1.1 Compute the Fourier expansions, the means and the variances of the following functions:
(a) $\mathrm{AND}_3 \colon \{-1,1\}^3 \to \{-1,1\}$;
(b) the inner product mod 2 function, $\mathrm{IP}_{2n} \colon \mathbb{F}^{2n} \to \{-1,1\}$ defined by $\mathrm{IP}_{2n}(x_1,\dots,x_n,y_1,\dots,y_n) = (-1)^{\sum_i x_i y_i}$;
(c) the selection function $\mathrm{Sel} \colon \{-1,1\}^3 \to \{-1,1\}$, which outputs x_2 if $x_1 = -1$ and outputs x_3 if $x_1 = 1$;
(d) the majority function $\mathrm{MAJ}_5 \colon \{-1,1\}^5 \to \{-1,1\}$.

1.2 Prove that there are no functions $f \colon \{-1,1\}^n \to \{-1,1\}$ with exactly 2 nonzero Fourier coefficients. What about exactly 3 nonzero Fourier coefficients?

1.3 For $f \colon \{-1,1\}^n \to \{-1,1\}$ show that

$$\mathbf{Var}[f] = 1 - \mathbf{E}[f]^2 = 4 \Pr_x[f(x) = 1] \Pr_x[f(x) = -1].$$

1.4 Prove that any $f \colon \{-1,1\}^n \to \{-1,1\}$ has at most one Fourier coefficient with magnitude exceeding $1/2$. Is this also true for any $f \colon \{-1,1\}^n \to \mathbb{R}$ with $||f||_2 = 1$?

1.5 Prove the equivalence of two definitions of linear functions.

Problems for homework **Due: January, 24, 2018**

1.6 Compute the Fourier expansion, the mean and the variance of the complete quadratic function, $\mathrm{CQ}_n \colon \mathbb{F}^n \to \{-1,1\}$ defined by $\mathrm{CQ}_n(x_1,\dots,x_n) = (-1)^{\sum_{i<j} x_i x_j}$;

1.7 Let $f \colon \{-1,1\}^n \to \{-1,1\}$. Suppose $\mathbf{W}^1[f] = 1$. Show that $f(x) = \pm\chi_S(x)$ for some $|S| = 1$.

1.8 Let $f \colon \{-1,1\}^n \to \{-1,1\}$. Then $2\epsilon \le \mathbf{Var}[f] \le 4\epsilon$ where

$$\epsilon = \min\{\mathsf{dist}(f,1), \mathsf{dist}(f,-1)\}.$$