

10.1 Show that for any prime p and any parameter l any $AC^0[p]$ circuit of depth d and size S can be approximated by a polynomial over \mathbb{F}_p of degree $((p-1)l)^d$ making an error in at most $S/2^l$ fraction of points.

10.2 For any prime p show that the function $PARITY_n$ cannot be approximated in 99/100 fraction of points by a polynomial over \mathbb{F}_p of degree at most $c\sqrt{n}$ for some constant c (depending on p).

10.3 Show that $PARITY_n$ can be computed by a polynomial size constant depth circuit consisting of AND, OR, NOT and MAJ gates (of unbounded fan-in).

10.4 Suppose $S_0 = \{x_1, \dots, x_n\}$. For each $i \leq C \log n$ (for large enough constant C) let S_i be constructed randomly from S_{i-1} by removing each variable with probability $1/2$ independently. Let $a_i = \sum_{x_j \in S_i} x_j$. Assume that for some $x \in \{0, 1\}^n$ we have $a_0 > 0$. Show that there is i with $a_i = 1$ with probability at least ε for some constant ε . (Possible choice of parameters is $C \log n = \log n + 1$ and $\varepsilon = 1/3$.)

10.5 Show that any AC^0 circuit of depth d and size s can be computed by a polynomial of degree $O((\log(s/\varepsilon) \log(s))^d)$ correctly on $1 - \varepsilon$ fraction of inputs.

10.6 Show that any $THR \circ AC^0$ circuit of depth $d + 1$ and size s can be computed by the sign of a polynomial of degree $O((\log(s/\varepsilon) \log(s))^d)$ correctly on $1 - \varepsilon$ fraction of inputs.

10.7 Suppose for $S \subseteq \{0, 1\}^n$ we have $|S| < \sum_{i=0}^k \binom{n}{i}$. Show that there is a polynomial q of degree at most $2k$ such that $q \neq 0$, $q(x) \geq 0$ for all $x \in \{0, 1\}^n$, and $q(x) = 0$ for all $x \in S$.

10.8 Show that the following statements are equivalent:

1. There is a degree- d polynomial p such that $PARITY_n(x) = \text{sgn}(p(x))$ for at least m inputs in $\{0, 1\}^n$;
2. $m \leq \sum_{0 \leq k \leq \frac{1}{2}(n+d+1)} \binom{n}{k}$.

10.9 Deduce that $PARITY$ cannot be computed by polynomial size $THR \circ AC^0$ circuit.