invertible sheaf L we can find j such that for the sheaf L^j the conditions a),b),c) are fulfilled.

References

[Ba] J. Backelin, *On the rates of growth of the homologies of Veronese subring*, Lecture Notes in Math., vol.1183, Springer-Verlag, 1985.

[Be] A. A. Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, Funkcionalniy analiz i ego pril., v.12, p.68-69 (1978) (in Russian).

[BBD] A. A. Beilinson, J. Bernstien, P. Deligne, *Faisceaux pervers*, Astérisque 100 (1982).

[BK] A. I. Bondal, M. M. Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser.Mat. 53 (1989), pp.1183-1205; English transl. in Math. USSR Izv. 35 (1990), pp.519-541.

[BO1] A. I. Bondal, D. O. Orlov, *Semiorthogonal decomposition for algebraic varieties*, Preprint MPI/95-15 (see also alg-geom9506012).

[BO2] A. I. Bondal, D. O. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Preprint MPI/97-36.

[GM] S. I. Gelfand, Yu. I. Manin, *Introduction to Cohomology Theory and Derived Categories*, Nauka, 1988 (in Russian).

[IM] S. P. Inamdar, V. B. Mehta, *Frobenius splitting of Schubert varieties and linear syzygies*, American Journal of Math., vol.116, N6 (1994), pp.1569-1586.

[LP] E. Looijenga, C. Peters, *Torelli theorem for Kähler K3 surfaces*, Compositio Math., v.58 (1981), pp.145-186.

[Mu1] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), pp.153–175.

[Mu2] S. Mukai, *On the moduli space of bundles on a K3 surface I*, Vector bundles on algebraic varieties, Tata Institute of Fundamental Research, Oxford University Press, Bombay and London, 1987.

[Ni] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, English trans., Math. USSR Izvestia, v.14 (1980), pp.103-167.

[V] J. L. Verdier, *Categories derivées*, in SGA 4 1/2, Lecture Notes in Math., v.569, Springer-Verlag, 1977.

Algebra section, Steklov Math. Institute, Vavilova 42, 117333 Moscow, RUSSIA

E-mail address: `orlov@genesis.mi.ras.ru`