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Nonexistence of Positive Solutions for Quasilinear Elliptic Problems on $\mathbb{R}^N$

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1. INTRODUCTION

The purpose of this paper is to prove some new nonexistence theorems for quasilinear elliptic equations and systems of quasilinear elliptic equations and inequalities on $\mathbb{R}^N$, $N \geq 1$. These theorems are proved by the method of integral relations; the comparison method and the maximum principle are not used.

The importance of nonexistence results in the theory of partial differential equations is well known and has its roots on the fundamental Liouville theorem for harmonic functions.

In the last twenty years a big deal has been devoted to prove the so-called "nonlinear Liouville theorems" for positive solutions of elliptic equations and systems (see, among other papers, [20, 21]). An important motivation to extend the "linear" Liouville theorems to nonlinear problems is that this kind of results can be used to prove existence results for the associated Dirichlet problem in bounded domain.[3] This is particularly useful if the problem under consideration is nonvariational (see, for example, [2, 3, 8, 11, 15]).

When dealing with superlinear or superhomogeneous problems in bounded domains, a method that can be used to prove the a priori bounds for positive solutions is the so called "blow-up method" (see [8, 15, 16]). This method relies on the application of the celebrated result due to Gidas and Spruck [17] concerning the absence of positive solutions for semilinear elliptic equations on $\mathbb{R}^N$.

In one of its simplest form this result can be stated as follows:

Let $N \geq 3$. If $1 < q < \frac{N+2}{N-2}$, then the problem

$$\begin{cases} -\Delta u = u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{on } \mathbb{R}^N, \\ u \in C^2(\mathbb{R}^N) \end{cases} \tag{1.1}$$

has no solutions.

One of the interesting features of this important theorem is that no *a priori* information on the behavior of the possible solutions at infinity is needed.

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[3]Another reason for studying nonexistence results relies on the fact that they may be used for proving "blow-up" estimates for parabolic problems (see [9, 18, 19] and the references therein).

Indeed, it is not difficult to show that the result holds even for wider class of equations under suitable decay conditions on the possible solutions to (1.1) at infinity.[4]

Another interesting fact is that the Gidas–Spruck result is sharp. This means that the statement does not hold for any $q \geq \frac{N+2}{N-2}$. It easily follows from an ODE argument (see, for example, [12] and [28]) applied to radial solutions to (1.1) with $q \geq \frac{N+2}{N-2}$. Indeed, if $q = \frac{N+2}{N-2}$, the Sobolev minimizer

$$u(x) := \frac{k}{(1+|x|^2)^{\frac{N-2}{2}}}, \qquad x \in \mathbb{R}^N,$$

with a suitable normalization constant $k > 0$ is the unique solution to (1.1) with $u(0) = k$, while for $q > \frac{N+2}{N-2}$ problem (1.1) possesses a unique solution for every fixed $u(0) > 0$.

A natural question that we address in this paper is the following: Is it possible to prove quasilinear versions of the Gidas–Spruck theorem? The first result in this direction has been proved by M.F. Bidaut-Veron [4], for the problem with the p-Laplacian of the form (2.1) in an exterior domain. The technique used by Bidaut-Veron is very interesting and it consists in a reduction procedure within the class of radial solutions. To the authors knowledge, all other available results (see [10, 14, 23, 27–29]) deal with radial solutions.

Let us state more precisely the kind of problems that we will study in this paper.

Let $\mathbb{A}$ be a second-order differential operator in the divergence form and let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. Find sufficient conditions on the pair $(\mathbb{A}, f)$ implying the nonexistence of positive solutions to

$$\mathbb{A}(u) = f(u) \qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $u \in S$. Here S is a suitable functional class that depends on the pair $(\mathbb{A}, f)$. Let us consider an important example (see [33]).

Let $N \geq 3$, $1 < q < \frac{N+2}{N-2}$,

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{on } \mathbb{R}^N, \\ u \in C^2(\mathbb{R}^N). \end{cases} \tag{1.3}$$

It is known [13] that, if we restrict our attention to radial solutions such that

$$0 < u(0) \leq (q+1)^{\frac{1}{q+1}} \left(\frac{N+2-q(N-2)}{N+q+1}\right)^{\frac{1}{1+q}}, \tag{1.4}$$

then (1.3) has no solutions. It is still an open question to prove that (1.3) has no solutions without assuming (1.4).

[4] A simple assumption that readily gives the result via a variational identity [31] is

$$u(x) \leq C|x|^{\frac{-2}{q-1}} \quad \forall x \neq 0.$$

Now, if instead of (1.3) we consider the corresponding differential inequality

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) \geq u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{on } \mathbb{R}^N, \\ u \in C^2(\mathbb{R}^N), \end{cases} \tag{1.5}$$

then, as a consequence of the method developed in this paper, the following result holds.

Theorem 1.1. *Let one of the following assumptions be satisfied:*

(i) $q \geq 0, \quad N = 1, 2;$

(ii) $1 < q \leq \dfrac{N}{N-2}, \quad N \geq 3.$

Then (1.5) *has no solutions.*[5]

Moreover, if $q > \frac{N}{N-2}$ *and* $\varepsilon > 0$ *is small, then*

$$u(x) = \frac{\varepsilon}{(1+|x|^2)^{\frac{1}{q-1}}}, \qquad x \in \mathbb{R}^N, \tag{1.6}$$

is a solution to (1.5).

The main goal of this paper is to show that, under reasonable assumptions on the pair $(\mathbb{A}, f)$, sharp nonexistence theorems hold for the problem

$$\begin{cases} \mathbb{A}(u) \geq f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in S. \end{cases} \tag{1.7}$$

Let us say now about functional spaces of solutions. For differential inequalities, in contrast to the case of equations, there is no theory of regular solutions. This is a principal difference, and that is why the definition of the class of solutions plays an essensial role. It is easy to construct examples of differential inequalities where solutions exist in one functional class and do not in another.

In this paper, we systematically use the local Sobolev spaces with nonlinear nonnegative (or positive) weights of the kind u^α or (for systems) $u^\alpha v^\beta$ defined by parameter $\alpha \in \mathbb{R}$ or $\alpha, \beta \in \mathbb{R}$, respectively. In noncoercive problems these parameters are nonpositive, in coercive ones—nonnegative.

It is clear that the use of nonlinear weights is substantiated only if the corresponding integrals exist on bounded sets in $\mathbb{R}^N$. In every particular case (theorem), these integrals are given explicitly, in the proofs. So, in nonexistence theorems we always assume that possible solutions lie in the local function space for which the corresponding integrals are defined. This is the meaning of the statement that a problem has no solutions in the corresponding class of locally bounded integrals.

This paper is organized as follows. In Section 2 we first study a model problem associated with the p-Laplacian operator and then discuss some generalizations. Section 3 deals with nonexistence results for a class of quasilinear inequalities involving (possibly) degenerate operators. In Section 4 we study a class of coercive inequalities and see how a simple modification of the method used in the

[5] If $u \geq 0$, it is understood that $u^0 \equiv 1$.

preceding sections can be used to prove nonexistence results in this case as well. Section 5 contains some extensions of the results of Section 2 to systems of quasilinear elliptic inequalities. We will study first a natural vector-valued version of Theorem 2.1 (see Section 2) to a (p,q)-Laplacian system and then present some generalizations to nonhomogeneous problems. Section 6 deals with quasilinear elliptic inequalities containing nonlinearities that strongly depend on the gradient of the solutions. We will first prove a nonexistence result for a model problem (see (6.3) below) and then indicate some extensions to more general elliptic inequalities.

Finally, in Sections 7 and 8 we analyze some systems of quasilinear inequalities with "weak" and "strong" coupling respectively.

Throughout this paper, all indefinite integrals are understood as taken on the support of a standard cut-off function φ that will be specified during the course; in addition, if $u \geq 0$ we will understand that $u^0 \equiv 1$.

2. A MODEL PROBLEM AND SOME GENERALIZATIONS

The aim of this section is to illustrate the applications of our method by means of a simple example. Then, we will present some generalizations. Let us begin with specifying the functional class of solutions that we will consider throughout.

Let $\alpha < 0$ be sufficiently small, $p > 1$ and $q \geq 0$. We define S (see the Introduction) by

$$S := W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) = \left\{u\colon \mathbb{R}^N \to \mathbb{R}_+, \quad u^{q+\alpha}, |Du|^p u^{\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\right\},$$

where $D_i u$, $i = 1, \dots, N$, are understood in the sense of distributions.

Our first result is the following

Theorem 2.1. *If either*

(i) $0 < p-1 < q \leq \dfrac{N(p-1)}{N-p}$ *and* $N > p$, *or*

(ii) $0 \leq q \leq p-1$, $p > 1$ *and* $N \geq 1$,

then the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) \geq u^q & \text{in } R^N, \\ u \geq 0, \ u \not\equiv 0 & \text{on } \mathbb{R}^N \end{cases} \tag{2.1}$$

has no solutions of class $W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$.

Proof. We shall organize the proof in case when (i) holds in two steps. The first one deals with the case $p-1 < q < \frac{N(p-1)}{N-p}$, while in the second one we consider the critical case $q = \frac{N(p-1)}{N-p}$.

First step. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a standard cut-off function that we will specify later. Let $\alpha < 0$ be a parameter (α will also be chosen later). Without loss of generality we may suppose that $u > 0$ on $\mathbb{R}^N$ (otherwise, we consider $u_\varepsilon := u + \varepsilon$ and let $\varepsilon \downarrow 0$ in the course). Multiplying (2.1) by $u^\alpha\varphi$ and integrating by parts, we obtain

$$\begin{aligned} \int u^{q+\alpha}\varphi\,dx &\leq \alpha \int |Du|^p u^{\alpha-1}\varphi\,dx + \int |Du|^{p-2}(Du, D\varphi)u^\alpha\,dx \\ &\leq \alpha \int |Du|^p u^{\alpha-1}\varphi\,dx + \int |Du|^{p-1}|D\varphi|u^\alpha\,dx \end{aligned}$$

and, by the Young inequality with parameter $\varepsilon > 0$,

$$\int u^{q+\alpha}\varphi\,dx + |\alpha|\int |Du|^p u^{\alpha-1}\varphi\,dx \\ \le \frac{\varepsilon^p(p-1)}{p}\int |Du|^p u^{\alpha-1}\,dx + \frac{1}{p\varepsilon^p}\int u^{\alpha+p-1}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\,. \tag{2.2}$$

By putting $\theta_\varepsilon = |\alpha| - \frac{\varepsilon^p(p-1)}{p} > 0$ and $\theta'_\varepsilon = \frac{1}{\varepsilon^p p}$, inequality (2.2) becomes

$$\int u^{q+\alpha}\varphi\,dx + \theta_\varepsilon\int |Du|^p u^{\alpha-1}\varphi\,dx \le \theta'_\varepsilon\int u^{\alpha+p-1}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\,. \tag{2.3}$$

Now, by choosing $\varkappa > 1$ such that $\frac{1}{\varkappa} + \frac{1}{\varkappa'} = 1$ and applying again the Young inequality, we obtain

$$c_\varepsilon\int u^{q+\alpha}\varphi\,dx + \theta_\varepsilon\int |Du|^p u^{\alpha-1}\varphi\,dx \le \theta''_\varepsilon\int u^{(\alpha+p-1)\varkappa}\varphi\,dx + \widetilde{\theta}_\varepsilon\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\,. \tag{2.4}$$

Choosing $\varkappa = \frac{\alpha+q}{\alpha+p-1}$ (note that it is possible by (i) and small $\alpha < 0$) we get

$$\theta_\varepsilon\int u^{q+\alpha}\varphi\,dx + \theta_\varepsilon\int |Du|^p u^{\alpha-1}\,dx \le \widetilde{\theta}_\varepsilon\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\,. \tag{2.5}$$

Next we estimate the term $\int u^q\varphi\,dx$. By multiplying (2.1) by φ and integrating by parts we find

$$\int u^q\varphi\,dx \le \int |Du|^{p-2}(Du, D\varphi)\,dx \le \int |Du|^{p-1}|D\varphi|\,dx \\ \le \int |Du|^{p-1}u^{\frac{(\alpha-1)(p-1)}{p}}\varphi^{\frac{1}{p}}u^{\frac{(1-\alpha)(p-1)}{p}}\varphi^{-\frac{1}{p}}|D\varphi|\,dx \\ \le \left(\int |Du|^p u^{\alpha-1}\varphi\,dx\right)^{\frac{p-1}{p}}\left(\int u^{(1-\alpha)(p-1)}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\right)^{\frac{1}{p}}. \tag{2.6}$$

By Hölder inequality with parameter $a > 1$ ($\frac{1}{a} + \frac{1}{a'} = 1$), the last term can be estimated as

$$\int u^q\varphi\,dx \le \left(\int |Du|^p u^{\alpha-1}\,dx\right)^{\frac{p-1}{p}}\left(\int u^{a(1-\alpha)(p-1)}\varphi\,dx\right)^{\frac{1}{ap}}\left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{a'p}}. \tag{2.7}$$

By choosing a such that $a(1-\alpha)(p-1) = q+\alpha$ (note that this choice is also possible by (i) for $q > p-1$) and combining (2.7) with (2.5), we obtain

$$\int u^q\varphi\,dx \le \left(\frac{\widetilde{\theta}_\varepsilon}{\theta_\varepsilon}\right)^{\frac{p-1}{p}}\left(\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\right)^{\frac{p-1}{p}}\left(\frac{\widetilde{\theta}_\varepsilon}{c_\varepsilon}\right)^{\frac{1}{ap}}\left(\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\right)^{\frac{1}{ap}}\left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{a'p}} \\ = \overline{C}_\varepsilon\left(\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\right)^{\frac{p-1}{p}+\frac{1}{ap}}\left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{a'p}}, \tag{2.8}$$

where $\overline{C}_\varepsilon = \left(\frac{\widetilde{\theta}_\varepsilon}{c_\varepsilon}\right)^{\frac{1}{ap}}\left(\frac{\widetilde{\theta}_\varepsilon}{\theta_\varepsilon}\right)^{\frac{p-1}{p}}$. Next we choose the function φ. Let $\xi \in C_0^\infty(\mathbb{R})$ and

$$\xi(x) = \xi_0\left(\frac{|x|}{R}\right), \qquad R > 0\,, \quad 0 \le \xi_0 \le 1\,, \tag{2.9}$$

with $\xi_0 \in C^\infty(\mathbb{R}_+)$ and

$$\xi_0(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & 2 \ge t. \end{cases}$$

We define φ by $\varphi := \xi^\lambda$ with λ chosen sufficiently large. With this choice of φ, by a standard change of variables, it follows from (2.8) that

$$\int_{B_R} u^q(x)\,dx \le \widetilde{C} R^\sigma\,, \tag{2.10}$$

where $\sigma = (N - p\varkappa')(\frac{p-1}{p} + \frac{1}{ap}) + (\frac{N-pa'}{pa'})$. Taking into account our choice of $\varkappa'$ and a, we find

$$\sigma = \frac{q(N-p) - N(p-1)}{q-p+1}\,. \tag{2.11}$$

Since $\sigma < 0$, inequality (2.10) implies

$$\int_{\mathbb{R}^N} u^q(x)\,dx = 0\,.$$

This contradicts our assumption that $u > 0$ thereby concluding the proof in the case $\sigma < 0$.

Second step. We pass to the case $q = \frac{N(p-1)}{N-p}$, i.e., $\sigma = 0$. By our choice of φ we have

$$\int_{B_R} \varphi u^q(x)\,dx \le \int_{S(D\varphi)} |Du|^{p-1}|D\varphi|\,dx$$

$$\le \left(\int_{S(D\varphi)} |Du|^p u^{(\alpha-1)}\varphi\,dx\right)^{\frac{p-1}{p}} \left(\int_{S(D\varphi)} u^{(1-\alpha)(p-1)}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_{S(D\varphi)} |Du|^p u^{(\alpha-1)}\varphi\,dx\right)^{\frac{p-1}{p}} \left(\int_{S(D\varphi)} u^{m(1-\alpha)(p-1)}\varphi\,dx\right)^{\frac{1}{pm}} \left(\int_{S(D\varphi)} \frac{|D\varphi|^{pm'}}{\varphi^{pm'-1}}\,dx\right)^{\frac{1}{pm'}}\,,$$

where $\frac{1}{m} + \frac{1}{m'} = 1$ and $S(D\varphi) = \operatorname{supp}(D\varphi)$. By choosing $m = \frac{q}{(1-\alpha)(p-1)} \ge 1$ (note that this is possible), using (2.5) and the definition of φ, we finally obtain

$$\int_{B_R} u^q\,dx \le C\left(\int_{S(D\varphi)} \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{S(D\varphi)} \frac{|D\varphi|^{pm'}}{\varphi^{pm'-1}}\,dx\right) \left(\int_{R\le|x|<2R} u^q\,dx\,dx\right)^{\frac{1}{pm}}\,. \tag{2.12}$$

Thus by the usual change of variable we get

$$\int_{B_R} u^q\,dx \le C_0 R^\tau \left(\int_{R<|x|<2R} u^q\,dx\right)^{\frac{1}{pm}}\,, \tag{2.13}$$

where, by taking into account that $q = \frac{N(p-1)}{p-1}$, we have

$$\tau = (N - p\varkappa')\frac{p-1}{p} + \frac{N-pm'}{pm'} = 0\,.$$

By (2.10) we know that

$$\int\limits_{\mathbb{R}^N} u^q\,dx < \infty\,.$$

Inequality (2.13) implies that there exists a sequence $\{R_k\}$ such that $R_k \to \infty$ and

$$\lim_{k\to\infty} \int\limits_{B_{R_k}} u^q\,dx = 0\,.$$

This completes the proof of the theorem in the case when (i) holds.

Next we consider case (ii). Let us first suppose that $0 \le q < p-1$. From (2.3) we know that

$$\int u^{q+\alpha}\varphi\,dx \le C\left(\int u^{(\alpha+p-1)\varkappa}\varphi\,dx\right)^{\frac{1}{\varkappa}}\left(\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\right)^{\frac{1}{\varkappa'}}, \tag{2.14}$$

where $\frac{1}{\varkappa}+\frac{1}{\varkappa'}=1$. Choosing $\alpha = q(N-p)-N(p-1) < 0$ (note that in this case $\alpha < 1-p < -q$ due to the choice of α) and

$$\varkappa = \frac{q+\alpha}{\alpha+p-1} \quad (>1),$$

we obtain

$$\int u^{q+\alpha}\varphi\,dx \le C\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\,. \tag{2.15}$$

Now, proceeding as in the proof of the first step of case (i), we arrive at

$$\int\limits_{B_R} u^{q+\alpha}\,dx \le CR^{N-p\varkappa'}$$

with

$$N-p\varkappa' = \frac{N(q-p+1)-pq-p\alpha}{q-p+1} < 0\,.$$

This completes the proof of case (ii) if the strict inequality in (ii) holds.

Suppose now that $q=p-1$. By choosing $\alpha = 1-p$ in (2.14) we get

$$\int \varphi\,dx \le \widetilde{C}\int \frac{|D\varphi|^{p\varkappa'}}{\varphi^{p\varkappa'-1}}\,dx\,.$$

Hence,

$$R^N \le CR^{N-p}\,,$$

which easily leads to a contradiction. $\square$

As remarked in the Introduction, for some problems we are able to prove sharp results. This is the case for problem (2.1). Indeed, suppose that the second inequality in (i) of Theorem 2.1 does not hold, i.e.,

$$q > \frac{N(p-1)}{N-p}\,, \qquad N>p\,.$$

It is very easy to verify that the function u defined by

$$u(x) := \frac{\varepsilon}{(1+|x|^{\frac{p}{p-1}})^{\frac{p-1}{q-p+1}}}\,, \qquad x\in\mathbb{R}^N\,,$$

is a positive solution of (2.1) provided that we choose ε such that

$$0 < \varepsilon < \left(\frac{p}{q-p+1}\right)^{\frac{p-1}{q-p+1}} \left(\frac{q(N-p)-N(p-1)}{q-p+1}\right)^{\frac{1}{q-p+1}}.$$

Let us now consider some generalizations of Theorem 2.1.

Let $A\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a given function and suppose that there exist $c_1, c_2 > 0$ and $p > 1$ such that we have

$$c_1 t^{p-1} \le A(t)t \le c_2 t^{p-1} \tag{2.16}$$

for any $t \ge 0$. The following result can be obtained by slightly adapting the proof of Theorem 2.1.

Theorem 2.2. *Suppose that $p > 1$ and A satisfies* (2.16). *If either*

(i) $p-1 < q \le \dfrac{N(p-1)}{N-p}$ *and* $N > p$, *or*

(ii) $0 \le q \le p-1$ *and* $N \ge 1$,

then the problem

$$\begin{cases} -\operatorname{div}(A(|Du|)Du) \ge u^q & \text{in } \mathbb{R}^N, \\ u \ge 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{2.17}$$

has no solutions.

Remark 2.1. It is not difficult to realize that, if the function A is a Caratheodory function such that for any $(x,s,t) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}_+$ we have

$$c_1 t^{p-1} \le A(x,s,t)t \le c_2 t^{p-1}, \tag{2.18}$$

where $c_1, c_2 > 0$ and $p > 1$, the corresponding problem

$$-\operatorname{div}(A(x,u,Du)Du) \ge u^q \quad \text{on } \mathbb{R}^N \tag{2.19}$$

has no solutions $u \ge 0$, $u \not\equiv 0$ on $\mathbb{R}^N$ which belong to $W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$ provided that (i) or (ii) of Theorem 2.2 are satisfied.

Another important class of operators that can be studied are the so-called "mean curvature type" operators.

Definition 2.1. Let $A\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Suppose that there exists $C > 0$ such that for any $t \ge 0$ we have

$$0 < A(t) \le C. \tag{2.20}$$

Then the operator T defined by

$$Tu = \operatorname{div}(A(|Du|)Du), \qquad u \in W^{1,2}_{\alpha,\mathrm{loc}}, \tag{2.21}$$

is called a *mean curvature type operator* associated with the function A.

Some important examples are [28, 29]

$$A(t) = \frac{1}{\sqrt{1+t^2}} \qquad (\textit{mean curvature operator}), \tag{2.22}$$

$$A(t) = \frac{1}{(1+|t|^k)^s}, \quad k, s > 0 \qquad (\textit{generalized mean curvature operator}). \tag{2.23}$$

A slight modification of the proof of Theorem 2.1 gives the following result.

Theorem 2.3. *Let A be a function satisfying* (2.20).
If either

(i) $1 < q \leq \dfrac{N}{N-2}$, $N > 2$, *or*

(ii) $0 \leq q \leq 1$, $N \geq 1$,

then the problem

$$\begin{cases} -\operatorname{div}(A(|Du|)Du) \geq u^q & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,2}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{2.24}$$

has no solutions.

Moreover, if $A(t) = \frac{1}{\sqrt{1+t^2}}$, $N > 2$, *and* $q > \frac{N}{N-2}$, *then the function u defined by*

$$u(x) := \frac{\varepsilon}{(1+|x|^2)^{\frac{1}{p-1}}}$$

is a solution to (2.24) *of class* $C^2(\mathbb{R}^N)$ *provided that $\varepsilon > 0$ is chosen sufficiently small.*

There are some other possible ways to generalize Theorem 2.1. A typical situation is when we replace the right-hand side of the inequality appearing in (2.19) with

$$f(x,u) = a(x)u^q\,, \qquad x \in \mathbb{R}^N\,,$$

where $a \geq 0$ is a function such that

$$a(x) \geq C_0|x|^\gamma \qquad \forall x\colon |x| \geq R_0 \geq 0\,, \quad \gamma \in \mathbb{R}. \tag{2.25}$$

The following result can be proved by repeating the proof of Theorem (2.1).

Theorem 2.4. *Let $N > p > 1$. Suppose that A and a satisfy, respectively,* (2.18) *and* (2.25).
If

$$p - 1 < q \leq \frac{(N+\gamma)(p-1)}{N-p}\,, \tag{2.26}$$

then the problem

$$\begin{cases} -\operatorname{div}(A(x,u,Du)Du) \geq a(x)u^q & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{2.27}$$

has no solutions.

Remark 2.2. The interested reader may formulate results similar to those of Theorem 2.4 when A generates an operator of mean curvature type, i.e., A satisfies (2.20) (see [26, 27]). Other possible generalizations can be obtained by considering the generating function A to be strongly dependent on its arguments. A simple example in this direction is given by

$$A(x,u,Du) := |x|^{\alpha_1}|u|^{q_1}|Du|^{p-2}\,. \tag{2.28}$$

In this case the nonexistence theorem for the inequality

$$\begin{cases} -\operatorname{div}(A(x,u,Du)Du) \geq a(x)u^q & \text{in } R^N, \\ u \geq 0,\ u \not\equiv 0 \end{cases} \tag{2.29}$$

will be formulated in terms of an algebraic condition involving N, α_1, q_1, p, γ, and q. We omit the details.

3. A GENERAL CLASS OF QUASILINEAR INEQUALITIES

In this section we will consider the nonexistence problem for a general class of quasilinear inequalities (see [32] for a related study).

First we state some assumptions. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers.

(i) Let $N > 1$ and $A_i \colon \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$, $i = 1, \dots, N$, be Caratheodory functions.

(ii) Let $f \colon \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ be a Caratheodory function.

(iii) Let $\gamma > 2$ with $\gamma \geq N$ be given and suppose that there exists $\alpha < 0$ (small) and $C_\alpha > 0$ such that for any $(x,u,\xi) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N$ we have

$$C_\alpha u^{2(\frac{\gamma-1+\alpha}{\gamma-2})} \mathcal{A}_0^{\frac{\gamma}{\gamma-2}}(x,u,\xi) \leq \mathcal{A}(x,u,\xi) + uf(x,u,\xi)\,, \tag{3.1}$$

where $\mathcal{A}_0$ and $\mathcal{A}$ are defined by

$$\mathcal{A}_0(x,u,\xi) := \sum_{i=1}^N A_i(x,u,\xi)\,, \tag{3.2}$$

$$\mathcal{A}(x,u,\xi) := \sum_{i=1}^N A_i(x,u,\xi)\xi_i^2\,. \tag{3.3}$$

Before stating our main result, let us specify the functional class that we will consider throughout this section. This functional class is related to the parameter α (see (iii)) and $\mathcal{A}$.

$$X_\alpha^{\mathcal{A}} = \left\{u \colon \mathbb{R}^N \to \mathbb{R},\ \mathcal{A}(x,u,Du)u^{\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\right\},$$

where $D_i u$ are understood as distributions;

$$X_\alpha^f = \left\{u \in X_\alpha^{\mathcal{A}} \colon\ f(x,u,Du)u^\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\right\}.$$

Theorem 3.1. *Suppose that the assumptions* (i)–(iii) *are satisfied. In addition suppose that for any* $(x,u,\xi) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N$ *we have*

$$\mathcal{A}(x,u,\xi) + uf(x,u,\xi) = 0 \quad \Rightarrow \quad u \equiv 0\,. \tag{3.4}$$

Then the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N D_i A_i(x,u,Du)\, D_i u \geq f(x,u,Du) & \textit{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \textit{on } \mathbb{R}^N \end{cases} \tag{3.5}$$

has no solutions that belong to X^f_α.

Proof. As in the preceding section, we will suppose that the possible solutions to (3.5) are positive on $\mathbb{R}^N$. By choosing $\varphi \in C_0^\infty(\mathbb{R}^N)$ as before and multiplying (3.5) by $u^\alpha\varphi$ with $\alpha < 0$ (small), we obtain

$$\int f(x,u,Du)u^\alpha\varphi\,dx + |\alpha|\int\sum_{i=1}^N A_i(x,u,Du)|D_iu|^2u^{\alpha-1}\,dx \le \int\sum_{i=1}^N A_i(x,u,Du)(D_iu,D_i\varphi)u^\alpha\,dx$$

and then

$$\int f(x,u,Du)u^\alpha\varphi\,dx + \frac{|\alpha|}{2}\int \mathcal{A}_0(x,u,Du)u^{\alpha-1}\varphi\,dx \le \frac{1}{2|\alpha|}\int \mathcal{A}_0(x,u,Du)\frac{|D\varphi|^2}{\varphi}u^{\alpha+1}\,dx\,. \quad (3.6)$$

Using the Young inequality we can bound the integral that appears on the right-hand side of (3.6) as follows:

$$\int \mathcal{A}_0(x,u,Du)u^{\alpha+1}\frac{|D\varphi|^2}{\varphi}\,dx$$
$$\le \left(\frac{\gamma-2}{\gamma}\right)C_1\int \mathcal{A}_0(x,u,Du)^{\frac{\gamma}{\gamma-2}}u^{(\alpha+1)\frac{\gamma}{\gamma-2}}\varphi\,dx + \frac{2}{\gamma}C_1^{\frac{2-\gamma}{\gamma}}\int\frac{|D\varphi|^\gamma}{\varphi^{\gamma-1}}\,dx\,, \quad (3.7)$$

where $C_1 > 0$ is an arbitrary constant.

Now, from our assumption (3.1) it follows that

$$\Big(\mathcal{A}_0(x,u,Du)u^{\alpha+1}\Big)^{\frac{\gamma}{\gamma-2}}u^{1-\alpha} \le \frac{1}{C_\alpha}\Big(\mathcal{A}(x,u,Du) + uf(x,u,Du)\Big)\,, \quad (3.8)$$

hence

$$\Big(\mathcal{A}_0(x,u,Du)u^{\alpha+1}\Big)^{\frac{\gamma}{\gamma-2}} \le \frac{1}{C_\alpha}\Big(\mathcal{A}(x,u,Du)u^{\alpha-1} + f(x,u,Du)u^\alpha\Big)\,,$$

and thus from (3.7) we have

$$\int \mathcal{A}_0(x,u,Du)u^{\alpha+1}\frac{|D\varphi|^2}{\varphi}\,dx$$
$$\frac{C_1^{\frac{\gamma-2}{\gamma}}}{C_\alpha}\int\Big(\mathcal{A}(x,u,Du)u^{\alpha-1} + uf(x,u,Du)u^\alpha\Big)\varphi\,dx + \frac{2C_1^{\frac{2-\gamma}{2}}}{\gamma}\int\frac{|D\varphi|^\gamma}{\varphi^{\gamma-1}}\,dx\,. \quad (3.9)$$

Now we choose $C_1 > 0$, i.e.,

$$C_1 = \frac{\gamma}{4(\gamma-2)}C_\alpha|\alpha|^2\,.$$

With this choice of C_1, from (3.6) it follows that

$$\frac{|\alpha|}{4}\int\Big(\mathcal{A}(x,u,Du) + uf(x,u,Du)\Big)u^{\alpha-1}\varphi\,dx \le \frac{2}{\gamma}C_1^{\frac{2-\gamma}{2}}\int\frac{|D\varphi|^\gamma}{\varphi^{\gamma-1}}\,dx\,, \quad (3.10)$$

and, by the same reasoning as in the preceding section, from (3.10)

$$\int_{B_R}\Big(\mathcal{A}(x,u,Du) + uf(x,u,Du)\Big)u^{\alpha-1} \le C_\gamma R^{N-\gamma}\,, \quad (3.11)$$

where $C_\gamma > 0$ is independent of R. Now, if $N < \gamma$, the last inequality implies

$$\int\limits_{\mathbb{R}^N} \Big(\mathcal{A}(x,u,Du) + uf(x,u,Du)\Big) u^{\alpha-1}\,dx = 0\,,$$

and then from (3.4) we conclude that $u \equiv 0$ on $\mathbb{R}^N$. This completes the proof in the case $N < \gamma$.

If $N = \gamma$, then (3.11) implies

$$\int\limits_{B_R} \Big(\mathcal{A}(x,u,Du) + uf(x,u,Du)\Big) u^{\alpha-1}\,dx \le C_1\,, \tag{3.12}$$

and by (3.9)

$$\int\limits_{B_R} \mathcal{A}_0(x,u,Du) u^{\alpha+1} \frac{|D\varphi|^2}{\varphi}\,dx \le C_1 \tag{3.13}$$

uniformly for every $R > 0$. From (3.13) we deduce that

$$\lim_{S\to\infty} \int\limits_{S\le|x|} \mathcal{A}_0(x,u,Du) u^{\alpha+1} \frac{|D\varphi|^2}{\varphi}\,dx = 0$$

and, consequently,

$$\lim_{R\to\infty} \int\limits_{\frac{R}{2}\le|x|} \mathcal{A}_0(x,u,Du) u^{\alpha+1} \frac{|D\varphi|^2}{\varphi}\,dx = 0\,.$$

Finally, by our choice of the test function φ we find

$$\int\limits_{B_R} \mathcal{A}_0(x,u,Du) u^{\alpha+1} \frac{|D\varphi|^2}{\varphi}\,dx = \int\limits_{\frac{R}{2}<|x|<R} \mathcal{A}_0(x,u,Du) u^{\alpha+1} \frac{|D\varphi|^2}{\varphi}\,dx \longrightarrow 0$$

as $R \to \infty$. By using (3.6) this implies that

$$\lim_{R\to\infty} \int\limits_{B_R} \Big(\mathcal{A}(x,u,Du)u^{\alpha-1} + f(x,u,Du)u^{\alpha}\Big)\,dx = 0\,.$$

Thus from (3.4) we conclude that $u \equiv 0$ on $\mathbb{R}^N$. This completes the proof of the theorem. $\square$

4. COERCIVE PROBLEMS

In this section we will consider some applications of the technique developed in Sections 2 and 3 to a class of coercive problems. A simple example of coercive problem that has been studied in the literature is the following:

$$\begin{cases} \Delta u = a(x)u^q & \text{in } \mathbb{R}^N, \\ u \ge 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in C^2(\mathbb{R}^N), \end{cases} \tag{4.1}$$

where $q > 1$, $N \ge 3$ and $a\colon \mathbb{R} \to \mathbb{R}_+$ is a nonnegative function satisfying

$$a(x) \ge c_0|x|^{-\gamma} \qquad \forall x\colon |x| \ge R_0 > 0\,, \quad c_0 > 0\,, \gamma \in \mathbb{R}\,. \tag{4.2}$$

It is known (see, for example, [1, 37] and the references therein for some generalizations of this result weakening (4.2)) that (4.1) has no solutions if $\gamma \le 2$. An important tool used in the proof of results of this nature is the invariance of the operator Δ with respect to the "spherical mean operator" [1, 37]. It seems that this kind of invariance cannot be used in dealing with quasilinear versions of (4.1).

Other techniques are available for the study of nonexistence theorems for coercive quasilinear problems, but to the author's knowledge all these methods are based on the maximum principle (see [22, 37]) reducing in this way the possibility to obtain nonexistence results for problems in which is possibly absent a maximum or a comparison principle.

In this section we prove some nonexistence results for quasilinear inequalities of the form (4.1).

Let $\alpha > 0$ be sufficiently large, $p > 1$, $q > 0$, and $a\colon \mathbb{R}^N \to \mathbb{R}_+$. We define S (see the Introduction) by

$$S := W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) = \left\{u\colon \mathbb{R}^N \to \mathbb{R},\quad a(x)u^{q+\alpha}, |Du|^p u^{\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\right\},$$

where $D_i u$, $i = 1, \dots, N$, are understood in the sense of distributions.

Our main result is the following.

Theorem 4.1. *Let $N > p > 1$ and $q > p-1$. Suppose that $A\colon \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ is a Caratheodory function satisfying condition* (2.18). *Let $a\colon \mathbb{R}^N \to \mathbb{R}_+$ be such that*

$$a(x) \ge c_0|x|^{-\gamma} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.3}$$

where $c_0 > 0$ and $\gamma < p$.

Then the problem

$$\begin{cases} \operatorname{div}(A(x,u,Du)Du) \ge a(x)u^q & \text{in } \mathbb{R}^N, \\ u \ge 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{4.4}$$

has no solutions.

Remark 4.1. It is not difficult to verify (see the proof of Theorem 4.1 below) that the same conclusion of Theorem 4.1 holds if instead of (4.3) we assume that "There exists $R_0 > 0$ such that for any $|x| \ge R_0$ we have

$$a(x) \ge c_0|x|^{-\gamma}, \tag{4.5}$$

where $c_0 > 0$ and $\gamma < p$.

Proof of Theorem 4.1. We will strictly follow the idea of the proof of Theorem 2.1. Let $\alpha > 0$, and suppose that u is a positive solution to (4.4). Let φ be a standard cut-off function (see Section 2). Multiplying (4.4) by $u^\alpha\varphi$ and integrating by parts, we obtain

$$\int a(x)u^{\alpha+q}\varphi\,dx \le -\alpha\int A(x,u,Du)|Du|^2\varphi\,dx - \int A(x,u,Du)u^\alpha(Du,D\varphi)\,dx.$$

Using (2.18) we find

$$\int a(x)u^{q+\alpha}\varphi\,dx + \alpha c_1\int |Du|^p u^{\alpha-1}\,dx \le c_2\int |Du|^{p-1}u^\alpha|D\varphi|\,dx. \tag{4.6}$$

Applying the Young inequality with suitable $\varepsilon > 0$ to the right-hand side of (4.6), we deduce that (for some $C_\varepsilon > 0$)

$$\int a(x)u^{q+\alpha}\varphi\,dx \le C_\varepsilon \int u^{\alpha+p-1}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\,,$$

and then

$$\int a(x)u^{q+\alpha}\varphi\,dx \le C_\varepsilon \int u^{\alpha+p-1}a(x)^{\frac{1}{y}}\frac{|D\varphi|^p}{\varphi^{p-1}}a(x)^{-\frac{1}{y}}\,dx\,, \tag{4.7}$$

where $y > 1$ has to be chosen.

Next from (4.7) it follows that

$$\int a(x)u^{q+\alpha}\varphi\,dx \le C_\varepsilon \left(\int u^{(\alpha+p-1)y}\varphi a(x)\,dx\right)^{\frac{1}{y}} \left(\int \frac{|D\varphi|^{py'-1}}{\varphi^{py'-1}}a(x)^{-\frac{y}{y'}}\right)^{\frac{1}{y'}} \tag{4.8}$$

with $\frac{1}{y}+\frac{1}{y'}=1$. We choose now y such that

$$q+\alpha = (\alpha+p-1)y\,,$$

i.e., $y = \frac{q+\alpha}{\alpha+p-1}$. We observe that $y > 1$ since by assumption $q > p-1$. Using (4.3), from (4.8) we get

$$\int a(x)u^{q+\alpha}\,dx \le \widetilde{C}_\varepsilon \int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}}|x|^{\frac{\gamma y'}{y}}\,dx\,. \tag{4.9}$$

By standard arguments, we finally obtain

$$\int\limits_{B_R} a(x)u^{q+\alpha}\,dx \le kR^\sigma\,, \tag{4.10}$$

where $\sigma = N - \frac{p(q+\alpha)}{q-p+1} + \gamma\left(\frac{\alpha+p-1}{q-p+1}\right)$. Choosing α sufficiently large and using the assumption $\gamma < p$ it follows that $\sigma < 0$ and, thus, from (4.10),

$$\int\limits_{\mathbb{R}^N} a(x)u^{q+\alpha}(x)\,dx = 0\,.$$

This is a contradiction with our assumptions on u and a. This completes the proof. $\square$

Remark 4.2. We conjecture that the conclusion of Theorem 4.1 holds even if $\gamma = p$.

As a simple application of Theorem 4.1 we have the following

Corollary 4.1. *Let $N > p > 1$ and $q > p-1$. Consider*

$$\begin{cases} \operatorname{div}(|Du|^{p-2}Du) \ge |x|^\gamma u^\alpha & \textit{in } \mathbb{R}^N \\ u \ge 0,\ u \not\equiv 0 & \textit{on } \mathbb{R}^N\,, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{4.11}$$

If $\gamma < p$ then (4.11) *has no solutions.*

It is of course possible to generalize Theorem 4.1 in several directions. Here, for the sake of brevity, we just consider the following result whose proof can be obtained by repeating the proof of Theorem 4.1 and leave to the interested reader the investigation of more general problems.

This result (concerning mean curvature type operators) can be obtained by repeating the proof of Theorem 4.1.

Theorem 4.2. *Let $N \geq 3$ and let A be a function satisfying* (2.20). *Let $a\colon \mathbb{R}^N \to \mathbb{R}_+$ be such that there exist $\gamma < 2$ and $R_0 > 0$ such that*

$$a(x) \geq c_0|x|^{-\gamma} \qquad \forall x:\ |x| \geq R_0\,.$$

Then the problem

$$\begin{cases} \operatorname{div}(A(|Du|)Du) \geq a(x)u^q & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,2}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{4.12}$$

has no solutions.

Remark 4.3. It would be interesting to know whether Theorem 4.2 holds in the case $\gamma = 2$.

5. NONEXISTENCE THEOREMS FOR SYSTEMS OF QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

In this section we will study some nonexistence theorems for systems of quasilinear differential equations. Some results concerning the nonexistence of positive radial solutions of such systems have been obtained recently in [24–27] (see also [30] for some results related to semilinear potential systems and [15, 23] for Hamiltonian elliptic systems).

The goal of this section is to prove quasilinear nonradial versions of those results. In what follows, we will consider the same functional class of solutions defined in Section 2.

We first consider a canonical problem (see [11, 34, 35]).

Theorem 5.1. *Let $N > p$, $q > 1$ and suppose that*

(i) $q - 1 < q_1$ *and* $p - 1 < p_1$.

If

$$\max\left\{\frac{qq_1 + p(q-1)}{p_1q_1 - (p-1)(q-1)} - \frac{N-p}{p-1}\,,\ \frac{pp_1 + q(p-1)}{p_1q_1 - (p-1)(q-1)} - \frac{N-q}{q-1}\right\} \geq 0\,, \tag{5.1}$$

then the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) \geq v^{q_1} & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|Du|^{q-2}Du) \geq u^{p_1} & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ v \geq 0,\ v \not\equiv 0 & \text{on } \mathbb{R}^N, \end{cases} \tag{5.2}$$

has no solutions $(u,v) \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \times W^{1,q}_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$.

Proof. Let $(u,v) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N) \times W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution to (5.2). Without loss of generality, we may suppose that $u > 0$ and $v > 0$ on $\mathbb{R}^N$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a standard cut-off function and let $\alpha < 0$ be sufficiently small to be chosen later. By multiplying the first and the second equations of (5.2) by $u^\alpha\varphi$ and, respectively, by $v^\alpha\varphi$ and integrating by parts, we find

$$\int u^{p_1}v^\alpha\varphi\,dx + c''_\varepsilon\int |Dv|^q v^{\alpha-1}\varphi\,dx \leq c'_\varepsilon\int \frac{|D\varphi|^q}{\varphi^{q-1}}v^{\alpha+q-1}\,dx\,, \tag{5.3}$$

$$\int v^{q_1}u^\alpha\varphi\,dx + d''_\varepsilon\int |Du|^p u^{\alpha-1}\varphi\,dx \leq d'_\varepsilon\int \frac{|D\varphi|^p}{\varphi^{p-1}}v^{\alpha+p-1}\,dx\,, \tag{5.4}$$

where the constants $c'_\varepsilon,\ c''_\varepsilon,\ d'_\varepsilon,\ d''_\varepsilon > 0$ depend only on $\alpha,\ p,\ q$ and $\varepsilon > 0$. Next, we multiply (5.2) by φ and obtain

$$\int v^{q_1}\varphi\,dx \le \left(\int |Du|^p u^{(\alpha-1)}\varphi\,dx\right)^{\frac{p-1}{p}} \left(\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{(1-\alpha)(p-1)}\,dx\right)^{\frac{1}{p}}, \tag{5.5}$$

$$\int u^{p_1}\varphi\,dx \le \left(\int |Dv|^q v^{(\alpha-1)}\varphi\,dx\right)^{\frac{q-1}{q}} \left(\int \frac{|D\varphi|^q}{\varphi^{q-1}} v^{(1-\alpha)(p-1)}\,dx\right)^{\frac{1}{q}}. \tag{5.6}$$

By using (5.3) and (5.4), the last estimate implies that

$$\int v^{q_1}\varphi\,dx \le D_\varepsilon \left(\int \frac{|D\varphi|}{\varphi^p} u^{\alpha+p-1}\,dx\right)^{\frac{p-1}{p}} \left(\int \frac{|D\varphi|^p u^{(1-\alpha)(p-1)}}{\varphi^{p-1}}\,dx\right)^{\frac{1}{p}}, \tag{5.7}$$

$$\int u^{p_1}\varphi\,dx \le E_\varepsilon \left(\int \frac{|D\varphi|}{\varphi^q} v^{\alpha+q-1}\,dx\right)^{\frac{q-1}{q}} \left(\int \frac{|D\varphi|^q v^{(1-\alpha)(p-1)}}{\varphi^{q-1}}\,dx\right)^{\frac{1}{q}}, \tag{5.8}$$

where $E_\varepsilon, D_\varepsilon > 0$ depend only on $p,\ q,\ \alpha$, and $\varepsilon > 0$.

Now we apply the Hölder inequality with parameters $a,\ a'$ to the first integral on the right-hand side of (5.7) and we get

$$\left[\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{\alpha+p-1}\,dx\right]^{\frac{p-1}{p}} \le \left(\int u^{(\alpha+p-1)a}\varphi\,dx\right)^{\frac{p-1}{pa}} \left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{(p-1)}{pa'}} \tag{5.9}$$

with $\frac{1}{a}+\frac{1}{a'}=1$. By choosing a so that

$$(\alpha+p-1)a = p_1\,,$$

from (5.7) we have

$$\int v^{q_1}\varphi\,dx \le D_\varepsilon \left(\int u^{p_1}\varphi\,dx\right)^{\frac{p-1}{pa}} \left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{p-1}{pa'}} \left(\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{(1-\alpha)(p-1)}\,dx\right)^{\frac{1}{p}}. \tag{5.10}$$

By repeating this procedure with parameters $y, y' > 1$ for the third integral in (5.10), we get

$$\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{(1-\alpha)(p-1)}\,dx \le \left(\int u^{y(1-\alpha)(p-1)}\varphi\,dx\right)^{\frac{1}{y}} \left(\int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{y'}}, \tag{5.11}$$

where $\frac{1}{y}+\frac{1}{y'}=1$.

By choosing $(1-\alpha)(p-1)y = p_1$ in (5.11) and taking into account (5.10), we deduce that

$$\int v^{q_1}\varphi\,dx \le D_\varepsilon \left(\int u^{p_1}\varphi\,dx\right)^{\frac{p-1}{pa}} \left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{p-1}{pa'}} \left(\int u^{p_1}\varphi\,dx\right)^{\frac{1}{py}} \left(\int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}, \tag{5.12}$$

that is,

$$\int v^{q_1}\varphi\,dx \le D_\varepsilon \left(\int u^{p_1}\varphi\,dx\right)^{\left(\frac{p-1}{pa}+\frac{1}{py}\right)} \left(\int \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{p-1}{pa'}} \left(\int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}, \tag{5.13}$$

where we have choosen the parameters a and y such that

$$\begin{cases} \dfrac{1}{y}+\dfrac{1}{y'}=1, & (1-\alpha)(p-1)y=p_1,\\[2mm] \dfrac{1}{a}+\dfrac{1}{a'}=1, & (\alpha+p-1)a=p_1. \end{cases} \tag{5.14}$$

We observe that this choice of a and y is admissible by our assumption (i) provided that $\alpha<0$ is chosen sufficiently small.

Introducing new parameters b and $\varkappa$ such that

$$\begin{cases} \dfrac{1}{\varkappa}+\dfrac{1}{\varkappa'}=1, & (1-\alpha)(q-1)\varkappa=q_1,\\[2mm] \dfrac{1}{b}+\dfrac{1}{b'}=1, & (\alpha+q-1)b=q_1, \end{cases} \tag{5.15}$$

and estimating now the left-hand side of (5.8) (using the same method as above), we obtain

$$\int u^{p_1}\varphi\,dx \le E_\varepsilon\left(\int v^{q_1}\varphi\,dx\right)^{\left(\frac{q-1}{qb}+\frac{1}{q\varkappa}\right)}\left(\int\frac{|D\varphi|^{qb'}}{\varphi^{qb'}}\,dx\right)^{\frac{1}{qb'}}\left(\int\frac{|D\varphi|^{q\varkappa'}}{\varphi^{q\varkappa'-1}}\,dx\right)^{\frac{1}{q\varkappa'}}. \tag{5.16}$$

Combining (5.13) and (5.16) we finally get

$$\begin{aligned}\left(\int v^{q_1}\varphi\,dx\right)^{1-mn} &\le D_\varepsilon E_\varepsilon^n\left(\int\frac{|D\varphi|^{qb'}}{\varphi^{qb'-1}}\,dx\right)^{\frac{n}{qb'}}\left(\int\frac{|D\varphi|^{q\varkappa'}}{\varphi^{q\varkappa'-1}}\,dx\right)^{\frac{n}{q\varkappa'}}\\ &\times\left(\int\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{pa-1}{pa'}}\left(\int\limits_{S(\varphi)}\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}\left(\int u^{p_1}\varphi\,dx\right)^{1-mn} &\le E_\varepsilon D_\varepsilon^m\left(\int\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{m}{pa'}}\left(\int\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{m}{py'}}\\ &\times\left(\int\frac{|D\varphi|^{qb'}}{\varphi^{qb'-1}}\,dx\right)^{\frac{q-1}{qb'}}\left(\int\frac{|D\varphi|^{q\varkappa'}}{\varphi^{q\varkappa'-1}}\,dx\right)^{\frac{1}{q\varkappa'}},\end{aligned} \tag{5.18}$$

where

$$n:=\frac{p-1}{pa}+\frac{1}{py}, \qquad m:=\frac{q-1}{qb}+\frac{1}{q\varkappa}. \tag{5.19}$$

An easy computation taking into account (5.14) and (5.15) gives the explicit values of m and n, that is

$$m=\frac{q-1}{q_1}, \qquad n=\frac{p-1}{p_1}. \tag{5.20}$$

Consequently, by our assumption it follows that the exponent appearing on the left-hand side of (5.17), (5.18) is

$$1-mn=\frac{p_1q_1-(p-1)(q-1)}{p_1q_1}>0.$$

Next, by the usual change of variable and choosing φ appropriately (for details see Section 2) it follows that

$$\int\limits_{B_R} v^{q_1}\,dx \le F_\varepsilon R^{\sigma_1}\,, \tag{5.21}$$

$$\int\limits_{B_R} u^{p_1}\,dx \le G_\varepsilon R^{\sigma_1}\,, \tag{5.22}$$

where F_ε and G_ε are positive and independent of R and

$$\sigma_1 = n\left\{\frac{N-qb'}{qb'} + \frac{N-q\varkappa'}{q\varkappa'}\right\} + \frac{(N-pa')}{pa'}(p-1) + \frac{N-py'}{py'}\,,$$
$$\sigma_2 = m\left\{\frac{N-pa'}{pa'} + \frac{N-py'}{py'}\right\} + \frac{(N-qb')}{qb'}(p-1) + \frac{N-q\varkappa'}{q\varkappa'}\,.$$

By (5.14), (5.15) the explicit values of σ_1 and σ_2 are given by

$$\begin{aligned}\sigma_1 = \sigma_v &:= N-p-(p-1)\left\{\frac{p(q-1)+qq_1}{p_1q_1-(p-1)(q-1)}\right\},\\ \sigma_2 = \sigma_u &:= N-q-(q-1)\left\{\frac{q(p-1)+pp_1}{p_1q_1-(p-1)(q-1)}\right\}.\end{aligned} \tag{5.23}$$

Suppose now that

$$\begin{aligned}\max&\left\{\frac{p(q-1)+qq_1}{p_1q_1-(p-1)(q-1)} - \frac{N-p}{p-1}\,, \frac{q(p-1)+pp_1}{p_1q_1-(p-1)(q-1)} - \frac{N-q}{q-1}\right\}\\ &= \frac{p(q-1)+qq_1}{p_1q_1-(p-1)(q-1)} - \frac{N-p}{p-1} > 0\,,\end{aligned} \tag{5.24}$$

thus from (2.22) it follows that

$$\int\limits_{\mathbb{R}^N} v^{q_1}\,dx = 0 \tag{5.25}$$

and then $v \equiv 0$. This contradicts our assumption that $v > 0$ on $\mathbb{R}^N$. This concludes the proof of the theorem if (5.24) holds. The case when the maximum appearing in (5.24) equals $\frac{q(p-1)+pp_1}{p_1q_1-(p-1)(q-1)} - \frac{N-q}{q-1} > 0$ is similar to the preceding one so that we will omit its proof.

Next we consider the critical case

$$\frac{p(q-1)+qq_1}{p_1q_1-(p-1)(q-1)} - \frac{N-p}{p-1} = 0$$

(the case $\frac{q(p-1)+pp_1}{p_1q_1-(p-1)(q-1)} - \frac{N-q}{q-1} = 0$ being similar).

By our choice of the function $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int\limits_{B_R} v^{q_1}\,dx \le \int\limits_{B_{2R}} v^{q_1}\varphi\,dx \le \left(\int\limits_{R<|x|<2R} |Du|^p u^{\alpha-1}\varphi\,dx\right)^{\frac{p-1}{p}} \left(\int\limits_{R<|x|<2R} u^{(1-\alpha)(p-1)}\frac{|D\varphi|^p}{\varphi^{p-1}}\,dx\right)^{\frac{1}{p}}, \tag{5.26}$$

and proceeding as in the first part of this proof, we deduce that

$$\int\limits_{B_R} v^{q_1}\,dx \le \widetilde{C}\left(\int\limits_{R<|x|<2R} v^{q_1}\,dx\right)^{mn} R^{\sigma_v}\,. \tag{5.27}$$

Since $\sigma_v = \sigma_1 = 0$, from (5.21) it follows that

$$\int\limits_{\mathbb{R}^N} v^{q_1}\,dx < \infty\,.$$

This implies that the right-hand side of (5.27) goes to zero as $R \to \infty$ and then

$$\int\limits_{\mathbb{R}^N} v^{q_1}\,dx = 0\,.$$

This contradiction concludes the proof. $\square$

Remark 5.1. Theorem 2.1 is sharp. Indeed, if we assume that

$$\max\left\{\frac{qq_1+p(q-1)}{p_1q_1-(p-1)(q-1)}-\frac{N-p}{p-1}\,,\,\frac{pp_1+q(p-1)}{p_1q_1-(p-1)(q-1)}-\frac{N-q}{q-1}\right\}<0\,,$$

then (5.2) has a positive solution. Consider the case $p=q$. For $\varepsilon>0$ and $x\in\mathbb{R}^N$ we define u and v by

$$u(x) := \varepsilon\Big(1+|x|^{\frac{p}{p-1}}\Big)^{(1-p)\left(\frac{q_1+p-1}{p_1q_1-(p-1)^2}\right)}\,,$$

$$v(x) := \varepsilon\Big(1+|x|^{\frac{p}{p-1}}\Big)^{(1-p)\left(\frac{p_1+p-1}{p_1q_1-(p-1)^2}\right)}\,.$$

It is easy to see that if

$$\varepsilon < \min\Bigg\{p^{\frac{p-1}{q_1-p+1}}\,(q_1+p-1)^{\frac{p-1}{q_1-p+1}}\,(p-1)^{\frac{p}{q_1-p+1}}\,\sigma_v^{\frac{1}{q_1-p+1}}\,,$$
$$p^{\frac{p-1}{p_1-p+1}}\,(p_1+p-1)^{\frac{p-1}{p_1-p+1}}\,(p-1)^{\frac{p}{p_1-p+1}}\,\sigma_u^{\frac{1}{p_1-p+1}}\Bigg\},$$

then (u,v) is a positive solution to (5.2) satisfying $u,v\in C^2(\mathbb{R}^N-\{0\})\cap C^1(\mathbb{R}^N)$. We leave to the interested reader to check that a slight modification of the definition of (u,v) above is still possible if $p\ne q$. This provides an explicit solution to (5.2) proving the sharpness of Theorem 5.1 in this case as well.

The next result can be easily proved by combining Theorem 2.1 and Theorem 5.1.

Theorem 5.2. *Let $p,q>1$ and $p_1,p_2,q_1,q_2>0$. Let one of the following conditions be satisfied:*

(i) $p-1<q_2,p_1,\quad q-1<p_2,q_1,\quad N>\max\{p,q\}$,

$$\max\Bigg\{\frac{p}{q_2-p+1}-\frac{N-p}{p-1}\,,\,\frac{qq_1+p(q-1)}{p_1q_1-(p-1)(q-1)}-\frac{N-p}{p-1}\,,$$
$$\frac{q}{p_2-q+1}-\frac{N-q}{q-1}\,,\,\frac{pp_1+q(p-1)}{p_1q_1-(p-1)(q-1)}-\frac{N-q}{q-1}\Bigg\}\ge 0\,,$$

(ii) $0 < q_2 \le p-1, \quad 0 < p_2 \le q-1, \quad N \ge 1.$

Then the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) \ge u^{q_2} + v^{q_1} & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|Dv|^{q-2}Dv) \ge u_1^p + v^{p_2} & \text{in } \mathbb{R}^N, \\ u \ge 0, \ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ v \ge 0, \ v \not\equiv 0 & \text{on } \mathbb{R}^N \end{cases} \tag{5.28}$$

has no solutions $(u,v) \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \times W^{1,q}_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$.

Remark 5.2. It is possible to prove nonexistence results similar to that of Theorem 5.2 by considering, instead of the (p,q) Laplacian operators, differential operators that satisfy condition (2.18).

We can also consider the right-hand side of (5.28) of the form

$$f(x,u,v) = |x|^{\sigma_2}u^{q_2} + |x|^{\sigma_1}v^{q_1},$$
$$g(x,u,v) = |x|^{\alpha_1}u^{p_1} + |x|^{\alpha_2}v^{p_2},$$

where $\sigma_1, \sigma_2, \alpha_1, \alpha_2 \in \mathbb{R}$ and q_1, q_2, p_1, p_2 satisfy a suitable algebraic condition. This kind of results has been proved in the radial case in [24, 27].

In addition, by using the same technique that we have applied in the proof of Theorem 5.1, we can handle also system of quasilinear elliptic inequalities with more than two unknowns and involving differential operators satisfying (2.18) or (2.20). Clearly, in this case the algebraic condition on the parameters N and the growth of the nonlinearities that implies the nonexistence of positive solutions becomes extremely complicated.

The following is a simple example in this direction.

Theorem 5.3. *Let* $N > \max\{p, 2\}$ *with* $p > 1$. *Let* A *and* B *be operators of* p*-Laplacian type* (*see* (2.18)) *and of "mean curvature type"* (*see* (2.20)), *respectively. If*

(i) $p-1 < q_2, p_1, \ 1 < p_2, q_1$ *and*

$$\max\left\{\frac{p}{q_2-p+1} - \frac{N-p}{p-1}, \frac{2q_1+p}{p_1q_1-p+1} - \frac{N-p}{p-1}, \frac{2}{p_2-1} - N - 2, \frac{pp_1+p-1}{p_1q_1-p+1} - N - 2\right\} \ge 0,$$

then the problem

$$\begin{cases} -\operatorname{div}(A(x,u,Du)Du) \ge u^{q_2} + v^{q_1} & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(B(|Dv|)Dv) \ge u^{p_1} + v^{p_2} & \text{in } \mathbb{R}^N, \\ u \ge 0, \ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ v \ge 0, \ v \not\equiv 0 & \text{on } \mathbb{R}^N \end{cases} \tag{5.29}$$

has no solutions $(u,v) \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \times W^{1,2}_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$.

6. PROBLEMS WITH THE RIGHT-HAND SIDE THAT DEPENDS ON THE GRADIENT OF THE SOLUTION

In this section we would like to show, by means of a simple model problem, how our technique can be applied to problems with the "gradient dependence" on the right-hand side. Similar problems have been considered for instance in [10]. Let $\alpha < 0$ be sufficiently small, $p > 1$, $q_1 > 0$, and $q_2 \geq 0$. We define S (see the Introduction) by

$$S := W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R},\ u^{q_1+\alpha}, |Du|^p u^{\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \right\},$$

where $D_i u$, $i = 1, \dots, N$, are understood in the sense of distributions.

Our main result is the following

Theorem 6.1. *Let $N > p > 1$, and let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ be a Caratheodory function satisfying*

$$f(t, s) \geq K_0 t^{q_1} |s|^{q_2} \tag{6.1}$$

for any $(t, s) \in \mathbb{R}_+ \times \mathbb{R}^N$, where $q_1 > 0$, $q_2 \geq 0$, $q_1 + q_2 > p - 1$, and $K_0 > 0$. If

$$q_1 + \frac{N-1}{N-p} q_2 \leq \frac{N(p-1)}{N-p}, \tag{6.2}$$

then the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2} Du) \geq f(u, Du) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{on } \mathbb{R}^N, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{6.3}$$

has only locally constant solutions.

The proof of the above theorem is a modification of the proof of Theorem 2.1. As we will see (see the proof below) the same results hold if we replace in (6.3) the p-Laplacian operator with the more general one $\operatorname{div}(A(x, u, Du) Du)$, where A satisfies (2.18).

Proof of Theorem 6.1. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a standard nonnegative cut-off function and let $\alpha < 0$ be sufficiently small. Multiplying (6.3) by $u^\alpha \varphi$ (here, without loss of generality, we suppose that $u > 0$ is a solution which is not locally constant) and integrating the resulting inequality on the support of φ we obtain

$$\int f(u, Du) u^\alpha \varphi \, dx + |\alpha| \int |Du|^p u^{\alpha-1} \varphi \, dx \leq \int |Du|^p u^\alpha |D\varphi| \, dx. \tag{6.4}$$

Using our assumption (6.1) in (6.4) we find

$$K_0 \int u^{q_1+\alpha} |Du|^{q_2} \, dx + |\alpha| \int |Du|^p u^{\alpha-1} \varphi \, dx \leq \int |Du|^{p-1} u^\alpha |D\varphi| \, dx. \tag{6.5}$$

Let $\varkappa := \frac{pq_1 + \alpha(p-q_2) + q_2}{q_1(p-1) + \alpha(p-1-q_2)}$. It is easy to see that, if we choose $\alpha > 1 - p$, then

$$\begin{cases} pq_1 + \alpha(p - q_2) + q_2 > 0, \\ q_1(p-1) + \alpha(p-1-q_2) > 0 \end{cases}$$

and $\varkappa > \frac{p}{p-1} > 1$.

Next we rewrite (6.5) as follows:

$$K_0\int u^{q_1+\alpha}|Du|^{q_2}\varphi\,dx+|\alpha|\int|Du|^p u^{\alpha-1}\varphi\,dx\le\int|Du|^{p-1}u^{\frac{\alpha-1}{\varkappa}}u^{\alpha+\frac{1-\alpha}{\varkappa}}|D\varphi||Du|^{\frac{p}{\varkappa}}|Du|^{-\frac{p}{\varkappa}}\,dx$$
$$\le\int|Du|^{\frac{p}{\varkappa}}u^{\frac{\alpha-1}{\varkappa}}\varphi^{\frac{1}{\varkappa}}|Du|^{\frac{\varkappa(p-1)-p}{\varkappa}}u^{\frac{\alpha\varkappa+(1-\alpha)}{\varkappa}}\frac{|D\varphi|}{\varphi^{\frac{1}{\varkappa}}}\,dx\,. \tag{6.6}$$

Applying the Young inequality with exponents $\varkappa,\varkappa'>1$ and parameter $\varepsilon>0$ to the right-hand side of (6.6) we get

$$K_0u^{q_1+\alpha}|Du|^{q_2}\varphi\,dx+C_\varepsilon\int|Du|^pu^{\alpha-1}\varphi\,dx\le D_\varepsilon\int u^{\frac{\alpha\varkappa-1+\alpha}{\varkappa}}|Du|^{\left[\frac{\varkappa(p-1)-p}{\varkappa}\right]\varkappa'}\frac{|D\varphi|^{\varkappa'}}{\varphi^{\varkappa'-1}}\,dx\,. \tag{6.7}$$

Now for $\varkappa'=\frac{pq_1+\alpha(p-q_2)+q_2}{q_1+q_2+\alpha}$ it follows that $\frac{1}{\varkappa}+\frac{1}{\varkappa'}=1$, hence the preceding application of the Young inequality is justified.

Next we apply the Hölder inequality to the left-hand side of (6.7) and obtain

$$\int u^{q_1+\alpha}|Du|^{q_2}\varphi\,dx\le\left(\int u^{\frac{[\alpha\varkappa+(1-\alpha)]}{\varkappa}\varkappa'y}|Du|^{\frac{[\varkappa(p-1)-p]}{\varkappa}y}\varphi\,dx\right)^{\frac{1}{y}}\left(\int\frac{D\varphi|^{y'\varkappa'}}{\varphi^{y'\varkappa'-1}}\,dx\right)^{\frac{1}{y'}} \tag{6.8}$$

with $\frac{1}{y}+\frac{1}{y'}=1$. Choosing y such that

$$\begin{cases} q_2=\dfrac{\varkappa'}{\varkappa}[\varkappa(p-1)-p]y,\\[2mm] q_1+\alpha=\dfrac{\varkappa'}{\varkappa}\left[\alpha\varkappa+(1-\alpha)\right]y,\end{cases} \tag{6.9}$$

it follows that $y>1$ and (6.9) are compatible. As a consequence from (6.8) we get

$$\int u^{q_1+\alpha}|Du|^{q_2}\varphi\,dx\le C\int\frac{|D\varphi|^{\varkappa'y'}}{\varphi^{\varkappa'y'-1}}\,dx$$

and by the usual scaling argument

$$\int\limits_{B_R}u^{q_1+\alpha}|Du|^{q_2}\,dx\le CR^\sigma\,, \tag{6.10}$$

where $\sigma=N-\varkappa'y'$.

Using the explicit values of $\varkappa$ and y and $\frac{1}{\varkappa}+\frac{1}{\varkappa'}=1$, $\frac{1}{y}+\frac{1}{y'}=1$ we easily deduce that

$$\sigma<0\quad\Leftrightarrow\quad q_1+\frac{N-1}{N-p}q_2<\frac{N(p-1)}{N-p}$$

and $\alpha<0$ is sufficiently small.

Thus, if the strict inequality holds in (6.2), (6.10) implies that

$$\int\limits_{\mathbb{R}^N}u^{q_1+\alpha}|Du|^{q_2}\,dx=0\,,$$

which is a contradiction. In the case

$$q+\frac{N-1}{N-p}q_2=\frac{N(p-1)}{N-p}\,,$$

we proceed exactly as in the final part of the proof of Theorem 2.1.

This completes the proof. $\square$

Combining the preceding theorem with Theorem 2.1 we obtain the following

Theorem 6.2. *Let $N > p > 1$. If $p - 1 < p_1, q_1$ and*

$$p_1 \leq \frac{N(p-1)}{N-p} \qquad or \qquad q_1 \leq \frac{N(p-1)}{N-1}\,,$$

then the problem

$$\begin{cases} -\operatorname{div}\left(|Du|^{p-2}Du\right) \geq u^{p_1} + |Du|^{q_1} & in\ \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & on\ \mathbb{R}^N, \\ u \in W^{1,p}_{\alpha,\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{6.11}$$

has (respectively) no solutions (only locally constant solutions).

The above result can be generalized in several directions following the ideas developed in the preceding sections. In particular, it is possible to obtain a vector-valued version of Theorems 6.1 and 6.2 for systems involving different kind of differential operators and containing more than two unknowns. This result will be published elsewhere.

7. NONEXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF WEAKLY COUPLED SYSTEMS

Our main result in this section is a vector-valued version of Theorem 3.1 (see Section 3). We first introduce some assumptions that explains the meaning of "weak coupling." As usual we denote by $\mathbb{R}_+$ the set of nonnegative real numbers.

(i) Let $N > 1$, $A_i, B_i \colon \mathbb{R}^N \times \mathbb{R}^2_+ \times (\mathbb{R}^N)^2 \to \mathbb{R}^+$, $i = 1, \ldots, N$, be Caratheodory functions.

(ii) Let $f, g \colon \mathbb{R}^N \times \mathbb{R}^2_+ \times (\mathbb{R}^N)^2 \to \mathbb{R}_+$ be Caratheodory functions.

(iii) Let $\gamma > 2$ with $\gamma \geq N$ be given and suppose that there exists $\alpha, \beta < 0$ (small) and $C_{\alpha,\beta} > 0$ such that for any $(x, u, \xi), (x, v, \eta) \in (\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N)^2$ we have

$$\begin{gathered}\Big(u^{\alpha+1}\mathcal{A}_0(x,u,v,\xi,\eta) + v^{\beta+1}\mathcal{B}_0(x,u,v,\xi,\eta)\Big)^{\frac{\gamma}{\gamma-2}} \\ \leq C_{\alpha,\beta}\left(u^{\alpha-1}\mathcal{A}(x,u,v,\xi,\eta) + v^{\beta-1}\mathcal{B}(x,u,v,\xi,\eta) + u^{\alpha}f(x,u,v,\xi,\eta) + v^{\alpha}g(x,u,v,\xi,\eta)\right),\end{gathered} \tag{7.1}$$

where

$$\mathcal{A}_0(x,u,v,\xi,\eta) := \sum_{i=1}^{N} A_i(x,u,v,\xi,\eta)\,, \tag{7.2}$$

$$\mathcal{A}(x,u,v,\xi,\eta) := \sum_{i=1}^{N} A_i(x,u,v,\xi,\eta)\xi_i^2 \tag{7.3}$$

and $\mathcal{B}_0$, $\mathcal{B}$ are defined similarly.

The functional class that we will consider is as follows. Of course, as in Section 3, this functional class depends on the parameters α and β:

$$\begin{aligned} X^{\mathcal{A},\mathcal{B}}_{\alpha,\beta} = \Big\{(u,v) \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+,\ & \mathcal{A}(x,u,v,Du,Dv)u^{\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \\ & \mathcal{B}(x,u,v,Du,Dv)v^{\beta-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\Big\}\,, \end{aligned}$$

where $D_i u$ and $D_i v$, $i=1,\dots,N$, exist in the sense of distributions;

$$X^{f,g}_{\alpha,\beta} = \left\{(u,v) \in X^{\mathcal{A},\mathcal{B}}_{\alpha,\beta} :\ f(x,u,v,Du,Dv)u^\alpha + g(x,u,v,Du,Dv)v^\beta \in L^1_{\rm loc}(\mathbb{R}^N)\right\}.$$

Theorem 7.1. *Suppose that the assumptions* (i)–(iii) *are satisfied. In addition, suppose that for any* $(x,u,v,\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$ *we have*

$$u^{\alpha-1}\mathcal{A}(x,u,v,\xi,\eta) + v^{\beta-1}\mathcal{B}(x,u,v,\xi,\eta) + u^\alpha f(x,u,\xi,\eta) + v^\beta g(x,u,v,\xi,\eta) = 0 \quad \Rightarrow \quad u \equiv v \equiv 0. \tag{7.4}$$

Then the problem

$$\begin{cases} -\sum_{i=1}^N D_i(A_i D_i u) \ge f(x,u,v,Du,Dv) & \text{in } \mathbb{R}^N, \\ -\sum_{i=1}^N D_i(B_i D_i v) \ge g(x,u,v,Du,Dv) & \text{in } \mathbb{R}^N, \\ (u,v) \in \mathbb{R}^2_+,\ u \not\equiv 0,\ v \not\equiv 0 \end{cases} \tag{7.5}$$

has no solutions of class $X^{f,g}_{\alpha,\beta}$.

Proof. As usual we will suppose that (u,v) is a solution of (7.5) and derive a contradiction. Moreover, without loss of generality, we may suppose that (u,v) is a positive solution. Proceeding as in the proof of Theorem 3.1 we obtain

$$\begin{aligned} &\int (fu^\alpha + gv^\beta)\varphi\,dx + |\alpha| \int \mathcal{A}u^{\alpha-1}\varphi\,dx + |\beta| \int \mathcal{B}v^{\beta-1}\varphi\,dx \\ &\quad \le \sum_{i=1}^N \int A_i(D_i u_i D\varphi)u^\alpha\,dx + \sum_{i=1}^N \int B_i(D_i v_i D\varphi)v^\beta\,dx\,. \end{aligned} \tag{7.6}$$

Now we estimate the right-hand side of (7.6) as follows:

$$\begin{aligned} &\sum_{i=1}^N \int A_i(D_i u_i D\varphi)u^\alpha\,dx + \sum_{i=1}^N \int B_i(D_i v_i D\varphi)v^\beta\,dx \le \frac{|\alpha|}{2}\sum_{i=1}^N \int A_i|D_i u|^2 u^{\alpha-1}\varphi\,dx \\ &+ \frac{1}{2|\alpha|}\sum_{i=1}^N \int A_i u^{\alpha+1}\frac{|D\varphi|^2}{\varphi}\,dx + \frac{|\beta|}{2}\sum_{i=1}^N \int B_i|D_i v|^2 v^{\beta-1}\varphi\,dx + \frac{1}{2|\beta|}\sum_{i=1}^N \int B_i v^{\beta+1}\frac{|D\varphi|^2}{\varphi}\,dx\,. \end{aligned} \tag{7.7}$$

Further we have

$$(\mathcal{A}_0 u^{\alpha+1} + \mathcal{B}_0 v^{\beta+1})\frac{|D\varphi|^2}{\varphi} \le \frac{\gamma-2}{\gamma}(\mathcal{A}_0 u^{\alpha+1} + \mathcal{B}_0 v^{\beta+1})^{\frac{\gamma}{\gamma-2}}\varphi + \frac{2}{\gamma}C_1^{\frac{2-\gamma}{2}}\frac{|D\gamma\varphi|^\gamma}{\varphi^{\gamma-1}}, \tag{7.8}$$

where $C_1 > 0$ is an arbitrary constant.

Let $\tilde\delta := \min\{|\alpha|, |\beta|, 2\}$. Choosing in a suitable way $C_1 > 0$, from (7.1) and (7.7), (7.8) we get

$$\frac{\tilde\delta}{4}\int (\mathcal{A}u^{\alpha-1} + \mathcal{B}v^{\beta-1} + u^\alpha f + v^\beta g)\varphi\,dx \le \frac{2}{\gamma}C_1^{\frac{2-\gamma}{2}}\int \frac{|D\varphi|^\gamma}{\varphi^{\gamma-1}}\,dx\,. \tag{7.9}$$

Now repeating the steps of the proof of Theorem 3.1 and using our assumption (7.4) we conclude the proof. $\square$

7.1. Some examples. In this subsection we will consider some examples which illustrate Theorem 7.1.

Example 7.1. Consider the following noncooperative system:

$$\begin{cases} -\Delta u \geq u^q - auv^{q-1} = f(x,u,v) & \text{in } \mathbb{R}, \\ -\Delta v \geq v^q - au^{q-1}v = g(x,u,v) & \text{in } \mathbb{R}, \\ u,v \geq 0, \ u,v \not\equiv 0, \end{cases} \tag{7.10}$$

where $q > 0$, $a > 0$, and $N \geq 1$. With the notation introduced above we have

$$\mathcal{A}_0 = N\,, \qquad \mathcal{A} = |\xi|^2\,, \qquad \mathcal{B}_0 = N\,, \qquad \mathcal{B} = |\eta|^2\,. \tag{7.11}$$

Now in order to apply Theorem 7.1 we need to check that inequality (7.1) holds. That is, we need to show that there exists $C_{\alpha,\beta} > 0$ such that for all $\alpha, \beta < 0$ sufficiently small we have

$$(u^{\alpha+1}\mathcal{A}_0 + v^{\beta+1}\mathcal{B})^{\frac{\gamma}{\gamma-2}} \leq C_{\alpha,\beta}\left\{u^{\alpha-1}\mathcal{A} + v^{\beta-1}\mathcal{B} + u^\alpha f(x,u,v) + v^\beta g(x,u,v)\right\}, \tag{7.12}$$

where $\gamma > 2$ and $\gamma \leq N$. Now by (7.10) and $\alpha = \beta < 0$ inequality (7.12) becomes

$$N^{\frac{\gamma}{\gamma-2}}\left(u^{\alpha+1} + v^{\beta+1}\right)^{\frac{\gamma}{\gamma-2}} \leq C_{\alpha,\beta}\left(u^{\alpha-1}|\xi|^2 + v^{\alpha-1}|\eta|^2 + u^{q+\alpha} + v^{q+\alpha} - au^{\alpha+1}v^{q-1} - av^{\alpha+1}u^{q-1}\right). \tag{7.13}$$

By the Young inequality we have

$$au^{\alpha+1}v^{q-1} \leq \frac{1}{2}u^{q+\alpha} + k_a v^{q+\alpha}\,, \tag{7.14}$$

$$av^{\alpha+1}u^{q-1} \leq \frac{1}{2}v^{q+\alpha} + k_a u^{\frac{q+\alpha}{q-1}}\,, \tag{7.15}$$

where

$$k_a = \frac{q-1}{q+\alpha}\left(2\frac{\alpha+1}{q+\alpha}\right)^{\frac{\alpha+1}{q-1}} a^{\frac{q+\alpha}{q-1}} \tag{7.16}$$

and $\alpha \in (-1, 0)$.

Let $\gamma = N > 2$. Consider the following inequality:

$$(u^{\alpha+1} + v^{\alpha+1})^{\frac{N}{N-2}} \leq C\Big\{\frac{1}{2}u^{q+\alpha} + \frac{1}{2}v^{q+\alpha} - k_a u^{q+\alpha} - k_a v^{q+\alpha}\Big\} = C\Big(\frac{1}{2} - k_a\Big)\big\{u^{q+\alpha} + v^{q+\alpha}\big\}, \tag{7.17}$$

where $k_a < \frac{1}{2}$.

Suppose for a moment that this inequality holds or, equivalently, that

$$(u^{\alpha+1} + v^{\alpha+1})^{\frac{N}{N-2}} \leq C\left\{u^{q+\alpha} + v^{q+\alpha}\right\}. \tag{7.18}$$

For $u, v > 0$ we have

$$(u^{\alpha+1} + v^{\alpha+1})^{\frac{N}{N-2}} \leq C\left\{u^{(\alpha+1)\frac{N}{N-2}} + v^{(\alpha+1)\frac{N}{N-2}}\right\}. \tag{7.19}$$

Consequently, if there exists $\alpha \in (-1, 0)$ such that

$$q + \alpha = (\alpha+1)\frac{N}{N-2}\,, \tag{7.20}$$

then (7.18) holds.

Clearly (7.20) is equivalent to

$$q = \frac{N}{N-2} + \alpha\frac{2}{N-2}\,. \tag{7.21}$$

This implies that for any $q \in (1, \frac{N}{N-2})$ there exists $\alpha \in (-1, 0)$ such that (7.18) holds. Now as a last step let us analyze the inequality $k_a < \frac{1}{2}$ used in obtaining (7.17). This inequality is equivalent to

$$\frac{q-1}{q-\alpha}\left(2\frac{\alpha+1}{q+\alpha}\right)^{\frac{\alpha+1}{q-1}} a^{\frac{q+\alpha}{q-1}} < \frac{1}{2}\,, \tag{7.22}$$

where $a > 0$ and by (7.21)

$$-1 < \alpha = \frac{(N-2)q - N}{2} < 0.$$

Substituting this value of α in (7.22) we finally get

$$\frac{2}{N}\left(2\frac{N-2}{N}\right)^{\frac{N-2}{2}} a^{\frac{N}{2}} < \frac{1}{2} \quad \Leftrightarrow \quad a < \frac{1}{2}\left(\frac{N-2}{N}\right)^{\frac{2}{N}} \frac{N}{N-2}\,.$$

We have then proved the following

Theorem 7.2. *Let* $1 < q < \frac{N}{N-2}$ *and*

$$0 < a < \frac{1}{2}\left(\frac{N-2}{2}\right)^{\frac{2}{N}} \frac{N}{N-2}\,. \tag{7.23}$$

Then (7.10) *has no solutions of class* $X^{f,g}_{\alpha,\alpha}$.

Example 7.2. Consider

$$\begin{cases} -\Delta u \ge u^q - avu^{q-1} = f(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v \ge v^q - auv^{q-1} = g(x,u,v) & \text{in } \mathbb{R}^N, \\ u, v > 0. \end{cases} \tag{7.24}$$

Here, as above we assume that $N > 2$, $q > 1$, and $a > 0$. Using the same notations as before we have

$$\mathcal{A}_0 = N\,, \qquad \mathcal{A} = |\xi|^2\,, \qquad \mathcal{B}_0 = N\,, \qquad \mathcal{B} = |\eta|^2\,.$$

As in the preceding example, in order to apply Theorem 7.1 we need to verify (7.12). This condition, with the above choice of $\mathcal{A}_0$, $\mathcal{A}$, $\mathcal{B}_0$, and $\mathcal{B}$, becomes ($\alpha = \beta < 0$)

$$N^{\frac{\gamma}{\gamma-2}}\left(u^{\alpha+1} + v^{\alpha+1}\right)^{\frac{\gamma}{\gamma-2}} \le C_{\alpha,\beta}\Big(u^{\alpha-1}|\xi|^2 + v^{\alpha-1}|\eta|^2 + u^{q+\alpha} + v^{q+\alpha} - avu^{q+\alpha-1} - auv^{q+\alpha-1}\Big)\,, \tag{7.25}$$

where $\gamma > 2$ and $C_{\alpha,\beta} > 0$.

Now by the Young inequality we have

$$\begin{cases} avu^{q+\alpha-1} \le \dfrac{1}{2}v^{q+\alpha} + m_a u^{q+\alpha}, \\ auv^{q+\alpha-1} \le \dfrac{1}{2}u^{q+\alpha} + m_a v^{q+\alpha}, \end{cases} \tag{7.26}$$

where

$$m_a := \frac{q+\alpha-1}{q+\alpha}\left(\frac{2}{q+\alpha}\right)^{\frac{1}{q+\alpha-1}} a^{\frac{q+\alpha}{q+\alpha-1}}$$

and $\alpha \in (-1,0),\ q+\alpha > 1$.

By the same reasoning as was used in the study of Example 7.1, we can easily see that the following result holds.

Theorem 7.3. *Let $N > 2$ and $1+\frac{2}{N} < q < \frac{N}{N-2}$. If*

$$0 < a < \left(\frac{N(q-1)}{4}\right)^{\frac{2}{N(q-1)}} \left(\frac{1}{2}\,\frac{q-1}{q-1-\frac{2}{N}}\right)^{\frac{q-1-\frac{2}{N}}{q-1}} \tag{7.27}$$

holds, then (7.24) *has no solutions of class $X^{f,g}_{\alpha,\alpha}$.*

8. STRONGLY COUPLED SYSTEMS

In this section we will prove a nonexistence result for systems which are "strongly coupled." The precise meaning of the strong coupling is clear from the assumptions below.

(i) Let $N > 1$ and $A\colon \mathbb{R}^N \times \mathbb{R}^2_+ \times (\mathbb{R}^2)^2 \to \mathbb{R}_+$ be a Caratheodory function.

(ii) Let $f, g\colon \mathbb{R}^N_+ \times (\mathbb{R}^N)^2 \to \mathbb{R}_+$ be Caratheodory functions.

(iii) Suppose that there exists $\gamma > 2$ with $\gamma \geq N$. Let $\alpha, \beta < 0$ and $\lambda, \mu > 0$ be such that

$$\lambda(\beta+1) + \mu(\alpha+1) = 0 \tag{8.1}$$

and

$$\lambda v f + \mu u g \geq 0 \tag{8.2}$$

for any $u, v > 0$.

(iv) There exists $c_0 > 0$ such that for any $(x,\xi,\eta) \in (\mathbb{R}^N)^3$ and $u, v > 0$ we have

$$(u^{\alpha+1}v^{\beta+1}A)^{\frac{\gamma}{\gamma-2}} \leq c_0(\lambda v^2|\xi|^2 + \mu u^2|\eta|^2)u^{\alpha-1}v^{\beta-1}A + (\lambda v f + \mu u g)u^\alpha v^\alpha\,. \tag{8.3}$$

(v) If $u, v \geq 0$, $\xi, \eta \in \mathbb{R}^N$ and

$$\left(\lambda v^2|\xi|^2 + \mu u^2|\eta|^2\right) u^{\alpha-1}v^{\beta-1}A + (\lambda v f + \mu u g)u^\alpha v^\beta = 0\,,$$

then $u \equiv v \equiv 0$.

Theorem 8.1. *Suppose that the assumptions* (i)–(v) *are satisfied. Then the problem*

$$\begin{cases} -\operatorname{div}(A(x,u,v,Du,Dv)Du) \geq f(x,u,v,Du,Dv) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(A(x,u,v,Du,Dv)Dv) \geq g(x,u,v,Du,Dv) & \text{in } \mathbb{R}^N, \\ u,v \geq 0,\ u,v \not\equiv 0 \end{cases} \tag{8.4}$$

has no solutions in the corresponding class of locally bounded integrals.

In the proof of Theorem 8.1 we use the same idea developed throughout this paper. The only difference is the introduction of "mixed multipliers," that is, functions of the form

$$u^a v^b \varphi\,,$$

where $a, b \in \mathbb{R}$ and φ is a standard cut-off function.

Proof of Theorem 8.1. As usual we shall proceed by supposing that (u, v) is a positive solution of the problem under consideration and derive a contradiction.

Let (u, v) be a positive solution of (8.1) and introduce

$$u^\alpha v^{\beta+1}\varphi \tag{8.5}$$

and

$$u^{\alpha+1} v^\beta \varphi\,, \tag{8.6}$$

where φ is a standard cut-off function.

By multiplying the first equation of (8.4) [respectively, the second equation of (8.4)] by (8.5) [by (8.6)] and integrating by parts we obtain

$$\begin{aligned} \int f u^\alpha v^{\beta+1}\varphi\,dx &\le \alpha \int u^{\alpha-1} v^{\beta+1} A|Du|^2 \varphi\,dx \\ + (\beta+1)\int u^\alpha v^\beta A(Du, Dv)\varphi\,dx &+ \int u^\alpha v^{\beta+1} A(Du, D\varphi)\,dx\,, \end{aligned} \tag{8.7}$$

$$\begin{aligned} \int g v^\beta u^{\alpha+1}\varphi\,dx &\le \beta \int v^{\beta-1} u^{\alpha+1} A|Dv|^2 \varphi\,dx \\ + (\alpha+1)\int u^\alpha v^\beta A(Du, Dv)\varphi\,dx &+ \int v^\beta u^{\alpha+1} A(Du, D\varphi)\,dx\,. \end{aligned} \tag{8.8}$$

Next, by multiplying (8.7) [(and, respectively, (8.8)] by λ [by μ] and summing up, we get

$$\begin{aligned} &\alpha\lambda\int u^{\alpha-1} v^{\beta+1} A|Du|^2\varphi\,dx + \beta\mu\int v^{\beta-1}u^{\alpha+1}A|Dv|^2\varphi\,dx \\ &+\int (\lambda(\beta+1)A + \mu(\alpha+1)A)u^\alpha v^\beta (Du, Dv)\varphi\,dx + \int \lambda u^\alpha v^{\beta+1} A(Du, D\varphi)\,dx \\ &+ \int \mu v^\beta u^{\alpha+1} A(Dv, D\varphi)\,dx \ge \int (\lambda u^\alpha v^{\beta+1} f + \mu v^\beta u^{\alpha+1} g)\varphi\,dx\,. \end{aligned} \tag{8.9}$$

Now by (iii), the last inequality becomes

$$\begin{aligned} &-\int (\alpha\lambda v^2|Du|^2 + \beta\mu u^2|Dv|^2)u^{\alpha-1}v^{\beta-1}A\varphi\,dx + \int (\lambda v f + \mu u g)u^\alpha v^\beta \varphi\,dx \\ &\le \int [\lambda v(Du, D\varphi) + \mu u(Dv, D\varphi)]u^\alpha v^\beta A\,dx\,. \end{aligned} \tag{8.10}$$

By using the Young inequality we obtain

$$v(Du, D\varphi)u^\alpha v^\beta \le v|Du|u^\alpha v^\beta|D\varphi| \le \frac{|\alpha|}{2}u^{\alpha-1}v^{\beta+1}|Du|^2\varphi + \frac{1}{2|\alpha|}u^{\alpha+1}v^{\beta+1}\frac{|D\varphi|^2}{\varphi}$$

and

$$u(Dv, D\varphi)u^\alpha v^\beta \le u|Dv|u^\alpha v^\beta|D\varphi| + \frac{|\beta|}{2}u^{\alpha+1}v^{\beta-1}|Dv|^2\varphi + \frac{1}{2|\beta|}u^{\alpha+1}v^{\beta+1}\frac{|D\varphi|^2}{\varphi}\,.$$

Thus, (8.10) becomes

$$\int \Big(\frac{|\alpha|}{2}\lambda v^2|Du|^2+\frac{|\beta|}{2}\mu u^2|Dv|^2\Big)u^{\alpha-1}v^{\beta-1}A\varphi\,dx+\int(\lambda vf+\mu ug)u^\alpha v^\beta\varphi\,dx \leq \frac{1}{2}\Big(\frac{\mu}{|\beta|}+\frac{\lambda}{|\alpha|}\Big)\int u^{\alpha+1}v^{\beta+1}A\frac{|D\varphi|^2}{\varphi}\,dx\,. \tag{8.11}$$

Applying again the Young inequality to the right-hand side of (8.11) we arrive at

$$\frac{1}{2}\int(|\alpha|\lambda v^2+|Du|^2+|\beta|\mu u^2|Dv|^2)u^{\alpha-1}v^{\beta-1}A\varphi\,dx+\int(\lambda vf+\mu ug)u^\alpha v^\beta\varphi\,dx \leq \frac{1}{2}\Big(\frac{\lambda}{|\alpha|}+\frac{\mu}{|\beta|}\Big)C_1\int(u^{\alpha+1}v^{\beta+1}A)^{\frac{\gamma}{\gamma-2}}\varphi\,dx+\frac{1}{2}\Big(\frac{\lambda}{|\alpha|}+\frac{\mu}{|\beta|}\Big)C_1^{\frac{2-\gamma}{2}}\int\frac{|D\varphi|^\gamma}{\varphi^{\gamma+1}}\,dx\,, \tag{8.12}$$

where $C_1>0$ is an arbitrary constant.

Let

$$\tilde\delta=\min\{|\alpha|,|\beta|,1\}\,.$$

By choosing $C_1>0$ (appearing in (8.12)) such that

$$C_1=\frac{1}{2}\tilde\delta^2\frac{1}{(\lambda+\mu)C_0}\,,$$

we see that (8.12) implies

$$\int\Big[(\lambda v^2|Du|^2+\mu u^2|Dv|^2)u^{\alpha-1}v^{\beta-1}A\,dx+\int(\lambda vf+\mu ug)u^\alpha v^\beta\Big]\varphi\,dx\leq C\int\frac{|D\varphi|^\gamma}{\varphi^{\gamma-1}}\,dx\,. \tag{8.13}$$

where $C>0$.

In order to complete the proof, we just need now to follow the same procedure used in the proof of Theorem 2.1. We omit the details. $\square$

8.1. Examples.

Example 8.1. Consider

$$\begin{cases} -\Delta u\geq \dfrac{u^p}{v^{q+1}} & \text{in } \mathbb{R}^N\,,\\ -\Delta v\geq \dfrac{u^{p-1}}{v^q} & \text{in } \mathbb{R}^N\,,\\ u,v\geq 0,\ u,v\not\equiv 0, \end{cases} \tag{8.14}$$

where $1<p,\ 0<q$, and $N>2$.

It is not difficult to verify that the assumptions of Theorem 8.1 are fulfilled with $\gamma=N>2$, thus we have

Theorem 8.2. *If* $1<p<\frac{N}{N-2}$ *and* $q>0$, *then* (8.14) *has no solutions such that*

(a) $\displaystyle\int_\Omega u^{-\frac{(N+2)-(N-2)p}{2}}v^{-\frac{(N-2)(q+1)}{2}}|Du|^2\,dx<\infty$,

(b) $\displaystyle\int_\Omega u^{\frac{(N-2)(p-1)}{2}}v^{-\frac{(N+2)+(N-2)q}{2}}|Dv|^2\,dx<\infty$,

(c) $\int_\Omega u^p v^{-q}\,dx < \infty$

for any bounded domain $\Omega \subset \mathbb{R}^N$.

Example 8.2. Consider

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(u\frac{\partial u}{\partial x_i}\Big) \geq u^p v^{q-1} \quad \text{in } \mathbb{R}^N, \\ -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(u\frac{\partial v}{\partial x_i}\Big) \geq u^{p-1} v^{q} \quad \text{in } \mathbb{R}^N, \\ u, v \geq 0,\ u, v \not\equiv 0 \end{cases} \tag{8.15}$$

where

(i) $1 < q < \frac{N}{N-2},\ N > 2$;

(ii) $1 < p < \frac{2(N-1)}{N-2},\ N > 2$.

Applying Theorem 8.1 we obtain

Theorem 8.3. *Suppose that* (i) *and* (ii) *hold. Then* (8.15) *has no solutions such that*

(a) $\int_\Omega u^{\frac{p(N-2)-2N}{2}} v^{\frac{(q-1)(N-2)}{2}} |Du|^2\,dx < \infty$,

(b) $\int_\Omega u^{\frac{(p-2)(N-2)}{2}} v^{\frac{q(N-2)-(N+2)}{2}} |Dv|^2\,dx < \infty$,

(c) $\int_\Omega u^{\frac{(p-2)N}{2}} v^{\frac{(q-1)N}{2}}\,dx < \infty$

for any bounded domain $\Omega \subset \mathbb{R}^N$.

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