Note. In this problem list you might want to use the Cauchy-Schwartz inequality $|\mathbf{E}[f(x)g(x)]| \leq \|f\|_2\|g\|_2$ and, more generally, Hölder's inequality $\|fg\|_1 \leq \|f\|_p\|g\|_q$ for $1/p + 1/q = 1$.

6.1 Show that for any $f\colon \{-1,1\}^n \to \mathbb{R}$ and any $k \in \mathbb{N}$ we have

$$\|\mathrm{T}_{1/\sqrt{3}} f^{=k}\|_4 \leq \|f^{=k}\|_2.$$

6.2 ((2, 4)**-Hypercontractivity**) Show that for any $f\colon \{-1,1\}^n \to \mathbb{R}$ we have

$$\|\mathrm{T}_{1/\sqrt{3}} f\|_4 \leq \|f\|_2.$$

6.3 ((4/3, 2)**-Hypercontractivity**) Show that for any $f\colon \{-1,1\}^n \to \mathbb{R}$ we have

$$\|\mathrm{T}_{1/\sqrt{3}} f\|_2 \leq \|f\|_{4/3}.$$

Deduce that

$$\mathbf{Stab}_{1/3}[f] \leq \|f\|_{4/3}^2.$$

6.4 Show that for any $A \subseteq \{-1,1\}^n$ with $\Pr_{\boldsymbol{x}}[x \in A] = \alpha > 0$ we have

$$\Pr_{\substack{\boldsymbol{x}\sim A \\ \boldsymbol{y}\sim N_{1/3}(\boldsymbol{x})}}[\boldsymbol{y} \in A] \leq \alpha^{1/2}.$$

6.5 Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Show that $\mathbf{Inf}_i^{(1/3)}[f] \leq \mathbf{Inf}_i[f]^{3/2}$ for all i (recall from Problem list 3 that by definition $\mathbf{Inf}_i^{(\rho)}[f] = \mathbf{Stab}_\rho[\mathrm{D}_i f]$).

6.6 Show that for any $A \subseteq \{-1,1\}^n$ with $\Pr_{\boldsymbol{x}}[x \in A] = \alpha > 0$ we have

$$\mathbf{W}^{\leq k}[f] \leq 3^k \alpha^{3/2}.$$

6.7 Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be independent, not necessarily identically distributed, random variables satisfying $\mathbf{E}[\boldsymbol{x}_i] = \mathbf{E}[\boldsymbol{x}_i^3] = 0$. Assume that each $\boldsymbol{x}_i$ is B-reasonable. Let $\boldsymbol{f} = F(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, where F is a multilinear polynomial of degree at most k. Show that $\boldsymbol{f}$ is $\max(B, 9)^k$-reasonable.

Problems for homework **Due: February, 21, 2019**

6.8 Suppose for $\boldsymbol{X} \not\equiv 0$ we have $\mathbf{E}[\boldsymbol{X}^6] \leq B\,\mathbf{E}[\boldsymbol{X}^2]^3$ for some constant B. Show that

$$\Pr[|\boldsymbol{X}| \geq t\|\boldsymbol{X}\|_2] \leq \frac{B}{t^6}.$$

6.9 For every $1 < b < B$ show that there is a b-reasonable random variable X such that $1 + X$ is not B-reasonable.

6.10 Show that if $\boldsymbol{X}$ is B-reasonable, then $\mathbf{E}[|X|^3] \leq \sqrt{B}\,\mathbf{E}[X^2]^{3/2}$.