LECTURES ON MIRROR SYMMETRY, DERIVED CATEGORIES, AND D-BRANES

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ABSTRACT. This paper is an introduction to Homological Mirror Symmetry, derived categories, and topological D-branes aimed mainly at a mathematical audience. In the paper we explain the physicists' viewpoint of the Mirror Phenomenon, its relation to derived categories, and the reason why it is necessary to enlarge the Fukaya category with coisotropic A-branes; we discuss how to extend the definition of Floer homology to such objects and describe mirror symmetry for flat tori. The paper consists of four lectures which were given at the Institute for Pure and Applied Mathematics (Los Angeles), March 2003, as part of a program on Symplectic Geometry and Physics.

1. MIRROR SYMMETRY FROM A PHYSICAL VIEWPOINT

The goal of the first lecture is to explain the physicists' viewpoint of the Mirror Phenomenon and its interpretation in mathematical terms proposed by Maxim Kontsevich in his 1994 talk at the International Congress of Mathematicians [25]. Another approach to Mirror Symmetry was proposed by A. Strominger, S-T. Yau, and E. Zaslow [41], but we will not discuss it here.

From the physi cal point of view, Mirror Symmetry is a relation on the set of 2d conformal field theories with $N = 2$ supersymmetry. A 2d conformal field theory is a rather complicated algebraic object whose definition will be sketched in a moment. Thus Mirror Symmetry originates in the realm of algebra. Geometry will appear later, when we specialize to a particular class of $N = 2$ superconformal field theories related to Calabi-Yau manifolds.

Let us start with 2d conformal field theory. The data needed to specify a 2d CFT consist of an infinite-dimensional vector space V (the space of states), three special elements in V (the vacuum vector $|vac\rangle$, and two more elements L and $\bar{L}$), and a linear map Y from V to the space of "formal fractional power series in $z, \bar{z}$ with coefficients in $\text{End}(V)$" (Y is called the state-operator correspondence). The precise definition of what a "formal fractional power series" means can be found in [19]; to keep things simple, one can pretend that Y takes values in the space of Laurent series in $z, \bar{z}$ with coefficients in $\text{End}(V)$, although such a definition is not sufficient for applications to Mirror Symmetry. These data must satisfy a number of axioms whose precise form can be found in [19]. Roughly speaking, they are

(i) $Y(|vac\rangle) = \text{id}_V$.

(ii) $Y(L) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad Y(\bar{L}) = \sum_{n \in \mathbb{Z}} \frac{\bar{L}_n}{\bar{z}^{n+2}}$ for some $L_n, \bar{L}_n \in \text{End}(V)$.

(iii) Both L_n and $\bar{L}_n$ satisfy the commutation relations of the Virasoro algebra, and all L_n commute with all $\bar{L}_m$.

(iv) $[L_{-1}, Y(v, z, \bar{z})] = \partial Y(v, z, \bar{z}), \quad [L_0, Y(v, z, \bar{z})] = z\partial Y(v, z, \bar{z}) + Y(L_0 v, z, \bar{z}),$ for any $v \in V$, and similar conditions obtained by replacing $L_n \to \bar{L}_n, \ \partial \to \bar{\partial}$.

(iv) $Y(v)|vac\rangle = e^{zL_{-1} + \bar{z}\bar{L}_{-1}} v$ for any $v \in V$.

(v) $Y(v, z, \bar{z})Y(v', z', \bar{z}')$ has only power-like singularities on the diagonals $z = z'$ and $\bar{z} = \bar{z}'$.

(vi) $Y(v, z, \bar{z})Y(v', z', \bar{z}') - Y(v', z', \bar{z}')Y(v, z, \bar{z})$ is a formal distribution supported on the diagonal.

Recall that the Virasoro algebra is an infinite-dimensional Lie algebra spanned by elements $L_m, m \in \mathbb{Z}$ and the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + c\frac{m^3 - m}{12}\delta_{m,-n}.$$

It is a unique central extension of the Witt algebra (the Lie algebra of vector fields on a circle). The constant c is called the central charge. The Virasoro algebras spanned by L_n and $\bar{L}_n$ are called right-moving and left-moving, respectively.

There are certain variations of this definition. The modification which we will need most amounts to replacing all spaces and maps by their $\mathbb{Z}/2$-graded versions, and the "commutativity" axiom (vi) with supercommutativity. From the physical viewpoint, this means that we allow both fermions and bosons in our theory. Another important property which must hold in any acceptable CFT is the existence of a non-degenerate bilinear form on V which is compatible, in a suitable sense, with the rest of the data. Finally, most CFTs of interest are "left-right symmetric." This means that exchanging z and $\bar{z}$, and L_n and $\bar{L}_n$, gives an isomorphic CFT. We will only consider left-right symmetric CFTs.

A more geometric approach to 2d CFT has been proposed by G. Segal [40]. In Segal's approach, one starts with a certain category whose objects are finite ordered sets of circles, and morphisms are Riemann surfaces with oriented and analytically parametrized boundaries. Composition of morphisms is defined by sewing Riemann surfaces along boundaries with compatible orientations. A 2d CFT is a projective functor from this category to the category of Hilbert spaces which satisfies certain properties which are listed in [40, 10]. (A projective functor from a category $\mathcal{C}$ to the category of Hilbert spaces is the same as a functor from $\mathcal{C}$ to a category whose objects are Hilbert spaces, and morphisms are equivalence classes of Hilbert space morphisms under the operation of multiplication by non-zero scalars.) One can show that any 2d CFT in the sense of Segal's definition gives rise to a 2d CFT in the sense of our algebraic definition (see e.g. [10]). For example, the vector space V which appears in our algebraic definition is the Hilbert space associated to a single circle in Segal's approach. The map Y comes from considering the morphism which corresponds to a Riemann sphere with three holes. Conversely, it appears that any "algebraic" 2d CFT which is left-right symmetric and is equipped with a compatible inner product gives rise to a "geometric" 2d CFT in genus zero (i.e. with Riemann surfaces restricted to have genus zero).

An $N = 1$ super-Virasoro algebra is a certain infinite-dimensional Lie super-algebra which contains the ordinary Virasoro as a subalgebra. Apart from the Virasoro generators $L_n, n \in \mathbb{Z}$, it contains odd generators $Q_n, n \in \mathbb{Z}$. The additional commutation relations read

$$[L_m, Q_n] = \left(\frac{m}{2} - n\right) Q_{m+n}, \quad [Q_m, Q_n] = \frac{1}{2}L_{m+n} + \frac{c}{12}m^2\delta_{m,-n}.$$

An $N = 1$ superconformal field theory (SCFT) is a 2d CFT with an action on V of two copies of the $N = 1$ super-Virasoro algebra which is compatible with other structures of the SCFT in a fairly obvious sense. [1]

An $N = 2$ super-Virasoro algebra is a further generalization of the Virasoro algebra. It is a certain infinite-dimensional Lie super-algebra which contains the $N = 1$ super-Virasoro as a subalgebra. The even generators are $L_n, J_n, \ n \in \mathbb{Z}$. The odd generators are $Q_n^\pm, n \in \mathbb{Z}$. The commutators read, schematically:

(1) $\quad [L, L] \sim L, \qquad\qquad [J, J] \sim central, \quad [L, J] \sim J, \qquad\qquad\qquad\qquad [Q^\pm, Q^\pm] = 0,$

(2) $\quad [L, Q^\pm] \sim Q^\pm, \qquad [J, Q^\pm] \sim \pm Q^\pm, \quad [Q^\pm, Q^\mp] \sim J + L + central.$

The precise form of the commutation relations ca be found in [19]. The $N = 1$ super-Virasoro subalgebra is spanned by L_n and $Q_n = Q_n^+ + Q_n^-$. The relation between $N = 1$ and $N = 2$ super-Virasoro is analogous to the relation between the de Rham and Dolbeault differentials on a complex manifold: Q_n are analogous to d, while Q_n^+ and Q_n^- are analogous to ∂ and $\bar{\partial}$.

An $N = 2$ SCFT is an $N = 1$ SCFT with an action of two copies of $N = 2$ super-Virasoro algebra which is compatible with the remaining structures of the SCFT. Thus we have a hierarchy of algebraic structures:

$$\text{Set of all CFTs} \supset \text{Set of all N} = 1 \text{ SCFTs} \supset \text{Set of all N} = 2 \text{ SCFTs}$$

In fact, there is an even more general notion: 2d quantum field theory, without the adjective "conformal." We will not discuss it in these lectures.

It is possible to give a definition of $N = 1$ and $N = 2$ superconformal field theories à la Segal. The role of Riemann surfaces is played by 2d supermanifolds equipped with $N = 1$ or $N = 2$ superconformal structure.

An isomorphism of (super-)conformal field theories is a 1-1 map $V \xrightarrow{\sim} V'$ which preserves all the relevant structures. Two $N = 2$ superconformal field theories can be isomorphic as $N = 1$ superconformal field theories without being isomorphic as $N = 2$ superconformal field theories. (When physicists say that two (super-)conformal field theories are "the same", they often neglect to specify which structures are preserved by the isomorphism; this is usually clear from the context.)

$N = 2$ super-Virasoro algebra has an interesting automorphism called the mirror automorphism:

$$\mathfrak{M} : L_n \mapsto L_n, \quad J_n \mapsto -J_n, \quad Q_n^\pm \mapsto Q_n^\mp.$$

Suppose we have a pair of $N = 2$ superconformal field theories which are isomorphic as $N = 1$ SCFTs. Let $f : V \xrightarrow{\sim} V'$ be an isomorphism. We say that f is a (right) mirror morphism of $N = 2$ CFTs if it acts as the identity on the "left-moving" $N = 2$

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[1]Strictly speaking, we are talking about the Ramond-Ramond sector of the SCFT here.

super-Virasoro, and acts by the mirror automorphism on the "right-moving" $N = 2$ super-Virasoro. This makes sense because the mirror automorphism acts as the identity on the $N = 1$ super-Virasoro subalgebra of $N = 2$ super-Virasoro algebra. Exchanging left and right, we get the notion of a left mirror morphism of $N = 2$ CFTs. Finally, if f acts as the mirror automorphisms on both left and right super-Virasoro algebras, we will say that f is a target-space complex conjugation.

By definition, two $N = 2$ SCFTs are mirror to each other if there is a (left or right) mirror morphism between them. Clearly, if two $N = 2$ SCFTs are both (left-) mirror to a third $N = 2$ SCFT, then the first two SCFTs are isomorphic (as $N = 2$ SCFTs). Thus the mirror relation (for example, left) is an involutive relation on the set of isomorphism classes of $N = 2$ SCFTs. We stress that if two $N = 2$ SCFTs are mirror to each other, then they are isomorphic as $N = 1$ SCFTs, but usually not as $N = 2$ SCFTs. Many explicit examples of mirror pairs of $N = 2$ SCFTs have been constructed in the physics literature; one can construct more complicated examples by means of tensor product, orbifolding, etc.

Now let us turn to the relation between $N = 2$ SCFTs and Calabi-Yau manifolds. A physicist's Calabi-Yau is a compact complex manifold with a trivial canonical class equipped with a Kähler class and a B-field (an element of $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$). It is believed that to any physicist's Calabi-Yau one can attach, in a natural way, an $N = 2$ SCFT which depends on these geometric data. One can give the following heuristic argument supporting the claim. First of all, to any physicist's Calabi-Yau one can naturally attach a classical field theory called the $N = 2$ sigma-model. Its Lagrangian is given by an explicit, although rather complicated, formula (see e.g. [36]). Infinitesimal symmetries of this classical field theory include two copies of $N = 2$ super-Virasoro algebra (with zero central charge). Second, one can try to quantize this classical field theory while preserving $N = 2$ superconformal invariance (up to an unavoidable central extenstion). The result of the quantization should be an $N = 2$ SCFT.

Except for a few special cases, it is not known how to quantize the sigma-model exactly. On the other hand, one has a perturbative quantization procedure which works when the volume of the Calabi-Yau is large. That is, if one rescales the metric by a parameter $t \gg 1$, $g_{\mu\nu} \to t^2 g_{\mu\nu}$, and considers the limit $t \to \infty$ (so called large volume limit), then one can quantize the sigma-model order by order in $1/t$ expansion. It is believed that the resulting power series in $1/t$ has a non-zero radius of convergence, and defines an actual $N = 2$ SCFT.

It is natural to ask if it is possible to reconstruct a Calabi-Yau starting from an $N = 2$ SCFT; as we will discuss shortly, the reconstruction problem does not have a unique answer. However, some numerical characteristics of the "parent" Calabi-Yau X can be determined rather easily. For example, the complex dimension of X is given by $c/3$, where c is the central charge of the $N = 2$ super-Virasoro algebra. One can also determine the Hodge numbers $h^{p,q}(X)$ in the following manner. Commutation relations of the $N = 2$ super-Virasoro imply that the operator $D_B = Q_0^+ + \bar{Q}_0^+$ on V squares to zero. D_B is known as a BRST operator (of type B, see below), and its cohomology is called the BRST cohomology. The BRST cohomology $\mathrm{Ker}\ D_B/\mathrm{Im}\ D_B$ is finite-dimensional in any reasonable $N = 2$

SCFT. It is graded by the eigenvalues of the operators J_0 and $\bar{J}_0$ known as left and right-moving R-charges. The Hodge number $h^{p,q}(X)$ is simply the dimension of the component with R-charges $p - \frac{n}{2}$ and $q - \frac{n}{2}$, where $n = \dim_{\mathbb{C}} X$. This means, incidentally, that not every $N = 2$ SCFT arises from a Calabi-Yau manifold: those which do, must have integral $n = c/3$ and integral spectrum of $J_0 + \frac{n}{2}$ and $\bar{J}_0 + \frac{n}{2}$ in BRST cohomology. It is believed that any $N = 2$ SCFT with integral $c/3$ and integral spectrum of $J_0 + \frac{n}{2}$ and $\bar{J}_0 + \frac{n}{2}$ is related to some Calabi-Yau manifold, if one allows certain kinds of singular Calabi-Yau manifolds, such as orbifolds.

One can show that if two Calabi-Yau are complex-conjugate, then the corresponding $N = 2$ SCFTs are related by target-space conjugation (in the sense explained above). This explains why the name "target-space complex conjugation" was attached to a particular kind of morphisms of $N = 2$ SCFTs. On the other hand, if two Calabi-Yau manifolds have $N = 2$ SCFTs related by target-space complex conjugation, this does not imply that the Calabi-Yau manifolds themselves are complex-conjugate; it merely implies that their $N = 2$ SCFTs are related in a simple way. Thus one obvious question is

Question 1. When do two Calabi-Yau manifolds produce isomorphic $N = 2$ SCFTs?

An answer to this question would interpret "quantum symmetries" of 2d SCFTs in geometric terms. Another question of this kind is

Question 2. When do two Calabi-Yau manifolds produce $N = 2$ SCFTs which are mirror to each other?

We say that two Calabi-Yau manifolds are related by mirror symmetry if the corresponding $N = 2$ SCFTs are mirror.

First non-trivial examples of mirror pairs of Calabi-Yau manifolds have been constructed by B. Greene and R. Plesser [14]. The simplest example in complex dimension three (this dimension is the most interesting one from the physical viewpoint) is the following: one of the Calabi-Yau manifolds is the Fermat quintic $x^5 + y^5 + z^5 + v^5 + w^5 = 0$ in $\mathbb{CP}^4$, while the other one is obtained by taking a quotient of the Fermat quintic by a certain action of $(\mathbb{Z}/5)^3$ and blowing up the fixed points. We will not try to explain why these two Calabi-Yau manifolds are mirror. (The original argument [14] relied on a conjectural equivalence between the $N = 2$ SCFT corresponding to the Fermat quintic and a certain integrable $N = 2$ SCFT constructed by D. Gepner. Later this issue has been studied in detail by E. Witten [44] and now has the status of a physical "theorem.")

The answer to the first question is highly non-trivial. This can be seen already in the case when X is a complex torus with a flat metric (Lecture 2). For example, the torus and its dual give the same $N = 2$ SCFT, even though they are usually not isomorphic as complex manifolds. The answer to Question 2 – characterization of the mirror relation in geometric terms – is the ultimate goal of the Mirror Symmetry program.

On the most basic level, a mirror relation between X and X' implies a relation between their Hodge numbers $h^{p,q}(X)$ and $h^{p,q}(X')$. To see how this comes about, note that along with the cohomology of $D_B = Q_0^+ + \bar{Q}_0^+$ we may also consider the cohomology of $D_A = Q_0^- + \bar{Q}_0^+$. It can be shown that in any $N = 2$ SCFT satisfying the integrality constraint these two cohomologies are isomorphic as bi-graded vector spaces [29]. Now note that if X is mirror to X', then the cohomology of $D_A(X)$ in bi-degree (α, β) is

isomorphic to cohomology of $D_B(X')$ in bi-degree $(-\alpha, \beta)$. Recalling the relation between the Hodge numbers of X and cohomology of $D_B(X)$, we infer an important relation

$$(3) \qquad\qquad h^{p,q}(X) = h^{n-p,q}(X').$$

If one plots the Hodge numbers of a complex manifold on a plane with coordinates $p - q$ and $p + q - n$, the resulting table has the shape of a diamond (the Hodge diamond). For any Calabi-Yau manifold the Hodge diamond is unchanged by a rotation by $180°$ degrees. The relation (3) means that the Hodge diamonds of mirror Calabi-Yau manifolds are related by a rotation by $90°$ degrees.

Of course, the existence of a mirror relation between X and X' implies much more than this. The most promising approach to the problem of characterizing the mirror relation in geometric terms has been proposed in 1994 by M. Kontsevich. In the remainder of this lecture we will sketch Kontsevich's proposal and its interpretation in physical terms.

A physicist's Calabi-Yau has both a complex structure and a symplectic structure (the Kähler form). One can gain a considerable insight into the Mirror Symmetry Phenomenon by focusing one of the two structures. More precisely, if the B-field is present, we combine the Kähler form ω and the $(1, 1)$ part of the B-field into a "complexified Kähler form." We will regard the latter as parametrizing an "extended symplectic moduli space" of X. Similarly, we regard the $(0, 2)$ part of the B-field and the complex structure moduli as parametrizing an "extended complex structure moduli space" of X. We would like to isolate some aspects of the $N = 2$ SCFT which depend either only on the extended complex moduli, or only on the extended symplectic moduli. The procedure for doing this was proposed by E. Witten [45, 46] and is known as topological twisting.

Witten's construction rests on the observation that many $N = 2$ SCFTs have finite-dimensional sectors which are topological field theories, i.e. do not depend on the 2d metric (the metric on the world-sheet, if we use string theory terminology.) In fact, for many $N = 2$ SCFTs there are two such sub-theories ; they are known as A- and B-models. $N = 2$ SCFTs related to Calabi-Yau manifolds belong precisely to this class of theories.

Let us recall some basic facts about 2d topological field theories. These theories are similar to, but much simpler than, 2d CFTs. They can be described by axioms similar to Segal's axioms [2]. One starts with a category whose objects are finite ordered sets of oriented and parametrized circles and morphisms are oriented 2d manifolds (without complex structure) bounding the circles. A 2d topological field theory is a functor from this category to the category of finite-dimensional (graded) vector spaces which satisfies certain requirements similar to Segal's. As for 2d CFTs, there is a purely algebraic reformulation of this definition. It turns out that the "topological" counterpart of the notion of a conformal field theory is the well-known notion of a super-commutative Frobenius algebra, i.e. a super-commutative algebra with an invariant metric (see e.g. [6]).

A detailed discussion of Witten's procedure for constructing a 2d TFT out of an $N = 2$ SCFT is beyond the scope of these lectures. Roughly speaking, one passes from the space V to its BRST cohomology. One can show that the state-operator correspondence Y descends to a super-commutative algebra structure on the BRST cohomology. The invariant metric on BRST cohomology comes from a metric on V.

Note that we have two essentially different choices of a BRST operator: D_A or D_B. (One can also consider $D'_A = Q_0^+ + \bar{Q}_0^-$ and $D'_B = Q_0^- + \bar{Q}_0^-$, but these can be trivially related to D_A and D_B by replacing X with its complex-conjugate.) Thus Witten's construction associates to any physicist's Calabi-Yau X two topological field theories, called the A-model and the B-model, respectively.

It turns out that the A-model does not change as one varies the extended complex structure moduli, while the B-model does not depend on the extended symplectic moduli. In other words, the A-model isolates the symplectic aspects of the Calabi-Yau, while the B-models isolates the complex ones. In fact, the state space of the A-model is naturally isomorphic to the de Rham cohomology $H^{*,*}(X)$, while the state space of the B-model is naturally isomorphic to the Dolbeault cohomology

$$H^*(\Lambda^* T^{1,0}X),$$

where $T^{1,0}X$ is the holomorphic tangent bundle of X. For a Calabi-Yau manifold, $H^q(\Lambda^p T^{1,0}X)$ is isomorphic to $H^{n-p,q}(X)$, but not canonically: the isomorphism depends on the choice of a holomorphic section of the canonical line bundle. Note also that the spaces of the A and B-models are bi-graded. From the physical point of view, the bi-grading comes from the decomposition of the state spaces into the eigenspaces of J_0 and $\bar{J}_0$.

The algebra structure in the B-case is the obvious one, while in the A-case it is a deformation of the obvious one which depends on the extended symplectic structure on X; $H^*(X)$ equipped with this deformed algebra structure is known as the quantum cohomology ring of X.

Mirror symmetry acts on A and B-models in a very simple way. It is easy to see that the mirror automorphism exchanges D_A and D_B. Thus if X and X' are a mirror pair of physicist's Calabi-Yau manifolds, then the A-model of X is isomorphic to the B-model of X', and vice versa. We will say that X and X' are weakly mirror if the A-model of X is isomorphic to the B-model of X', and vice-versa. The notion of weak mirror symmetry is easier to work with, since one can define A and B-models of a Calabi-Yau directly, without appealing to the ill-defined quantization of the sigma-model. But clearly a lot of information is lost in the course of topological twisting, and one would like to find some richer objects associated to an $N = 2$ SCFT.

M. Kontsevich proposed that a suitable enriched version of the B-model is the bounded derived category of coherent sheaves on X, which we will denote $\mathbf{D}^b(X)$, while the enriched version of the A-model is some version of the derived Fukaya category of X, which will be denoted $\mathbf{D}\mathcal{F}_0(X)$. In other words, he conjectured that if X and X' are mirror, then $\mathbf{D}^b(X)$ is equivalent to $\mathbf{D}\mathcal{F}_0(X')$ and vice versa. This is known as the Homological Mirror Conjecture (HMC). If the converse statement were true, this would completely answer Question 1 and 2.

Let us sketch the definitions of these two categories. Let X be a smooth complex manifold. An object of the bounded derived category on X is a bounded complex of holomorphic vector bundles on X, i.e. a finite sequence of holomorphic vector bundles and morphisms between them

$$0 \to \cdots \to E_{n-1} \to E_n \to E_{n+1} \to \cdots \to 0,$$

so that the composition of any two successive morphisms is zero. We remind the reader that the cohomology of such a complex is a sequence of coherent sheaves on X. To define morphisms in the derived category, we first consider the category of bounded complexes $\mathbf{C}^b(X)$, where morphisms are defined as chain maps between complexes. A morphism in this category is called a quasi-isomorphism if it induces an isomorphism on the cohomology of complexes. The idea of the derived category is to identify all quasi-isomorphic complexes. That is, the bounded derived category $\mathbf{D}^b(X)$ is obtained from $\mathbf{C}^b(X)$ by formally inverting all quasi-isomorphisms. In this definition, one can replace holomorphic vector bundles by arbitrary coherent sheaves; the resulting derived category is unchanged. Lecture 3 will discuss derived categories in more detail.

While coherent sheaves and their complexes are very familiar creatures and are the basic tool of algebraic geometry, the derived Fukaya category $\mathbf{D}\mathcal{F}_0(X)$ is a much more recent invention. It is obtained by a rather complicated algebraic procedure from a certain geometrically defined category called the Fukaya category $\mathcal{F}(X)$. The latter has been introduced by K. Fukaya in [8]. Actually, $\mathcal{F}(X)$ is not quite a category: there are additional structures on morphisms (multiple compositions and the differential), and the composition of morphisms is associative only up to a chain homotopy. Such a structure is called an A_∞ category. The Fukaya category depends only on the extended symplectic structure on X. Objects of the Fukaya category are, roughly speaking, triples (Y, E, ∇), where Y is a Lagrangian submanifold of X, E is a complex vector bundle on Y with a Hermitian metric, and ∇ is a flat unitary connection on E. This definition is only approximate, for several reasons. First of all, not every Lagrangian submanifold Y is allowed: the so-called Maslov class of Y must vanish (the Maslov class is a class in $H^1(Y, \mathbb{Z})$). Second, Y has to be a graded Lagrangian submanifold (this notion was defined in 1968 by J. Leray for the case of Lagrangian submanifolds in a symplectic vector space, and generalized by Kontsevich to the Calabi-Yau case). Third, it is not completely clear if the flat connection ∇ has to be unitary. There are some indications that it might be necessary to relax this condition and require instead the eigenvalues of the monodromy representation to have unit absolute value. Fourth, in the presence of the B-field ∇ must be projectively flat rather than flat [19].

The space of morphisms in the Fukaya category is defined by means of the Floer complex. This will be discussed in Lecture 2.

The relation between the Homological Mirror Conjecture and the A and B-models is the following [25]. Given a triangulated category (more precisely, an "enhanced triangulated category" [3]), one can study its deformations. Information about deformations is encoded in the Hochschild cohomology of the category in question. In the case of the derived category of coherent sheaves, Hochschild cohomology seems to coincide with the cohomology of the exterior algebra of the holomorphic tangent bundle, i.e. the state space of the B-model.[2] Kontsevich conjectured that the Hochschild cohomology of the derived Fukaya category is the quantum cohomology ring of X, i.e. the state space of the A-model [25]. Thus the equivalence of $\mathbf{D}^b(X)$ and $\mathbf{D}\mathcal{F}_0(X')$ is likely to imply that the B-model of X is isomorphic

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[2]We say "seems", because there is no complete proof of this yet.

(as a 2d TFT) to the A-model of X'. In other words, Homological Mirror Symmetry seems to imply weak mirror symmetry.

Homological Mirror Symmetry Conjecture also has a clear physical meaning. The physical idea is to generalize the notion of a 2d TFT to allow the 2d world-sheet to have boundaries (see e.g. [47]). This generalization also makes sense in the full $N = 2$ SCFT and leads to the notion of a D-brane, which plays a very important role in string theory [37]. A D-brane is a nice boundary condition for the SCFT. It is not completely clear what this means in the quantum case, so let us discuss this notion using classical field theory. A classical 2d field theory is defined by an action which is an integral of a local Lagrangian over the 2d world-sheet. So far we took the world-sheet to be a cylinder whose noncompact direction was parametrized by the "time" variable. Thus the space was topologically a circle. Now let us take the space to be an interval I. The world-sheet becomes $\mathbb{R} \times I$. In order for the classical field theory to be well-defined, we require the the Cauchy problem for the Euler-Lagrange equations to have a unique solution, at least locally. This requires imposing a suitable boundary condition on the fields and their derivatives on the boundary of the world-sheet. For example, one can impose Dirichlet boundary conditions (i.e. vanishing) on some scalar fields which appear in the Lagrangian.[3] A classical D-brane is simply a choice of such a boundary condition.

If the classical field theory has some symmetries, it is reasonable to require the boundary condition to preserve this symmetry. For example, $N = 1$ sigma-models have two copies of $N = 1$ super-Virasoro algebra as their symmetries. It is not possible to preserve both of them, but there exist many boundary conditions which preserve the diagonal subalgebra. Such boundary conditions are ordinary D-branes of superstring theory [37]. In the $N = 2$ case, we have two copies of $N = 2$ super-Virasoro, and we may require the boundary condition to preserve the diagonal $N = 2$ super-Virasoro. Such boundary conditions are called D-branes of type B, or simply B-branes, because they are related to the B-model (see below). One can also exploit the existence of the mirror automorphism $\mathfrak{M}$ and consider boundary conditions which preserve a different $N = 2$ super-Virasoro subalgebra, namely the one spanned by

$$L_n + \bar{L}_n, \quad -J_n + \bar{J}_n, \quad Q_n^- + \bar{Q}_n^+, \quad Q_n^+ + \bar{Q}_n^-, \quad n \in \mathbb{Z}.$$

The corresponding branes are called D-branes of type A, or simply A-branes.

Given a classical D-brane, we can try to quantize a classical field theory on $\mathbb{R} \times I$ with boundaries, which related to this D-brane, so that the quantized theory has one copy of $N = 2$ super-Virasoro as its symmetry algebra. If such a quantization is possible, we say that the classical D-brane is quantizable, and the classical D-brane together with its quantization will be called a quantum D-brane. This is not a very satisfactory way to define quantum D-branes, and it remains an interesting problem to find a satisfactory and fully quantum definition of a boundary condition for a 2d SCFT.

Now let us turn to the relation of A and B-branes with A and B-models. A and B-models are obtained from the $N = 2$ SCFT by topological twisting. Roughly speaking, twisting amounts to truncating the theory to the cohomology of D_A or D_B. Now note

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[3]The letter D in the word "D-brane" actually came from "Dirichlet."

that D_A (resp. D_B) sits in the $N = 2$ super-Virasoro which is preserved by the A-type (resp. B-type) boundary condition. The significance of this is that the A-twist is consistent with A-type boundary conditions, while the B-twist is consistent with B-type boundary conditions. Thus an A-brane (resp. B-brane) gives rise to a consistent boundary condition for the A-model (resp. B-model).[4]

One can show that the set of A-branes (or B-branes) has the structure of a category. The space of morphisms between two branes A and A' is simply the space of states of the 2d TFT on $\mathbb{R} \times I$, with boundary conditions on the two ends corresponding to A and A'. Equivalently, one considers the state space of the $N = 2$ SCFT on an interval, and computes its BRST cohomology with respect to D_A. The composition of morphisms can be defined with the help of the state-operator correspondence Y (or rather, its analogue in the case of a 2d SCFT with boundaries).

To summarize, to any physicist's Calabi-Yau we can attach two categories: the categories of A-branes and B-branes. One can argue that the category of A-branes (resp. B-branes) does not depend on the extended complex (resp. extended symplectic) moduli [47]. One can think of these categories as the enriched versions of the A and B-models: while the A-model is a TFT on a world-sheet without boundaries, the totality of A-branes corresponds to the same 2d TFT on a world-sheet with boundaries and with all possible boundary conditions. Further, it is obvious that if two Calabi-Yau manifolds are related by a mirror morphism, then the A-brane category of the first manifold is equivalent to the B-brane category of the second one, and vice versa. Obviously, if two $N = 2$ SCFTs related two Calabi-Yau manifolds are isomorphic, then the corresponding categories of A-branes (and B-branes) are simply equivalent.

The Homological Mirror Conjecture would follow if we could prove that the category of A-branes (resp. B-branes) is equivalent to $\mathbf{D}\mathcal{F}_0(X)$ (resp. $\mathbf{D}^b(X)$). Alas, we cannot hope to prove this, because we do not have an honest definition of a D-brane. What we do know is that holomorphic vector bundles are examples of B-branes, and objects of the Fukaya category are examples of A-branes [47]. Furthermore, Witten showed that in this special case morphisms in the category of B-branes and A-branes agree with morphisms in the categories $\mathbf{D}^b(X)$ and $\mathbf{D}\mathcal{F}_0(X)$, respectively [47]. This computation served as a motivation for Kontsevich's conjecture. More recently it was shown that more general coherent sheaves, as well as complexes of coherent sheaves, are also valid B-branes. On the other hand, it has been shown recently that the Fukaya category is only a full subcategory of the category of A-branes, that is, for some X there exist A-branes which are not isomorphic to any object of $\mathbf{D}\mathcal{F}_0(X)$ [20]. This means that the symplectic side of Kontsevich's conjecture needs substantial modification. This will be discussed in more detail in Lecture 4.

From the mathematical point of view, the Homological Mirror Conjecture is not well-defined, because it is not clear how to quantize the sigma-model for an arbitrary Calabi-Yau manifold. But there is a class of Calabi-Yaus for which the sigma-model can be quantized, and the corresponding $N = 2$ SCFTs can be described quite explicitly. These are complex tori with a flat Kähler metric and arbitrary B-field. One can easily determine which pairs

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[4]Axioms for boundary conditions in 2d TFTs have been discussed by G. Moore and G. Segal [31] and C.I. Lazaroiu [27].

of such Calabi-Yaus give mirror $N = 2$ SCFTs; the resulting criterion can be expressed in terms of linear algebra [19]. (This will be discussed in Lecture 2). Thus in the case of flat tori the Homological Mirror Conjecture is mathematically well-defined (modulo the issues related to the precise definition of the Fukaya category). In [39] A. Polishchuk and E. Zaslow proved it for tori of real dimension two. But there are strong arguments showing that for higher-dimensional tori the Homological Mirror Conjecture cannot hold, unless one substantially enlarges the Fukaya category by adding new objects. This will be discussed in Lecture 4.

For more general Calabi-Yaus, one can take the Homological Mirror Conjecture as an attempt to give a *mathematical definition* of the mirror relation. Then the main issues are the precise definition of the Fukaya category, and the verification that the numerous mirror pairs proposed by physicists and mathematicians are in fact mirror in the sense of the Homological Mirror Conjecture.

Another approach to mirror symmetry has been proposed in [41] and is known as the SYZ Conjecture. According to this conjecture, mirror Calabi-Yau manifolds admit fibrations by special Lagrangian tori which in some sense are "dual" to each other. Recently a relation between the Homological Mirror Conjecture and the SYZ Conjecture has been studied in the paper [26].

2. Mirror Symmetry for Flat Complex Tori

In this lecture we describe $N = 2$ superconformal field theories (SCFT) related to complex tori T endowed with a flat Kähler metric G and a constant 2-form B (the B-field). We will give a criterion when two such data (T, G, B) and (T', G', B') produce isomorphic $N = 2$ SCFT and when they produce $N = 2$ SCFT which are mirror to each other.

As explained in Lecture 1, to define an $N = 2$ superconformal field theory we need to specify an infinite-dimensional $\mathbb{Z}_2$-graded vector space of states V, a vacuum vector $|vac\rangle$, a state-operator correspondence Y from V to the space of "formal fraction power series in $z, \bar{z}$ with coefficients in $\operatorname{End}(V)$" and, finally, the super-Virasoro elements $L, \bar{L}, Q^{\pm}, \bar{Q}^{\pm}, J, \bar{J}$ which enter into the definition of the superconformal structure (see Lecture 1).

We start with some notations. Let $\Gamma \cong \mathbb{Z}^{2d}$ be a lattice in a real vector space U of dimension $2d$, and let $\Gamma^* \subset U^*$ be the dual lattice. Consider real tori $T = U/\Gamma$, $T^* = U^*/\Gamma^*$. Let G be a metric on U, i.e. a positive symmetric bilinear form on U, and let B be a real skew-symmetric bilinear form on U. Denote by l the natural pairing $\Gamma \times \Gamma^* \to \mathbb{Z}$. (The natural pairing $U \times U^* \to \mathbb{R}$ will be also denoted as l.) Choose generators $e_1, \ldots, e_{2d} \in \Gamma$. The components of an element $w \in \Gamma$ in this basis will be denoted by $w^i, i = 1, \ldots, 2d$. The components of an element $m \in \Gamma^*$ in the dual basis will be denoted by $m_i, i = 1, \ldots, 2d$. We also denote by G_{ij}, B_{ij} the components of G, B in this basis. It will be apparent that the superconformal field theory which we construct does not depend on the choice of basis in Γ. In the physics literature Γ is sometimes referred to as the winding lattice, while Γ^* is called the momentum lattice.

Consider a triple (T, G, B). With any such triple we associate an $N = 2$ superconformal field theory $\mathcal{V}$ which may be regarded as a quantization of the supersymmetric sigma-model.

The state space of the SCFT $\mathcal{V}$ is

$$V = \mathcal{H}_b \otimes_{\mathbb{C}} \mathcal{H}_f \otimes_{\mathbb{C}} \mathbb{C} \left[\Gamma \oplus \Gamma^* \right].$$

Here $\mathcal{H}_b$ and $\mathcal{H}_f$ are bosonic and fermionic Fock spaces defined below, while $\mathbb{C} \left[\Gamma \oplus \Gamma^* \right]$ is the space of the group algebra of $\Gamma \oplus \Gamma^*$ over $\mathbb{C}$.

To define $\mathcal{H}_b$, consider an algebra over $\mathbb{C}$ with generators $\alpha^i_s, \bar{\alpha}^i_s, \ i = 1, \ldots, 2d, s \in \mathbb{Z}\backslash 0$ and relations

(4) $$[\alpha^i_s, \alpha^j_p] = s\left(G^{-1}\right)^{ij} \delta_{s,-p}, \quad [\bar{\alpha}^i_s, \bar{\alpha}^j_p] = s\left(G^{-1}\right)^{ij} \delta_{s,-p}, \quad [\alpha^i_s, \bar{\alpha}^j_p] = 0.$$

If s is a positive integer, α^i_{-s} and $\bar{\alpha}^i_{-s}$ are called left and right bosonic creators, respectively, otherwise they are called left and right bosonic annihilators. Either creators or annihilators are referred to as oscillators.

The space $\mathcal{H}_b$ is defined as the space of polynomials of even variables $a^i_{-s}, \bar{a}^i_{-s}, i = 1, \ldots, 2d, s = 1, 2, \ldots$. The bosonic oscillator algebra acts on the space $\mathcal{H}_b$ via

$$\begin{aligned} \alpha^i_{-s} &\mapsto a^i_{-s}\cdot, & \bar{\alpha}^i_{-s} &\mapsto \bar{a}^i_{-s}\cdot, \\ \alpha^i_s &\mapsto s\left(G^{-1}\right)^{ij} \frac{\partial}{\partial a^j_{-s}}, & \bar{\alpha}^i_s &\mapsto s\left(G^{-1}\right)^{ij} \frac{\partial}{\partial \bar{a}^j_{-s}}, \end{aligned}$$

for all positive s. This is the Fock-Bargmann representation of the bosonic oscillator algebra. The vector $1 \in \mathcal{H}_b$ is annihilated by all bosonic annihilators and will be denoted $|vac_b\rangle$.

The space $\mathcal{H}_b$ will be regarded as a $\mathbb{Z}_2$-graded vector space with a trivial (purely even) grading. It is clear that $\mathcal{H}_b$ can be decomposed as $\mathfrak{H}_b \otimes \bar{\mathfrak{H}}_b$, where $\mathfrak{H}_b$ (resp. $\bar{\mathfrak{H}}_b$) is the bosonic Fock space defined using only the left (right) bosonic oscillators.

To define $\mathcal{H}_f$, consider an algebra over $\mathbb{C}$ with generators $\psi^i_s, \bar{\psi}^i_s, \ i = 1, \ldots, 2d, s \in \mathbb{Z}+\frac{1}{2}$ subject to relations

(5) $$\{\psi^i_s, \psi^j_p\} = \left(G^{-1}\right)^{ij} \delta_{s,-p}, \quad \{\bar{\psi}^i_s, \bar{\psi}^j_p\} = \left(G^{-1}\right)^{ij} \delta_{s,-p}, \quad \{\psi^i_s, \bar{\psi}^j_p\} = 0.$$

If s is positive, ψ^i_{-s} and $\bar{\psi}^i_{-s}$ are called left and right fermionic creators respectively, otherwise they are called left and right fermionic annihilators. Collectively they are referred to as fermionic oscillators.

The space $\mathcal{H}_f$ is defined as the space of skew-polynomials of odd variables $\theta^i_{-s}, \bar{\theta}^i_{-s}, i = 1, \ldots, 2d, s = 1/2, 3/2, \ldots$. The fermionic oscillator algebra (5) acts on $\mathcal{H}_f$ via

$$\begin{aligned} \psi^i_{-s} &\mapsto \theta^i_{-s}\cdot, & \bar{\psi}^i_{-s} &\mapsto \bar{\theta}^i_{-s}\cdot, \\ \psi^i_s &\mapsto \left(G^{-1}\right)^{ij} \frac{\partial}{\partial \theta^j_{-s}}, & \bar{\psi}^i_s &\mapsto \left(G^{-1}\right)^{ij} \frac{\partial}{\partial \bar{\theta}^j_{-s}}, \end{aligned}$$

for all positive $s \in \mathbb{Z} + \frac{1}{2}$. This is the Fock-Bargmann representation of the fermionic oscillator algebra. The vector $1 \in \mathcal{H}_f$ is annihilated by all fermionic annihilators and will be denoted $|vac_f\rangle$. The fermionic Fock space has a natural $\mathbb{Z}_2$ grading such that $|vac_f\rangle$ is even. It can be decomposed as $\mathfrak{H}_f \otimes \bar{\mathfrak{H}}_f$, where $\mathfrak{H}_f$ (resp. $\bar{\mathfrak{H}}_f$) is constructed using only the left (right) fermionic oscillators.

For $w \in \Gamma$, $m \in \Gamma^*$ we will denote the vector $w \oplus m \in \mathbb{C}\left[\Gamma \oplus \Gamma^*\right]$ by (w, m). We will also use a shorthand $|vac, w, m\rangle$ for

$$|vac_b\rangle \otimes |vac_f\rangle \otimes (w, m).$$

To define $\mathcal{V}$, we have to specify the vacuum vector, $T, \bar{T}$, and the state-operator correspondence Y. But first we need to define some auxiliary objects. We define the operators $W : V \to V \otimes \Gamma$ and $M : V \to V \otimes \Gamma^*$ as follows:

$$W^i : b \otimes f \otimes (w, m) \mapsto w^i(b \otimes f \otimes (w, m)), \quad M_i : b \otimes f \otimes (w, m) \mapsto m_i(b \otimes f \otimes (w, m)).$$

We also set

$$\partial X^j(z) = \frac{1}{z}\left(G^{-1}\right)^{jk} P_k + {\sum_{s=-\infty}^{\infty}}' \frac{\alpha_s^j}{z^{s+1}}, \qquad \bar{\partial} X^j(\bar{z}) = \frac{1}{\bar{z}}\left(G^{-1}\right)^{jk} \bar{P}_k + {\sum_{s=-\infty}^{\infty}}' \frac{\bar{\alpha}_s^j}{\bar{z}^{s+1}},$$

$$\psi^j(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \frac{\psi_r^j}{z^{r+1/2}}, \qquad \bar{\psi}^j(\bar{z}) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \frac{\bar{\psi}_r^j}{\bar{z}^{r+1/2}},$$

where a prime on a sum over s means that the term with $s = 0$ is omitted, and P_k and $\bar{P}_k$ are defined by

$$P_k = \frac{1}{\sqrt{2}}(M_k + (-B_{kj} - G_{kj}) W^j), \quad \bar{P}_k = \frac{1}{\sqrt{2}}(M_k + (-B_{kj} + G_{kj}) W^j).$$

Note that we did not define $X^j(z, \bar{z})$ themselves, but only their derivatives. The reason is that the would-be field $X^j(z, \bar{z})$ contains terms proportional to $\log z$ and $\log \bar{z}$, and therefore is not a "fractional power series."

The vacuum vector of $\mathcal{V}$ is defined by

$$|vac\rangle = |vac, 0, 0\rangle.$$

The general formula for the state-operator correspondence Y is complicated and can be found in [19]. We will only list a few special cases of the state-operator correspondence. The states $\alpha_{-s}^j|vac, 0, 0\rangle$ and $\bar{\alpha}_{-s}^j|vac, 0, 0\rangle$ are mapped by Y to

$$\frac{1}{(s-1)!}\partial^s X^j(z), \qquad \frac{1}{(s-1)!}\bar{\partial}^s X^j(\bar{z}).$$

The states $\psi_{-s}^j|vac, 0, 0\rangle$ and $\bar{\psi}_{-s}^j|vac, 0, 0\rangle$ are mapped to

$$\frac{1}{(s-\frac{1}{2})!}\partial^{s-1/2}\psi^j(z), \qquad \frac{1}{(s-\frac{1}{2})!}\bar{\partial}^{s-1/2}\bar{\psi}^j(\bar{z}).$$

To define an $N = 2$ superconformal structure on $\mathcal{V}$, we need to choose a complex structure I on U with respect to which G is a Kähler metric. Let $\omega = GI$ be the corresponding Kähler form. Then the left-moving vectors are defined as follows:

$$L = \frac{1}{2}G\left(a_{-1}, a_{-1}\right) - \frac{1}{2}G\left(\theta_{-1/2}, \theta_{-3/2}\right),$$

$$Q^{\pm} = \frac{-i}{4\sqrt{2}}\left(G \mp i\omega\right)\left(\theta_{-1/2}, a_{-1}\right),$$

$$J = -\frac{i}{2}\omega\left(\theta_{-1/2}, \theta_{-1/2}\right).$$

The right-moving vectors $\bar{L}$, $\bar{Q}^\pm$ and $\bar{J}$ are defined by the same expressions with a replaced by $\bar{a}$ and θ replaced by $\bar{\theta}$.

An isomorphism of $N = 2$ SCFTs is an isomorphism of underlying vector spaces $f : V \xrightarrow{\sim} V'$, which preserves the state-operator correspondence $Y'(f(a))f(b) = f(Y(a)b)$ and acts on the generators of both left and right super-Virasoro algebras as the identity map.

A mirror morphism between two $N = 2$ SCFTs is an isomorphism between the underlying $N = 1$ SCFTs which induces the following map on the generators of left/right super-Virasoro algebras:

$$f(L) = L', \ f(Q^+) = Q^{-'}, \ f(Q^-) = Q^{+'}, \ f(J) = -J',$$
$$f(\bar{L}) = \bar{L}', \ f(\bar{Q}^+) = \bar{Q}^{+'}, \ f(\bar{Q}^-) = \bar{Q}^{-'}, \ f(\bar{J}) = \bar{J}'.$$

A composition of two mirror morphisms is an isomorphism of $N = 2$ SCFTs.

Now we can describe when two different quadruples (Γ, I, G, B) and (Γ', I', G', B') yield isomorphic $N = 2$ SCFTs and when they are mirror symmetric.

The natural pairing $l : \Gamma \oplus \Gamma^* \to \mathbb{Z}$ induces a natural $\mathbb{Z}$-valued symmetric bilinear form q on $\Gamma \oplus \Gamma^*$ defined by

$$(6) \qquad q((w_1, m_1), (w_2, m_2)) = l(w_1, m_2) + l(w_2, m_1), \qquad w_{1,2} \in \Gamma, \ m_{1,2} \in \Gamma^*.$$

Given G, I, B, we can define two complex structures on $T \times T^*$:

$$(7) \qquad \mathcal{I}(I, B) = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix},$$

$$(8) \qquad \mathcal{J}(G, I, B) = \begin{pmatrix} -IG^{-1}B & IG^{-1} \\ GI - BIG^{-1}B & BIG^{-1} \end{pmatrix}.$$

The notation here is as follows. We regard $\mathcal{I}$ and $\mathcal{J}$ as endomorphisms of $U \oplus U^*$, and write the corresponding matrices in the basis in which the first $2d$ vector span U, while the remaining vectors span U^*. In addition, G and B are regarded as elements of $\mathrm{Hom}(U, U^*)$, and I^t denotes the endomorphism of U^* conjugate to I.

It is easy to see that $\mathcal{J}$ depends on G, I only in the combination $\omega = GI$, i.e. it depends only on the symplectic structure on T and the B-field. There is also a third natural complex structure $\tilde{\mathcal{I}}$ on $T \times T^*$, which is simply the complex structure that $T \times T^*$ gets because it is a Cartesian product of two complex manifolds:

$$\tilde{\mathcal{I}} = \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}.$$

This complex structure will play only a minor role in what follows. Note that $\mathcal{I}$ coincides with $\tilde{\mathcal{I}}$ if and only if $B^{(0,2)} = 0$.

Theorem 2.1. [19] SCFT(Γ, I, G, B) *is isomorphic to* SCFT(Γ', I', G', B') *if and only if there exists an isomorphism of lattices* $\Gamma \oplus \Gamma^*$ *and* $\Gamma' \oplus \Gamma'^*$ *which takes* q *to* q', $\mathcal{I}$ *to* $\mathcal{I}'$, *and* $\mathcal{J}$ *to* $\mathcal{J}'$.

The second theorem describes when (T, I, G, B) is mirror symmetric to (T', I', G', B').

Theorem 2.2. [19] $\mathrm{SCFT}(\Gamma, I, G, B)$ *is mirror to* $\mathrm{SCFT}(\Gamma', I', G', B')$ *if and only if there is an isomorphism of lattices* $\Gamma \oplus \Gamma^*$ *and* $\Gamma' \oplus \Gamma'^*$ *which takes* q *to* q', $\mathcal{I}$ *to* $\mathcal{J}'$, *and* $\mathcal{J}$ *to* $\mathcal{I}'$.

This theorem allows to give many examples of mirror pairs of tori. Suppose that we are given a complex torus (T, I) with a constant Kähler form ω, and suppose that $T = A \times B$, where A and B are Lagrangian sub-tori with respect to ω. In particular, the lattice Γ decomposes as $\Gamma_A \oplus \Gamma_B$. Let $\hat{A}$ be the dual torus for A, and let $T' = \hat{A} \times B$. The lattice corresponding to T' is $\Gamma' = \Gamma_A^* \oplus \Gamma_B$, and there is an obvious isomorphism from $\Gamma \oplus \Gamma^*$ to $\Gamma' \oplus \Gamma'^*$ which takes q to q'. We let $\mathcal{I}'$ and $\mathcal{J}'$ to be the image of $\mathcal{J}$ and $\mathcal{I}$, respectively, and invert the relationship between $\mathcal{I}', \mathcal{J}'$ and I', ω', B' to find the complex structure, the symplectic form, and the B-field on T'. It is easy to check that this procedure always produces a complex torus with a flat Kähler metric and a B-field which is mirror to the original one. This recipe is a special case of the physical notion of T-duality [36].

We will soon see how these results can be used to test the Homological Mirror Conjecture for flat tori. First, let us recall the formulation of the conjecture.

A physicist's Calabi-Yau (X, G, B) is both a complex manifold and a symplectic manifold (the symplectic form being the Kähler form $\omega = GI$). We can associate to each such manifold a pair of triangulated categories: the bounded derived category of coherent sheaves $\mathbf{D}^b(X)$ and the derived Fukaya category $\mathbf{D}\mathcal{F}_0(X)$. The former depends only on the complex structure of X, while the latter depends only on its symplectic structure. The Homological Mirror Conjecture (HMC) asserts that if two algebraic Calabi-Yaus (X, G, B) and (X', G', B') are mirror to each other, then $\mathbf{D}^b(X)$ is equivalent to $\mathbf{D}\mathcal{F}_0(X')$, and vice versa $\mathbf{D}^b(X')$ is equivalent to $\mathbf{D}\mathcal{F}_0(X)$.

Next, we need to recall the definitions of these two categories. We begin with the Fukaya category $\mathcal{F}(X)$. An object of the Fukaya category is a triple (Y, E, ∇_E) where Y is a graded Lagrangian submanifold of X, E is a complex vector bundle on Y with a Hermitian metric, and ∇_E is a flat unitary connection of E. The only term which needs to be explained here is "graded Lagrangian submanifold." This notion was introduced by J. Leray in 1968 in the case of Lagrangian submanifolds in a symplectic affine space, and generalized by Kontsevich to the Calabi-Yau case in [25]. Let us first recall the definition of the Maslov class of a Lagrangian submanifold. Let us choose a holomorphic section Ω of the canonical line bundle (which is trivial for Calabi-Yaus). Restricting it to Y, we obtain a nowhere vanishing n-form. On the other hand, we also have a volume form vol on Y, which comes from the Kähler metric on X. This is also a nowhere vanishing n-form on Y, and therefore $\Omega|_Y = h \cdot vol$, where h is a nowhere vanishing complex function on Y. The function h can be thought of as an element of $H^0(Y, \mathcal{C}_Y^{\infty *})$, where $\mathcal{C}_Y^{\infty *}$ is the sheaf of $\mathbb{C}^*$-valued infinitely smooth functions on Y. The standard exponential exact sequence gives a homomorphism from $H^0(Y, \mathcal{C}_Y^{\infty *})$ to $H^1(Y, \mathbb{Z})$, and the Maslov class of Y is defined as the image of h under this homomorphism. (Explicitly, the Cech cocycle representing the Maslov class is constructed as follows: choose a good cover of Y, take the logarithm of h on each set of the cover, divide by $2\pi i$, and compare the results on double overlaps). Although the definition of the Maslov class seems to depend both on the complex

and symplectic structures on X, in fact it is independent of the choice of complex structure. Note also that if the Maslov class vanishes, the logarithm of h exists as a function, and is unique up to addition of $2\pi im, m \in \mathbb{Z}$. A graded Lagrangian submanifold Y is a Lagrangian submanifold in X with a vanishing Maslov class and a choice of the branch of $\log h$.

Morphisms are defined as follows. Suppose we are given two objects (Y_1, E_1, ∇_1) and (Y_2, E_2, ∇_2). We will assume that Y_1 and Y_2 intersect transversally at a finite number of points; if this is not the case, we should deform one of the objects by flowing along a Hamiltonian vector field, until the transversality condition is satisfied. Let $\{e_i, i \in I\}$ be the set of intersection points of Y_1 and Y_2. Now we consider the Floer complex. As a vector space, it is a direct sum of vector spaces

$$V_i = \operatorname{Hom}(E_1(e_i), E_2(e_i)), \quad i \in I.$$

The grading is defined as follows. At any point $p \in Y$ the space T_pY defines a point q in the Grassmannian of Lagrangian subspaces in the space T_pX. Let us denote by $\widetilde{Lag}_p$ the universal cover of the Lagrangian Grassmannian of the space T_pX. On a Calabi-Yau variety X, this spaces fit into a fiber bundle over X denoted by $\widetilde{Lag}$ [25]. Grading of Y provides a canonical lift of q to $\widetilde{Lag}_p$ for all p; this lifts assemble into a section of the restriction of $\widetilde{Lag}$ to Y [25]. Thus for each intersection point e_i we have a pair of points $q_1, q_2 \in \widetilde{Lag}_{e_i}$. The grade of the component of the Floer complex corresponding to e_i is the Maslov index of q_1, q_2 (see [4] for a definition of the Maslov index.) Finally, we need to define the differential. Let e_i and e_j be a pair of points whose grade differs by one. The component of the Floer differential which maps V_i to V_j is defined by counting holomorphic disks in X with two marked points on the boundary, so that the two marked points are e_i and e_j (the Maslov index of e_j is the Maslov index of e_i plus one), and the two intervals which make up the boundary of the disks are mapped to Y_1 and Y_2. Note that in order to compute the differential one has to choose an (almost) complex structure J on X such that the form $\omega(\cdot, J\cdot)$ is a Hermitian form on the tangent bundle of X. For a precise definition of the Floer differential, see [9]. The space of morphisms in the Fukaya category is defined to be the Floer complex. The composition of morphisms can be defined using holomorphic disks with three marked points and boundaries lying on three Lagrangian submanifolds.

The definition sketched above is only approximate. First, in order to define the Floer differential one has to fix a relative spin structure on Y [9]. Second, the Floer differential does not square to zero in general, so the Floer "complex" is not really a complex. A related difficulty is that the composition of morphisms is associative only up to homotopy, which depends on certain ternary product of morphisms. Actually, there is an infinite sequence of higher products in the Fukaya category, which are believed to satisfy the identities of an A_∞ category (see [24] for a review of A_∞ categories). It is also believed that changing the almost complex structure J gives an equivalent A_∞ category, so that the equivalence class of the Fukaya category is a symplectic invariant. For a detailed discussion of these issues see [9].

If the B-field is non-zero, one has to modify the definition of the Fukaya category as follows. Objects are triples (Y, E, ∇_E), where Y is a graded Lagrangian submanifold, E is a vector bundle on Y with a Hermitian metric, and ∇_E is a Hermitian connection on E such that its curvature satisfies

$$\nabla_E^2 = 2\pi i B|_Y.$$

Thus the connection is projectively flat rather than flat.

Morphisms are modified in the following way: all occurences of the symplectic form ω in the definition of the Floer complex and higher products are replaced with $\omega + iB$. The modified Fukaya category of a symplectic manifold X with a B-field B will be denoted $\mathcal{F}(X, B)$.

The Fukaya category $\mathcal{F}(X, B)$ is not a true category, but an A_∞ category with a translation functor. The set of morphisms between two objects in an A_∞–category is a differential graded vector space. To any A_∞–category one can associate a true category which has the same objects but the space of morphisms between two objects is the 0–th cohomology group of the morphisms in the A_∞–category. Applying this construction to $\mathcal{F}(X, B)$, we obtain a true category $\mathcal{F}_0(X, B)$ which is also called the Fukaya category. Considering twisted complexes over $\mathcal{F}(X, B)$ M.Kontsevich [25] also constructs a certain triangulated category $\mathbf{D}\mathcal{F}_0(X, B)$. We will call it the derived Fukaya category. The category $\mathbf{D}\mathcal{F}_0(X, B)$ contains $\mathcal{F}_0(X, B)$ as a full subcategory.

In the next lecture we will discuss the derived category of coherent sheaves, and use its properties to test the Homological Mirror Conjecture.

3. Derived Categories of Coherent Sheaves and a Test of the Homological Mirror Conjecture

Let X be a complex algebraic variety (or a complex manifold). Denote by $\mathcal{O}_X$ the sheaf of regular functions (or the sheaf of holomorphic functions). Recall that a coherent sheaf is a sheaf of $\mathcal{O}_X$–modules that locally can be represented as a cokernel of a morphism of algebraic (holomorphic) vector bundles. Coherent sheaves form an abelian category which will be denoted by $\text{coh}(X)$.

Next we recall the definition of a derived category and describe some properties of derived categories of coherent sheaves on smooth projective varieties. There is a lot of texts where introductions to the theory of derived and triangulated categories are given, we can recommend [43, 15, 12, 21, 23].

Let $\mathcal{A}$ be an abelian category. We denote by $\mathbf{C}^b(\mathcal{A})$ the category of bounded differential complexes

$$M^\bullet = (0 \longrightarrow \cdots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \longrightarrow \cdots \longrightarrow 0), \quad M^p \in \mathcal{A}, \quad p \in \mathbb{Z}, \quad d^2 = 0.$$

Morphisms $f : M^\bullet \longrightarrow N^\bullet$ between complexes are sets of morphisms $f^p : M^p \longrightarrow N^p$ in the category $\mathcal{A}$ which commute with the differentials, i.e.

$$d_N f^p - f^{p+1} d_M = 0 \qquad \text{for all} \quad p.$$

A morphism of complexes $f : M^\bullet \longrightarrow N^\bullet$ is null-homotopic if $f^p = d_N h^p + h^{p+1} d_M$ for all $p \in \mathbb{Z}$ for some family of morphisms $h^p : M^{p+1} \longrightarrow N^p$. We define the homotopy
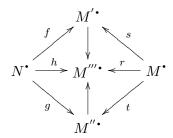
category $\mathbf{H}^b(\mathcal{A})$ as a category which has the same objects as $\mathbf{C}^b(\mathcal{A})$, whereas morphisms in $\mathbf{H}^b(\mathcal{A})$ are the equivalence classes of morphisms of complexes $\bar{f}$ modulo the null-homotopic morphisms.

For any complex $N^\bullet$ and for any $p \in \mathbb{Z}$ we define the cohomology $H^p(N^\bullet) \in \mathcal{A}$ as the quotient $\operatorname{Ker} d^p / \operatorname{Im} d^{p-1}$. Hence for any p there is a functor $H^p : \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{A}$, which assigns to the complex $N^\bullet$ the cohomology $H^p(N^\bullet) \in \mathcal{A}$. We define a quasi-isomorphism to be a morphism of complexes $s : N^\bullet \to M^\bullet$ such that the induced morphisms $H^p s : H^p(N^\bullet) \to H^p(M^\bullet)$ are invertible for all $p \in \mathbb{Z}$. We denote by Σ the class of all quasi-isomorphisms. This class of morphisms enjoys good properties which are similar to the Ore conditions in the localization theory of rings.
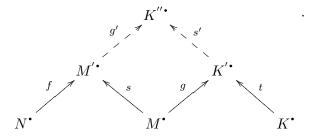
The bounded derived category $\mathbf{D}^b(\mathcal{A})$ is defined as the localization of $\mathbf{H}^b(\mathcal{A})$ with respect to the class Σ of all quasi-isomorphisms. This means that the derived category has the same objects as the homotopy category $\mathbf{H}^b(\mathcal{A})$, and that morphisms in the derived category from $N^\bullet$ to $M^\bullet$ are given by left fractions $s^{-1} \circ f$, i.e. equivalence classes of diagram

$$N^\bullet \xrightarrow{f} M'^\bullet \xleftarrow{s} M^\bullet, \qquad s \in \Sigma,$$

where pairs (f, s) and (g, t) are considered equivalent iff there is a commutative diagram in $\mathbf{H}^b(\mathcal{A})$

$$
\begin{array}{ccccc}
 & & M'^\bullet & & \\
 & {}^f\nearrow & \downarrow & \nwarrow^s & \\
N^\bullet & \xrightarrow{h} & M'''^\bullet & \xleftarrow{r} & M^\bullet \\
 & {}_g\searrow & \uparrow & \nearrow_t & \\
 & & M''^\bullet & &
\end{array}
$$

such that $r \in \Sigma$. Composition of morphisms (f, s) and (g, t) is a morphism $(g'f, s't)$ which is defined using the commutative diagram:

$$
\begin{array}{ccccccc}
 & & & K''^\bullet & & & \\
 & & {}^{g'}\nearrow & & \nwarrow^{s'} & & \\
 & M'^\bullet & & & & K'^\bullet & \\
 & {}^f\nearrow \quad \nwarrow^s & & & {}^g\nearrow \quad \nwarrow^t & & \\
N^\bullet & & & M^\bullet & & & K^\bullet
\end{array}
$$

Such a diagram always exists, and one can check that the composition law is associative.

We have a canonical functor $\mathbf{H}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$ sending a morphism $f : N^\bullet \to M^\bullet$ to the pair (f, id_M). This functor makes all quasi-isomorphisms invertible and is universal among functors with this property. The abelian category $\mathcal{A}$ can be considered as a full subcategory of $\mathbf{D}^b(\mathcal{A})$ identifying an object $A \in \mathcal{A}$ with the complex $\cdots \to 0 \to A \to 0 \to \cdots$ having A in degree 0. If $N^\bullet$ is an arbitrary complex, we denote by $N^\bullet[1]$ the complex with components $N^\bullet[1]^p = N^{p+1}$ and the differential $d_{N[1]} = -d_N$. This correspondence gives a functor on the derived category $\mathbf{D}^b(\mathcal{A})$ which is an autoequivalence and is called the translation functor.

Any derived category $\mathbf{D}^b(\mathcal{A})$ has a structure of a triangulated category [43]. This means that the following data are specified:

   a) a translation functor $[1] : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$ which is an additive autoequivalence,

   b) a class of distinguished (or exact) triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

that satisfies a certain set of axioms (for details see [43, 12, 21, 23]).

To define a triangulated structure on the derived category $\mathbf{D}^b(\mathcal{A})$ we introduce the notion of a standard triangle as a sequence

$$N \xrightarrow{Qi} M \xrightarrow{Qp} K \xrightarrow{\partial\varepsilon} N[1],$$

where $Q : \mathbf{C}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$ is the canonical functor,

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{p} K \longrightarrow 0$$

is a short exact sequence of complexes, and $\partial\varepsilon$ is a certain morphism in $\mathbf{D}^b(\mathcal{A})$. The morphism $\partial\varepsilon$ is the fraction $s^{-1} \circ j$, where j is the inclusion of the subcomplex K into the complex $C(p)$ with components $K^n \oplus M^{n+1}$ and differential

$$d_{C(p)} = \begin{pmatrix} d_K & p \\ 0 & -d_M \end{pmatrix},$$

and the quasi-isomorphism $s : N[1] \longrightarrow C(p)$ is the morphism $(0, i)$.

A *distinguished triangle* in $\mathbf{D}^b(\mathcal{A})$ is a sequence in $\mathbf{D}^b(\mathcal{A})$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

which is isomorphic to a standard triangle.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor which is left (resp. right) exact. The functor F induces a functor between the categories of differential complexes and a functor $\bar{F} : \mathbf{H}^b(\mathcal{A}) \to \mathbf{H}^b(\mathcal{B})$ obtained by applying F componentwise. If F is not exact it does not transform quasi-isomorphisms into quasi-isomorphisms. Nevertheless, often we can define its right (resp. left) derived functor $\mathbf{R}F$ (resp. $\mathbf{L}F$) between the corresponding derived categories. The derived functor $\mathbf{R}F$ (resp. $\mathbf{L}F$) will be *exact* functor between triangulated categories in the following sense: it commutes with the translation functors and takes every distinguished triangle to a distinguished triangle. We will not give here the definition of the derived functors, but the idea is to apply the functor F componentwise to well-selected representatives of classes of quasi-isomorphic complexes (see [43, 15, 12, 21, 23]).

For example, let us consider the derived categories of coherent sheaves on smooth projective (or proper algebraic) varieties. We denote $\mathbf{D}^b(\mathrm{coh}(X))$ by $\mathbf{D}^b(X)$. Every morphism $f : X \to Y$ induces the inverse image functor $f^* : \mathrm{coh}(Y) \longrightarrow \mathrm{coh}(X)$. This functor is right exact and has the left derived functor $\mathbf{L}f^* : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(X)$. To define it we have to replace a complex on Y by a quasi-isomorphic complex of locally free sheaves, and apply the functor f^* componentwise to this locally free complex. Similarly, for any complex $\mathcal{F} \in \mathbf{D}^b(X)$ we can define an exact functor $\overset{\mathbf{L}}{\otimes} \mathcal{F} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X)$ replacing $\mathcal{F}$ by a quasi-isomorphic locally free complex.

The morphism $f : X \longrightarrow Y$ induces also the direct image functor $f_* : \operatorname{coh}(X) \longrightarrow \operatorname{coh}(Y)$ which is left exact and has the right derived functor $\mathbf{R}f_* : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$. To define it we have to include the category of coherent sheaves into the category of quasi-coherent sheaves and replace a complex by a quasi-isomorphic complex of injectives. After that we can apply the functor f_* componentwise to the complex of injectives. The functor $\mathbf{R}f_*$ is right adjoint to $\mathbf{L}f^*$. This means that there is a functorial isomorphisms

$$\operatorname{Hom}(A, \mathbf{R}f_*B) \cong \operatorname{Hom}(\mathbf{L}f^*A, B).$$

for all A, B. This property can also be regarded as a definition of the functor $\mathbf{R}f_*$.

Using these functors one can introduce a larger class of exact functors. Let X and Y be smooth projective (or proper algebraic) varieties. Consider the projections

$$X \xleftarrow{\ p\ } X \times Y \xrightarrow{\ q\ } Y.$$

Every object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ defines an exact functor $\Phi_{\mathcal{E}} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y)$ by the following formula:

$$(9) \qquad\qquad \Phi_{\mathcal{E}}(\cdot) := \mathbf{R}q_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p^*(\cdot)).$$

Obviously, the same object defines another functor $\Psi_{\mathcal{E}} : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(X)$ by a similar formula

$$\Psi_{\mathcal{E}}(\cdot) := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(\cdot)).$$

Thus there is a reasonably large class of exact functors between bounded derived categories of smooth projective varieties that consists of functors having the form $\Phi_{\mathcal{E}}$ for some complex $\mathcal{E}$ on the product variety. This class is closed under composition of functors. Indeed, let X, Y, Z be three smooth projective varieties and let

$$\Phi_I : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y), \qquad \Phi_J : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(Z)$$

be two functors, where I and J are objects of $\mathbf{D}^b(X \times Y)$ and $\mathbf{D}^b(Y \times Z)$ respectively. Denote by p_{XY}, p_{YZ}, p_{XZ} the projections of $X \times Y \times Z$ on the corresponding pair of factors. The composition $\Phi_J \circ \Phi_I$ is isomorphic to Φ_K, where $K \in \mathbf{D}^b(X \times Z)$ is given by the formula

$$K \cong \mathbf{R}p_{XZ*}(p_{YZ}^*(J) \otimes p_{XY}^*(I)).$$

Presumably, the class of exact functors described above encompasses all exact functors between bounded derived categories of coherent sheaves on smooth projective varieties. We do not know if it is true or not. However it is definitely true for exact equivalences.

Theorem 3.1. ([33], also [35]) *Let X and Y be smooth projective varieties. Suppose $F : \mathbf{D}^b(X) \xrightarrow{\sim} \mathbf{D}^b(Y)$ is an exact equivalence. Then there exists a unique (up to an isomorphism) object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ such that the functor F is isomorphic to the functor $\Phi_{\mathcal{E}}$.*

Now we consider the bounded derived categories of coherent sheaves on abelian varieties. There are examples of different abelian varieties which have equivalent derived categories of coherent sheaves. Moreover, one can completely describe classes of abelian varieties with equivalent derived categories of coherent sheaves.

Let A be an abelian variety of dimension n over $\mathbb{C}$. This means that A is a complex torus $(U/\Gamma, I)$ which is algebraic, i.e. it has an embedding to the projective space. Let $\widehat{A}$ be the dual abelian variety, i.e. the dual torus $(U^*/\Gamma^*, -I^t)$. It is canonically isomorphic to Picard group $\mathrm{Pic}^0(A)$. It is well-known that there is a unique line bundle P on the product $A \times \widehat{A}$ such that for any point $\alpha \in \widehat{A}$ the restriction P_α on $A \times \{\alpha\}$ represents an element of $\mathrm{Pic}^0 A$ corresponding to α, and, in addition, the restriction $P\big|_{\{0\} \times \widehat{A}}$ is trivial. Such P is called the Poincare line bundle.

The Poincare line bundle gives an example of an exact equivalence between derived categories of coherent sheaves of two non-isomorphic varieties. Let us consider the projections

$$A \xleftarrow{\ p\ } A \times \widehat{A} \xrightarrow{\ q\ } \widehat{A}$$

and the functor $\Phi_P : \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(\widehat{A})$, defined as in (9), i.e. $\Phi_P(\cdot) = \mathbf{R}q_*(P \otimes p^*(\cdot))$. It was proved by Mukai [32] that the functor $\Phi_P : \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(\widehat{A})$ is an exact equivalence, and there is an isomorphism of functors:

$$\Psi_P \circ \Phi_P \cong (-1_A)^*[n],$$

where (-1_A) is the inverse map on the group A.

Now let A_1 and A_2 be two abelian varieties of the same dimension. We denote by Γ_{A_1} and Γ_{A_2} the first homology lattices $H_1(A_1, \mathbb{Z})$ and $H_1(A_2, \mathbb{Z})$. Every map $f : A_1 \longrightarrow A_2$ of abelian varieties induces a map $\bar{f} : \Gamma_{A_1} \longrightarrow \Gamma_{A_2}$ of the first homology groups.

For any abelian variety A the first homology lattice of the variety $A \times \widehat{A}$ coincides with $\Gamma_A \oplus \Gamma_A^*$ and hence it has the canonical symmetric bilinear form q_A defined by Equation (6). Consider an isomorphism $f : A_1 \times \widehat{A}_1 \xrightarrow{\sim} A_2 \times \widehat{A}_2$ of abelian varieties. We call such map *isometric* if the isomorphism $\bar{f} : \Gamma_{A_1} \oplus \Gamma_{A_1}^* \xrightarrow{\sim} \Gamma_{A_2} \oplus \Gamma_{A_2}^*$ identifies the forms q_{A_1} and q_{A_2}.

Now we can formulate a criterion for two abelian varieties to have equivalent derived categories of coherent sheaves.

Theorem 3.2. ([34]) *Let A_1 and A_2 be abelian varieties. Then the derived categories $\mathbf{D}^b(A_1)$ and $\mathbf{D}^b(A_2)$ are equivalent as triangulated categories if and only if there exists an isometric isomorphism*

$$f : A_1 \times \widehat{A}_1 \xrightarrow{\sim} A_2 \times \widehat{A}_2,$$

i.e $\bar{f}$ identifies the forms q_{A_1} and q_{A_2} on $\Gamma_{A_1} \oplus \Gamma_{\widehat{A}_1}$ and $\Gamma_{A_2} \oplus \Gamma_{\widehat{A}_2}$.

Using Theorems 2.1 and 3.2 we can now make a check of the Homological Mirror Conjecture for tori. Suppose the tori (T_1, I_1, G_1, B_1) and (T_2, I_2, G_2, B_2) are both mirror to (T', I', G', B'). Then $\mathrm{SCFT}(\Gamma_1, I_1, G_1, B_1)$ is isomorphic to $\mathrm{SCFT}(\Gamma_2, I_2, G_2, B_2)$, and by Theorem 2.1 there is an isomorphism of lattices $\Gamma_1 \oplus \Gamma_1^*$ and $\Gamma_2 \oplus \Gamma_2^*$ which intertwines q_1 and q_2, $\mathcal{I}_1$ and $\mathcal{I}_2$, and $\mathcal{J}_1$ and $\mathcal{J}_2$.

On the other hand, if we now assume that both complex tori (T_1, I_1) and (T_2, I_2) are algebraic, then HMC implies that $\mathbf{D}^b((T_1, I_1))$ is equivalent to $\mathbf{D}^b((T_2, I_2))$. The criterion for this equivalence is given in Theorem 3.2: it requires the existence of an isomorphism of $\Gamma_1 \oplus \Gamma_1^*$ and $\Gamma_2 \oplus \Gamma_2^*$ which intertwines q_1 and q_2, and $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$. Clearly, since $\mathcal{I} \neq \tilde{\mathcal{I}}$ in general, we get two conditions that contradict to each other. However, since $\mathcal{I}$ coincides with $\tilde{\mathcal{I}}$ under condition $B^{(0,2)} = 0$ we obtain the following result.

Corollary 3.3. *If* $\mathrm{SCFT}(\Gamma_1, I_1, G_1, B_1)$ *is isomorphic to* $\mathrm{SCFT}(\Gamma_2, I_2, G_2, B_2)$, *both* (T_1, I_1) *and* (T_2, I_2) *are algebraic, and both* B_1 *and* B_2 *are of type* $(1,1)$, *then* $\mathbf{D}^b((T_1, I_1))$ *is equivalent to* $\mathbf{D}^b((T_2, I_2))$.

Let (T_1, I_1, G_1, B_1) be a complex torus equipped with a flat Kähler metric and a B-field of type $(1,1)$ and let (T_2, I_2) be another complex torus. Suppose there exists an isomorphism of lattices $g : \Gamma_1 \oplus \Gamma_1^* \to \Gamma_2 \oplus \Gamma_2^*$ mapping q_1 to q_2 and $\tilde{\mathcal{I}}_1$ to $\tilde{\mathcal{I}}_2$. We can prove that in this case there exists a Kähler metric G_2 and a B-field B_2 of type $(1,1)$ on T_2 such that $\mathrm{SCFT}(\Gamma_1, I_1, G_1, B_1)$ is isomorphic to $\mathrm{SCFT}(\Gamma_2, I_2, G_2, B_2)$ as an $N = 2$ superconformal field theory.

Combining this with Theorem 2.1 and the criterion for the equivalence of $\mathbf{D}^b((T_1, I_1))$ and $\mathbf{D}^b((T_2, I_2))$, we obtain a result converse to Corollary 3.3.

Corollary 3.4. *Let* (T_1, I_1, G_1, B_1) *be an algebraic torus equipped with a flat Kähler metric and a B-field of type* $(1,1)$. *Let* (T_2, I_2) *be another algebraic torus. Suppose* $\mathbf{D}^b((T_1, I_1))$ *is equivalent to* $\mathbf{D}^b((T_2, I_2))$. *Then on* T_2 *there exists a Kähler metric* G_2 *and a B-field* B_2 *of type* $(1,1)$ *such that* $\mathrm{SCFT}(\Gamma_1, I_1, G_1, B_1)$ *is isomorphic to* $\mathrm{SCFT}(\Gamma_2, I_2, G_2, B_2)$ *as an* $N = 2$ *superconformal field theory.*

If $\dim_{\mathbb{C}} T = 1$, then the B-field is automatically of type $(1,1)$. Therefore the HMC passes the check in this special case. Of course, this is what we expect, since the HMC is known to be true for the elliptic curve [39]. On the other hand, for $\dim_{\mathbb{C}} T > 1$ we seem to have a problem.

Not all is lost however, and a simple modification of the HMC passes our check. The modification involves replacing $\mathbf{D}^b((T, I))$ with a derived category of $\beta(B)$-twisted sheaves, where $\beta(B)$ is an element of $H^2((T, I), \mathcal{O}_T^*)$ depending on the B-field $B \in H^2(X, \mathbb{R})$.

Let X be an algebraic variety over $\mathbb{C}$, and let $B \in H^2(X, \mathbb{R})$. Consider the homomorphism $\beta : H^2(X, \mathbb{R}) \to H^2(X, \mathcal{O}_X^*)$ induced by the canonical map $\mathbb{R} \longrightarrow \mathcal{O}_X^*$ from the following commutative diagram of sheaves:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp(2\pi i \cdot)} & \mathcal{O}_X^* & \longrightarrow & 0
\end{array}
$$

Any element $a \in H^2(X, \mathcal{O}_X^*)$ gives us an $\mathcal{O}_X^*$ gerbe $\mathcal{X}_a$ over X. Consider the category of weight-1 coherent sheaves $\mathrm{coh}_1(\mathcal{X}_a)$ on the gerbe $\mathcal{X}_a$. Now our triangulated category can be defined as the derived category $\mathbf{D}^b(\mathrm{coh}_1(\mathcal{X}_{\beta(B)}))$ which will be denoted as $\mathbf{D}^b(X, B)$. Recall that weight-1 coherent sheaves on the gerbe $\mathcal{X}_a$ can be described as twisted coherent sheaves on X in the following way. Choose an open cover $\{U_i\}_{i \in I}$ of X such that the element $a \in H^2(X, \mathcal{O}_X^*)$ is represented by a Čech 2-cocycle $a_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X^*)$ where $U_{ijk} = U_i \cap U_j \cap U_k$. Now an a-twisted sheaf can be defined as a collection of coherent sheaves $\mathcal{F}_i$ on U_i for all $i \in I$ together with isomorphisms $\phi_{ji} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$ for all $i, j \in I$ (s.t. $\phi_{ij} = \phi_{ji}^{-1}$) satisfying the twisted cocycle condition $\phi_{ij}\phi_{jk}\phi_{ki} = a_{ijk}\mathrm{id}$.

When $\beta(B)$ is a torsion element of $H^2(X, \mathcal{O}_X^*)$, the abelian category of twisted sheaves is equivalent to the abelian category of coherent sheaves of modules over an Azumaya

algebra $\mathcal{A}_B$ which corresponds to this element. This implies that the corresponding derived categories are also equivalent.

Let us remind the definition and basic facts about Azumaya algebras. Let $\mathcal{A}$ be an $\mathcal{O}_X$–algebra which is coherent as a sheaf $\mathcal{O}_X$–modules. Recall that $\mathcal{A}$ is called an Azumaya algebra if it is locally free as a sheaf of $\mathcal{O}_X$–modules, and for any point $x \in X$ the restriction $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_X} \mathbb{C}(x)$ is isomorphic to a matrix algebra $M_r(\mathbb{C})$. A trivial Azumaya algebra is an algebra of the form $\mathcal{E}nd(E)$ where E is a vector bundle. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are called similar (or Morita equivalent) if there exist vector bundles E and E' such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd(E) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathcal{E}nd(E').$$

Denote by $\mathrm{coh}(\mathcal{A})$ the abelian category of sheaves of (right) $\mathcal{A}$–modules which are coherent as sheaves of $\mathcal{O}_X$–modules, and by $\mathbf{D}^b(X, \mathcal{A})$ the bounded derived category of $\mathrm{coh}(\mathcal{A})$. It is easy to see that if the Azumaya algerbas $\mathcal{A}$ and $\mathcal{A}'$ are similar, then the categories $\mathrm{coh}(\mathcal{A})$ and $\mathrm{coh}(\mathcal{A}')$ are equivalent, and therefore the derived categories $\mathbf{D}^b(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A}')$ are equivalent as well.

Azumaya algebras modulo Morita equivalence form a group with respect to tensor product. This group is called the Brauer group of the variety X and is denoted by $Br(X)$. There is a natural map

$$Br(X) \longrightarrow H^2(X, \mathcal{O}_X^*).$$

This map is an embedding and its image is contained in the torsion subgroup $H^2(X, \mathcal{O}_X^*)_{tors}$. The latter group is denoted by $Br'(X)$ and called the cohomological Brauer group of X. The well-known Grothendieck conjecture asserts that the natural map $Br(X) \longrightarrow Br'(X)$ is an isomorphism for smooth varieties. This conjecture has been proved for abelian varieties [16].

Suppose now that $\beta(B)$ is a torsion element of $H^2((T, I), \mathcal{O}_T^*)$, and consider an Azumaya algebra $\mathcal{A}_B$ which corresponds to this element. The derived category $\mathbf{D}^b((T, I), \mathcal{A}_B)$ does not depend on the choice of $\mathcal{A}_B$ because all these algebras are Morita equivalent. It can be shown that the derived category $\mathbf{D}^b((T, I), B)$ is equivalent to the derived category $\mathbf{D}^b((T, I), \mathcal{A}_B)$.

A sufficient condition for the equivalence of $\mathbf{D}^b((T_1, I_1), B_1)$ and $\mathbf{D}^b((T_2, I_2), B_2)$ for the case of algebraic tori is provided by the following theorem [38].

Theorem 3.5. ([38]) *Let (T_1, I_1) and (T_2, I_2) be two algebraic tori. Let $B_1 \in H^2(T_1, \mathbb{R})$ and $B_2 \in H^2(T_2, \mathbb{R})$, and suppose β maps both B_1 and B_2 to torsion elements. If there exists an isomorphism of lattices $\Gamma_1 \oplus \Gamma_1^*$ and $\Gamma_2 \oplus \Gamma_2^*$ which maps q_1 to q_2, and $\mathcal{I}_1$ to $\mathcal{I}_2$, then $\mathbf{D}^b((T_1, I_1), B_1)$ is equivalent to $\mathbf{D}^b((T_2, I_2), B_2)$.*

It appears plausible that this is also a necessary condition for $\mathbf{D}^b((T_1, I_1), B_1)$ to be equivalent to $\mathbf{D}^b((T_2, I_2), B_2)$. Combining Theorem 3.5 with our Theorem 2.1, we obtain the following result.

Corollary 3.6. *Suppose $\mathrm{SCFT}(\Gamma_1, I_1, G_1, B_1)$ is isomorphic to $\mathrm{SCFT}(\Gamma_2, I_2, G_2, B_2)$, both (T_1, I_1) and (T_2, I_2) are algebraic, and both B_1 and B_2 are mapped by β to torsion elements. Then $\mathbf{D}^b((T_1, I_1), B_1)$ is equivalent to $\mathbf{D}^b((T_2, I_2), B_2)$.*

This corollary suggests that we modify the HMC by replacing $\mathbf{D}^b(X)$ with $\mathbf{D}^b(X,B)$. Recall that in the presence of a B-field the definition of the Fukaya category is modified, and that the modified category is denoted $\mathbf{D}\mathcal{F}_0(X,B)$. The modified HMC asserts that if (X,G,B) is mirror to (X',G',B'), then $\mathbf{D}^b(X,B)$ is equivalent to $\mathbf{D}\mathcal{F}_0(X',B')$. Corollary 3.6 shows that this conjecture passes the check which the original HMC fails.

In the case of the elliptic curve the modified HMC is not very different from the original one. Since $h^{0,2}=0$ in this case, the complex side is unaffected by the B-field, while on the symplectic side its only effect is to complexify the symplectic form (replacing ω with $\omega+iB$). For true Calabi-Yaus (the ones whose holonomy group is strictly $SU(n)$ and not some subgroup) $h^{0,2}$ also vanishes, and the complex side is again unmodified, but on the symplectic side the effects of the B-field can be rather drastic. For example, flat connections on Lagrangian submanifolds must be replaced with projectively flat ones, and this has the tendency to reduce the number of A-branes. But for complex tori of dimension higher than one the B-field has important effects on both A-branes and B-branes.

4. The Category of A-branes and the Fukaya Category

In this lecture we will discuss topological D-branes of type A (A-branes) on Calabi-Yau manifolds. As it was stated above, the set of A-branes has the structure of an additive category, and if X is mirror to X', then the category of A-branes on X should be equivalent to the category of B-branes on X', and vice versa. There is a lot of evidence that the category of B-branes on X is equivalent to $\mathbf{D}^b(X)$ [5] [7, 28, 1, 5, 22]. The Homological Mirror Conjecture is essentially equivalent to the statement that the category of A-branes on X is equivalent to the derived Fukaya category of X. As we will see below, this is not true for some X, so the Homological Mirror Conjecture needs to be modified.

In the case when X is an elliptic curve, the Homological Mirror Conjecture, with some relatively minor modifications, has been proved by Polishchuk and Zaslow in [39]. On the other hand, in [20] it was shown that in general the Fukaya category is only a full subcategory of the category of A-branes. In the case when X is a torus of dimension higher than two with a constant symplectic form, we have constructed in [20] some examples of A-branes which are represented by coisotropic, rather than Lagrangian submanifolds. So far we do not have a proposal how to define the category of A-branes mathematically. The goal of this lecture is to explain the results of [20] and discuss the many unresolved issues.

To show that in general the category of A-branes on X is "bigger" than $\mathbf{D}\mathcal{F}_0(X)$, we will exhibit a certain mirror pair X and X' such that the group $K^0(\mathbf{D}^b(X))$ is strictly bigger than K^0 of the Fukaya category X'. In fact, to simplify life, we will tensor $K^0(\mathbf{D}^b(X))$ with $\mathbb{Q}$ and use the Chern character to map the rational K-theory to the cohomology of X. In the case of the derived category of coherent sheaves, the Chern character ch takes values in the intersection of $H^*(X,\mathbb{Q})$ and $\oplus_{p=0}^n H^{p,p}(X)$, which are both subgroups of $H^*(X,\mathbb{C})$. (The Hodge conjecture says that the image of ch should coincide with this intersection.) In the case of the Fukaya category, one can say the following. Mirror symmetry induces an isomorphism of $H^*(X,\mathbb{C})$ and $H^*(X',\mathbb{C})$, therefore the analogue of the Chern character for the Fukaya category should take values in some subgroup of $H^*(X',\mathbb{C})$. A

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[5]In this lecture we will assume that the B-field is trivial, for simplicity.

natural candidate for the Chern character of an object (Y, E, ∇) of the Fukaya category is the Poincaré dual of the corresponding Lagrangian submanifold (taken over $\mathbb{Q}$). This conjecture can also be physically motivated, and we will assume it in what follows.

Let E be an elliptic curve, e be an arbitrary point of E, and $\mathrm{End}_e(E)$ be the ring of endomorphisms of E which preserve e. For a generic E we have $\mathrm{End}_e(E) = \mathbb{Z}$, but for certain special E $\mathrm{End}_e(E)$ is strictly larger than $\mathbb{Z}$. Such special E's are called elliptic curves with complex multiplication. It is not difficult to check that E has complex multiplication if and only if its Teichmüller parameter τ is a root of a quadratic polynomial with integral coefficients. Let E be an elliptic curve with complex multiplication. Consider an abelian variety $X = E^n$, $n \geq 2$. One can show that for such a variety the dimension of the image of the Chern character

$$\mathrm{ch} : K^0(\mathbf{D}^b(X)) \otimes \mathbb{Q} \longrightarrow H^*(X, \mathbb{Q})$$

coincides with the intersection

$$H^*(X, \mathbb{Q}) \bigcap \left(\bigoplus_{p=0}^{n} H^{p,p}(X) \right)$$

and has the dimension equal to

$$\dim_{\mathbb{Q}} \mathrm{Im}(\mathrm{ch}) = \binom{2n}{n}.$$

Further, X is related by mirror symmetry to a symplectic torus X' of real dimension $2n$. Cohomology classes Poincaré-dual to Lagrangian submanifolds in X' lie in the kernel of the map

(10) $$H^n(X', \mathbb{R}) \xrightarrow{\wedge \omega} H^{n+2}(X', \mathbb{R}).$$

This map is an epimorphism, and therefore the dimension of the kernel is equal to $\binom{2n}{n} - \binom{2n}{n+2}$. Thus the image of the Chern character map for the Fukaya category of X' has dimension less or equal to $\binom{2n}{n} - \binom{2n}{n+2}$. On the other hand, the mirror relation between X and X' induces an isomorphism of their cohomology groups [13]. If we make a reasonable assumption that this isomorphism is compatible with the equivalence of the categories of B-branes on X and A-branes on X', we infer that the derived Fukaya category $\mathbf{D}\mathcal{F}_0(X')$ cannot be equivalent to the category of A-branes on X'. On the other hand, we expect on the physical grounds that the former is a full sub-category of the latter.

This leaves us with a question: if not all A-branes are Lagrangian submanifolds, how can one describe them geometrically? On the level of cohomology, if the Chern character of A-branes does not take values in the kernel of the map (10), where *does* it take values? In the case of flat tori, we can answer the second question. In this case we know that a mirror torus is obtained by dualizing a Lagrangian sub-torus, and can infer how the cohomology classes transform under this operation. The answer is the following [13, 18]. Suppose the original torus is of the form $X = A \times B$, where A and B are Lagrangian real sub-tori, and the mirror torus is $X' = \hat{A} \times B$. Consider a torus $Z = A \times \hat{A} \times B$. It has two obvious projections π and π' to X and X'. On $A \times \hat{A}$ we also have the Poincare line bundle P whose Chern character will be denoted $\mathrm{ch}(P)$. Using an obvious projection from Z to $A \times \hat{A}$, we may regard $\mathrm{ch}(P)$ as a cohomology class on Z. Given a cohomology class

$\alpha \in H^*(X, \mathbb{Q})$, let us describe its image under mirror symmetry [13]. We pull α back to Z using $\pi : Z \to X$, tensor with $\mathrm{ch}(P)$, and then push forward to X' using $\pi' : Z \to X'$. This gives a cohomology class $\alpha' \in H^*(X', \mathbb{Q})$ which is mirror to α. The requirement that α be in the intersection of $H^*(X, \mathbb{Q})$ and $\oplus_p H^{p,p}(X)$ implies the following condition on α' :

$$\iota_{\omega^{-1}} \alpha' - \omega \wedge \alpha' = 0.$$

Here $\iota_{\omega^{-1}}$ is the operator of interior multiplication by the bi-vector ω^{-1}. Cohomology classes dual to Lagrangian submanifolds satisfy this condition, but there are other solutions as well. For example, it is easy to construct some solutions of the form $\alpha' = e^a$, where $a \in H^2(X, \mathbb{Z})$. This suggests that there exist line bundles on X' which can be regarded as A-branes. We will see below that this guess is correct.

To make further progress in understanding A-branes, we need to rely on physical arguments. As explained in Lecture 1, a classical A-brane is a boundary condition for a sigma-model with preserves $N = 2$ super-Virasoro algebra. In [20] we analyzed this condition assuming that an A-brane is described by the following geometric data: a submanifold Y in X, a Hermitian line bundle E on Y, and a unitary connection ∇_E on E. We showed that in order for a triple (Y, E, ∇_E) to be a classical A-brane, the following three conditions are necessary and sufficient.

(i) Y must be a coisotropic submanifold of X. This means that the restriction of the symplectic form ω to Y must have a constant rank, and its kernel is an integrable distribution $\mathcal{L}Y \subset TY$. (An equivalent definition: Y is coisotropic if and only if for any point $p \in Y$ the skew-orthogonal complement of $TY_p \subset TX_p$ is contained in TY_p.) We will denote by $\mathcal{N}Y$ the quotient bundle $TY/\mathcal{L}Y$. By definition, the restriction of ω to Y descends to a symplectic form σ on the vector bundle $\mathcal{N}Y$.

(ii) The curvature 2-form $F = (2\pi i)^{-1}\nabla_E^2$, regarded as a bundle map from to TY to TY^*, annihilates $\mathcal{L}Y$. (The factor $(2\pi i)^{-1}$ is included to make F a real 2-form with integral periods). This implies that F induces on $\mathcal{N}Y$ a skew-symmetric pairing which we will denote f.

(iii) The forms σ and f, regarded as maps from $\mathcal{N}Y$ to $\mathcal{N}Y^*$, satisfy $(\sigma^{-1}f)^2 = -\mathrm{id}_{\mathcal{N}Y}$. This means that $\mathcal{J} = \sigma^{-1}f$ is a complex structure on the bundle $\mathcal{N}Y$.

Let us make some comments on these three conditions. The condition (i) implies the existence of a foliation of Y whose dimension is equal to the codimension of Y in X. It is known as the characteristic foliation of Y. If the characteristic foliation happens to be a fiber bundle $p : Y \to Z$ with a smooth base Z, then $\mathcal{N}Y$ is simply the pull-back of TZ to Y, i.e. $\mathcal{N}Y = p^*TZ$, and the form σ is a pull-back of a symplectic form on Z. In general, $\mathcal{N}Y$ is a foliated vector bundle over the foliated manifold Y, and the space of leaves Z is not a manifold, or even a Hausdorff topological space. It still makes sense to talk about the sheaf of local sections of $\mathcal{N}Y$ locally constant along the leaves of the foliation. This sheaf plays the role of the pull-back of the sheaf of sections of the generally non-existent tangent bundle to Z. In the same spirit, the 2-form σ should be interpreted as a symplectic form on Z. One can summarize the situation by saying that σ is a transverse symplectic structure on a foliated manifold Y. The bundle $\mathcal{N}Y$ is usually

called the normal bundle of the foliated manifold Y (not to be confused with the normal bundle of the submanifold Y itself).

The condition (ii) says that for any section v of $\mathcal{L}Y$ we have $\iota_v F = 0$. Since $dF = 0$, this implies that the Lie derivative of F along such v vanishes, i.e. F is constant along the leaves of the foliation. In the case when the characteristic foliation is a fiber bundle with a smooth base Z, this is equivalent to saying that f is a pull-back of a closed 2-form on Z. In general, one can say that f is a transversely-closed form on a foliated manifold Y.

The condition (iii) implies, first of all, that f is non-degenerate. Thus f is a transverse symplectic structure on Y, just like σ. Second, the condition (iii) says that the ratio of the two transverse symplectic structures is a transverse almost complex structure on the foliated manifold Y. If the characteristic foliation is a fiber bundle with a smooth base Z, then $\mathcal{J} = \sigma^{-1} f$ is simply an almost complex structure on Z.

An easy consequence of these conditions is that the dimension of Y must be $n + 2k$, where $n = \frac{1}{2} \dim_{\mathbb{R}} X$, and k is a non-negative integer. The number k has the meaning of "transverse complex dimension" of Y. If $k = 0$, then the condition (i) says that Y is a Lagrangian submanifold, and then the condition (ii) forces F to vanish. (The condition (iii) is vacuous in this case). Another interesting special case occurs when $Y = X$ (this is possible only if n is even). In this case the leaves of the characteristic foliation are simply points, the conditions (i) and (ii) are trivially satisfied, and the condition (iii) says that $\omega^{-1} F$ is an almost complex structure on X.

A less obvious property is that the transverse almost complex structure $\mathcal{J}$ is integrable [20]. This follows easily from the well-known Gelfand-Dorfman theorem which plays an important role in the theory of integrable systems [11]. Thus Y is a transverse complex manifold. It is also easy to see that both f and σ have type $(0,2) + (2,0)$ with respect to $\mathcal{J}$. In fact, $f + i\sigma$ is a transverse holomorphic symplectic form on the transverse complex manifold Y.

The somewhat mysterious condition (iii) can be rewritten in several equivalent forms. For example, an equivalent set of conditions is

$$\wedge^r (f + i\sigma) \neq 0, \quad r < k, \quad \wedge^k (f + i\sigma) = 0.$$

Here k is related to the dimension of Y as above.

Our attempts to generalize the conditions (i)-(iii) to A-branes which carry vector bundles of rank higher than one have been only partially successful. The first two conditions (i) and (ii) remain unchanged, but the condition (iii) causes problems. Both physical and mathematical arguments indicate that the correct generalization of (iii) looks as follows:

$$(\sigma^{-1} f)^2 = -\mathrm{id}_{E \otimes \mathcal{N}Y}.$$

Here the "transverse" curvature 2-form f is regarded as a section of $\mathrm{End}(E) \otimes \Lambda^2 (\mathcal{N}Y^*)$. This condition on f does not lead to a transverse complex structure on Y, and its geometric significance is unclear. This leads to problems when one tries to understand morphisms between such A-branes (see below).

So far our discussion of A-branes was classical. One the quantum level $N = 2$ super-Virasoro can be broken by anomalies. The absence of such anomalies is an important additional condition on A-branes. Let us focus our attention on the R-current J whose

Fourier modes we denoted by J_n and $\bar{J}_n$ previously. Its conservation can be spoiled only by non-perturbative effects on the world-sheet, i.e. by Riemann surfaces in X whose boundaries lie in Y and which cannot be continuously deformed to a point. In the case when Y is Lagrangian, the conditions for the absence of anomalies have been analyzed by K. Hori in [17]. The result is that there are no anomalies if and only if the Maslov class of Y vanishes. This provides a physical interpretation of the vanishing of the Maslov class for objects of the Fukaya category.

For coisotropic Y the condition for anomaly cancellation has been obtained in [30]. Let F be the curvature 2-form of the line bundle on Y, and let the dimension of Y be $n+2k$, as before. Let Ω be a holomorphic trivialization of the canonical line bundle on X. One can show that the $n + 2k$-form $\Omega|_Y \wedge F^k$ is nowhere vanishing, and therefore we have $\Omega|_Y \wedge F^k = h \cdot vol$, where vol is the volume form, and h is a smooth nowhere-vanishing complex-valued function on Y. We may regard h as an element of $H^0(Y, \mathcal{C}_Y^{\infty*})$, where $\mathcal{C}_Y^{\infty*}$ is the sheaf of smooth $\mathbb{C}^*$-valued functions on Y. Let $\alpha_h \in H^1(Y, \mathbb{Z})$ be the image of h under the Bockstein homomorphism. The anomaly of the R-current is absent if and only if $\alpha_h = 0$.

In the case when X is a torus with a constant symplectic form, Y is an affine subtorus, and the curvature 2-form F is constant, one can quantize the sigma-model and verify directly that the $N = 2$ super-Virasoro algebra is preserved on the quantum level. This shows that non-Lagrangian A-branes exist on the quantum level.

We hope that we have given convincing arguments that A-branes are not necessarily associated to Lagrangian submanifolds, and that the Fukaya category should be enriched with more general coisotropic A-branes. Unfortunately, we do not have a definite proposal for what should replace the Fukaya category. In the remainder of this lecture we will describe a heuristic idea which could help to solve this problem.

In these lectures we have encountered two kinds of A-branes. First, we have discussed objects of the Fukaya category, i.e. triples (Y, E, ∇_E) where Y is a graded Lagrangian submanifold, E is a vector bundle on Y with a Hermitian metric, and ∇_E is a flat unitary connection on E. Second, we have triples (Y, E, ∇_E), where E is a Hermitian *line* bundle on Y, ∇_E is a unitary connection on E, and the conditions (i)-(iii) are satisfied. For objects of the Fukaya category we know in principle how to compute spaces of morphisms and their compositions. Let us try to guess what the recipe should be for objects of the second type.

We need to generalize the Floer complex to coisotropic A-branes. To guess the right construction, it is useful to recall the intuition which underlies the definition of the Floer complex. Consider the space of smooth paths in X, which we will denote PX. This space is infinite-dimensional, but let us treat it as if it were a finite-dimensional manifold. We have a natural 1-form α on PX obtained by integrating the symplectic form ω on X along the path. More precisely, if $\gamma : I \to X$ is a path, and β is a tangent vector to PX at point γ (i.e. a vector field along the image of γ), then the value of α on β is defined to be

$$\int_I \omega(\dot{\gamma}(t), \beta(t)) dt.$$

Note that the space PX has two natural projections to X which we denote π_1 and π_2. It is easy to see that $d\alpha = \pi_2^*\omega - \pi_1^*\omega$. Thus if we consider a submanifold in PX consisting of paths which begin and end on fixed Lagrangian submanifolds in X, then the restriction of α to such a submanifold will be closed.

Let Y_1 and Y_2 be Lagrangian submanifolds in X, and let $PX(Y_1, Y_2)$ be the submanifold of PX consisting of paths which begin at Y_1 and end at Y_2. Then the operator $Q = d + 2\pi\alpha$ on the space of differential forms on $PX(Y_1, Y_2)$ satisfies $Q^2 = 0$, and we may try to compute its cohomology (in the finite-dimensional case this complex is called the twisted de Rham complex.) Since $PX(Y_1, Y_2)$ is infinite-dimensional, it is not easy to make sense of the twisted de Rham cohomology. A. Floer solved this problem by a formal application of Morse theory. That is, if we formally apply the Morse-Smale-Witten-Novikov theory to the computation of the cohomology of Q, we get precisely the Floer complex for the pair Y_1, Y_2.

This construction ignores the bundles E_1 and E_2, but it is easy to take them into account. Consider the bundles $\pi_1^* E_1^*$ and $\pi_2^* E_2$ on $PX(Y_1, Y_2)$. Both vector bundles have natural unitary connections obtained by pulling back $\nabla_{E_1^*}$ and ∇_{E_2}. We tensor them, and then add the 1-form $2\pi\alpha$ to the connection on the tensor product. Since $d\alpha = 0$, the resulting connection is still flat, but no longer unitary. If we formally use Morse theory to compute the cohomology of the resulting twisted de Rham complex on $PX(Y_1, Y_2)$, we get the Floer complex for a pair of objects of the Fukaya category (Y_1, E_1, ∇_{E_1}) and (Y_2, E_2, ∇_{E_2}).

Now consider a pair of coisotropic A-branes, instead of a pair of Lagrangian A-branes. We assume in addition that the bundles E_1 and E_2 are line bundles. By $PX(Y_1, Y_2)$ we still denote the space of smooth paths in X beginning at Y_1 and ending on Y_2. The first difficulty is that the restriction of α to $PX(Y_1, Y_2)$ is not closed, so we cannot use it to define a twisted de Rham complex. The second difficulty is that connections $\pi_1^*(\nabla_{E_1^*})$ and $\pi_2^*(\nabla_{E_2})$ on the bundles $\pi_1^* E_1^*$ and $\pi_2^* E_2$ are also not flat. However, thanks to conditions (ii) and (iii) these two difficulties "cancel" each other, as we will see in a moment. So let us proceed as in the Lagrangian case: tensor the line bundles $\pi_1^* E_1^*$ and $\pi_2^* E_2$, and add $2\pi\alpha$ to the connection. The resulting connection is not flat, but it has the following interesting property. Note that since Y_1 and Y_2 are foliated manifolds, $PX(Y_1, Y_2)$ also has a natural foliation. A leaf of this foliation consists of all smooth paths in X which begin on a fixed leaf in Y_1 and end on a fixed leaf of Y_2. The codimension of the foliation is finite and equal to the sum of codimensions of characteristic foliations of Y_1 and Y_2. Further, since Y_1 and Y_2 have natural transverse complex structures, the foliated manifold $PX(Y_1, Y_2)$ also has one. The connection on $\pi_1^* E_1^* \otimes \pi_2^* E_2$ has the following properties:

(A) it is flat along the leaves of the foliation;
(B) its curvature $\pi_1^*(F_1 + i\omega) - \pi_2^*(F_2 + i\omega)$ has type $(2, 0)$ in the transverse directions.

Thus it makes sense to consider a sheaf of sections of the line bundle $\pi_1^* E_1^* \otimes \pi_2^* E_2$ which are covariantly constant along the leaves and holomorphic in the transverse directions. It is natural to propose the cohomology of this sheaf as the candidate for the space of morphisms between the A-branes (Y_1, E_1, ∇_{E_1}) and (Y_2, E_2, ∇_{E_2}). This proposal can be formally justified by considering the path integral quantization of the topologically twisted σ-model

on an interval with boundary conditions corresponding to the A-branes (Y_1, E_1, ∇_{E_1}) and (Y_2, E_2, ∇_{E_2}). To get a rigorous definition of the spaces of morphisms, our formal proposal must be properly interpreted. In the case of Lagrangian A-branes, the above sheaf becomes the sheaf of covariantly constant sections of a flat line bundle on $PX(Y_1, Y_2)$, and one can interpret its cohomology using Morse-Smale-Witten-Novikov theory. We do not know how to make sense of our formal proposal in general.

The difficulty of generalizing the Floer complex to coisotropic A-branes suggests that perhaps the geometric description of A-branes by means of submanifolds and vector bundles on them is not the right way to proceed. Let us explain what we mean by this using an analogy from complex-analytic geometry. There exists a general notion of a holomorphic vector bundle on a complex manifold, whose special case is the notion of a holomorphic line bundle. One can study line bundles in terms of their divisors, but this approach does not extend easily to higher rank bundles. Perhaps objects of the Fukaya category, as well as coisotropic A-branes of rank one, are symplectic analogues of divisors, and in order to make progress one has to find a symplectic analogue of the notion of a holomorphic vector bundle (or a coherent sheaf). This analogy is strengthened by the fact that both divisors and geometric representations of A-branes by means of Lagrangian or coisotropic submanifolds provide a highly redundant description of objects in the respective categories: a line bundles does not change if one adds to the divisor a principal divisor, while objects of the Fukaya category are unchanged by flows along Hamiltonian vector fields. We believe that a proper definition of the category of A-branes will be very useful for understanding Mirror Symmetry, and perhaps also for symplectic geometry as a whole.

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