3.1 Show that $\mathrm{T}_{\rho_1}\mathrm{T}_{\rho_2} = \mathrm{T}_{\rho_1\rho_2}$.

For $f\colon \{-1,1\}^n \to \mathbb{R}$, $\rho \in [0,1]$ and $i \in [n]$, the *ρ-stable influence of i on f* is

$$\mathbf{Inf}_i^{(\rho)}[f] = \mathbf{Stab}_\rho[\mathrm{D}_i f].$$

Let $\mathbf{Inf}^{(\rho)}[f] = \sum_{i=1}^n \mathbf{Inf}_i^{(\rho)}[f]$.

3.2 Show that

$$\mathbf{Inf}_i^{(\rho)}[f] = \sum_{S \ni i} \rho^{|S|-1}\widehat{f}(S)^2.$$

3.3 Show that

$$\mathbf{Inf}^{(\rho)}[f] = \frac{d}{d\rho}\mathbf{Stab}_\rho[f] = \sum_{k=1}^n k\rho^{k-1}\mathbf{W}_k[f].$$

3.4 Show that for all $k \in \mathbb{N}_+$ and $0 < \delta \le 1$ we have $k(1-\delta)^{k-1} \le 1/\delta$.

3.5 Suppose $f\colon \{-1,1\}^n \to \mathbb{R}$ has $\mathbf{Var}[f] \le 1$. Given $0 < \delta, \varepsilon \le 1$, let $J = \{i \in [n] \mid \mathbf{Inf}_i^{1-\delta}[f] \ge \varepsilon\}$. Show that $|J| \le \frac{1}{\delta\varepsilon}$.

3.6 Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ has all $\widehat{f}_i$ equal. Show that $\mathbf{W}^1[f] \le 2/\pi + o(1)$.

3.7 Show that Arrow's theorem also holds if three different unanimous functions $f, g, h\colon \{-1,1\}^n \to \{-1,1\}$ are used in the three pairwise elections.

3.8 Define the *average influence* of $f\colon \{-1,1\}^n \to \mathbb{R}$ to be $\mathcal{E}[f] = \frac{1}{n}\mathbf{Inf}[f]$. For $f\colon \{-1,1\}^n \to \{-1,1\}$ show that

$$\mathcal{E}[f] = \Pr_{\boldsymbol{x}\sim\{-1,1\}^n, \boldsymbol{i}\in[n]}[f(\boldsymbol{x}) \ne f(\boldsymbol{x}^{\oplus \boldsymbol{i}})]$$

and

$$\frac{1-e^{-2}}{2}\mathcal{E}[f] \le \mathbf{NS}_{1/n}[f] \le \mathcal{E}[f].$$

Problems for homework **Due: February, 7, 2019**

3.9 Suppose $f_1, \ldots, f_s\colon \{-1,1\}^n \to \{-1,1\}$ satisfy $\mathbf{NS}_\delta[f_i] \le \varepsilon_i$. Let $g\colon \{-1,1\}^s \to \{-1,1\}$ and define $h\colon \{-1,1\}^n \to \{-1,1\}$ by $h = g(f_1, \ldots, f_s)$. Show that $\mathbf{NS}_\delta[h] \le \sum_{i=1}^s \varepsilon_i$.

3.10 Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ has

$$\min(\Pr[f=1], \Pr[f=-1]) = \alpha.$$

Show that $\mathbf{NS}_\delta[f] \le 2\alpha$ for all $\delta \in [0,1]$.