WORKSHOP ON BIRATIONAL GEOMETRY
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ABSTRACTS

Claudio Arezzo (ICTP)
Canonical metric on algebraic manifolds

Metrics of constant curvature have always been thought as the best Riemannian metrics we can equip a given manifold with. In the first part of the talk I will explain few classical reasons supporting this belief and some of the subtleties we face proving their existence. In particular real and complex manifolds behave very differently depending on which curvature (sectional, Ricci, scalar, ...) one tries to make constant. In the second part of the talk I will focus on the complex case, introducing some fundamental obstructions and analysing if and when few basic geometric operations such as Galois coverings, blow ups and resolutions of singularities preserve the existence of such metrics. A number of open problems will be discussed.

Alexey Golota (HSE)
On equivariant K-stability for Fano varieties with infinite automorphism groups

In recent years the algebro-geometric notions of K-polystability and uniform K-stability attracted a lot of attention thanks to their connection to Kähler–Einstein problem and moduli theory for Fano varieties. The stronger property of uniform K-stability can be checked using the recently established valuative criteria and computing a so-called delta-invariant. However, a uniformly K-stable Fano variety nesessarily has a finite automorphism group. To treat the case of a Fano variety with an action of an infinite group G, it is desirable to generalize these notions (uniform K-stabilty, valuative criterion, delta-invariant) to G-equivariant setting. In my talk I will survey a few recent works in this direction (independently, by Ziwen Zhu, Chi Li, and myself). Also I will consider some examples, such as spherical Fano varieties and G-varieties of complexity one.

Marta Pieropan (EPFL)
On rationally connected varieties over C_1 fields of characteristic 0

In the 1950s Lang studied the properties of C_1 fields, that is, fields over which every hypersurface of degree at most n in a projective space of dimension n has a rational point. Later he conjectured that every smooth proper rationally connected variety over a C_1 field has a rational point. The conjecture is proven for finite fields (Esnault) and function fields of curves over algebraically closed fields (Graber–Harris–de Jong–Starr), but it is still open for the maximal unramified extensions of p-adic fields. I use birational geometry in characteristic 0 to reduce the conjecture to the problem of finding rational points on Fano varieties with terminal singularities.

Vasily Rogov (HSE)
Non-algebraic deformations of flat Kähler manifolds

Consider a compact complex manifold admitting a Kähler metric with everywhere vanishing curvature tensor. Such manifolds can be viewed as natural generalisations of complex tori. For complex tori of dimension greater then one it is well-known that most of them cannot be obtained as analytifications of abelian varieties, and algebraic reduction of a complex torus can be realised as a quotient by a subtorus. I am going to describe nice holomorphic models for algebraic reductions of flat Kähler manifolds and give some conditions for existence of a non-algebraic flat Kähler manifold in given deformation class.

Sergey Rybakov (IITP and HSE)
Zeta functions of supersingular $K3$ surfaces over finite fields

Let X be a supersingular $K3$ surface over a finite field. The Neron–Severi group of X over an algebraic closure of the finite field is of rank 22 and has a semisimple linear Frobenius action. The zeta function of X is uniquely determined by the characteristic polynomial of this action. I will speak on a possible classification of zeta functions for supersingular $K3$ surfaces, and give some examples.

Dmitri Timashev (MSU)
Characterizing unipolar flag manifolds by their varieties of minimal rational tangents

It is well known that rational curves play a key role in the geometry of projective algebraic varieties, especially of Fano manifolds. In particular, on Fano manifolds of Picard number one, which are sometimes called unipolar, one may consider rational curves of minimal degree passing through general points. Tangent directions of minimal rational curves through a general point form a projective subvariety in the projectivized tangent space, called the variety of minimal rational tangents (VMRT).

In 90-s J.-M. Hwang and N. Mok developed a philosophy declaring that the geometry of a unipolar Fano manifold is governed by the geometry of its VMRT at a general point, as an embedded projective variety. In support of this thesis, they proposed a program of characterizing unipolar flag manifolds in the class of all unipolar Fano manifolds by their VMRT. In the following decades a number of partial results were obtained by Mok, Hwang, and their collaborators.

Recently the program was successfully completed (J.-M. Hwang, Q. Li, and the speaker). The main result states that a unipolar Fano manifold X whose VMRT is isomorphic to the one of a unipolar flag manifold Y is itself isomorphic to Y. Interestingly, the proof of the main result involves a bunch of ideas and techniques from "pure" algebraic geometry, differential geometry, structure and representation theory of simple Lie groups and algebras, and theory of spherical varieties.

Juanyong Wang (Université Paris-Sud 11)
On the Iitaka Conjecture $C_{n,m}$ for Kähler Fibre Spaces

For a holomorphic line bundle L over a complex variety X one defines its Kodaira(–Iitaka) dimension $\kappa(L)$ as the dimension of (the closure of) the image of the meromophic mapping given by the linear series $|\nu^* L^{\otimes m}|$ for m sufficiently large and divisible, where $\nu\colon X^\nu \to X$ stands for the normalization of X (the dimension of an empty set is by convention defined to be $-\infty$). Equivalently, $\kappa(L)$ is the (unique) integer $\kappa \in \{-\infty, 0, 1, \cdots, \dim X\}$ such that there exist $C_1, C_2 > 0$ independent of m satisfying

$$C_1 \cdot m^\kappa \leq \mathrm{h}^0(X, L^{\otimes m}) \leq C_2 \cdot m^\kappa.$$

The Kodaira dimension measures the positivity of a line bundle. In particular, the Kodaira dimension of X is defined as the Kodaira dimension of the canonical bundle of a smooth model of X, and is a bimeromorphic invariant.

The Iitaka conjecture $C_{n,m}$ (Ueno, 1975) predicts that the Kodaira dimension is sup-additive with respect to algebraic fibre spaces (that is, a surjective projective morphism with connected fibres). $C_{n,m}$ is intimately related to the birational classification of complex algebraic varieties (MMP); in fact, it can be seen as a consequence of the Abundance conjecture (Kawamata, Matsuda & Matsuki, 1987). $C_{n,m}$ is already known in lower dimensions ($\dim X \leq 6$, Birkar; $\dim Y = 1$, Kawamata; $\dim Y = 2$, Cao). As for higher dimensions, with the help of the method of the positivity of direct images developped by Griffiths, Fujita, Kawamata, Viehweg, Berndtsson, Păun, Takayama, etc., the conjecture is proved in the following three cases:

(1) Y is of general type (Kawamata, 1981; Viehweg, 1983);
(2) there exists an $m > 0$ such that the determinant bundle $\det f_*(mK_X)$ is big over Y (Viehweg, 1983);
(3) Y is an Abelian variety (Cao & Păun, 2017).

In this talk, I will explain how to generalize the three results above on $C_{n,m}$ to the (log/orbifold) Khler case. In fact, $C_{n,m}$, as well as the MMP and the Abundance, is considered as still hold for Kähler varieties (e.g. the 3-dimensional case is proved in the following three articles: Höring & Peternell, 2015; Höring & Peternell, 2016; Campana, Höring & Peternell, 2016). The proof of these results follows the mainstream of considering the positivity of the direct images of pluricanonical bundles (cf. Păun & Takayama, 2018; Deng, Wang, Zhang & Zhou, 2018), and is based on the Ohsawa–Takegoshi type extension theorem with optimal estimation for Khler fibre spaces (Cao, 2014) and a Green–Lazarsfeld–Simpson type result on the cohomology jumping loci over Kähler manifolds (Wang, 2016).

Grigory Yurgin (HSE)
Rational points on surfaces

The talk will be devoted to the proof of the following result. Let X be a smooth proper geometrically rational surface over a perfect field $\mathbf{k}$, suppose that X has no rational points and degree of X is at least 6. Also let $\mathbf{L}/\mathbf{k}$ be a field extention whose degree is not divisible by 2 and 3. Then the surface $X \times \mathrm{Spec}(\mathbf{L})$ has no rational points. The proof consists of the case of del Pezzo surfaces and the case of conic bundles; we will pay significant attention to both cases.