The diagonal of quartic 5-folds
(j.w. Nebojsa Pavic)

1. Introduction

X a variety $/k$.

Def. X retract rat'l $\Leftrightarrow$ $\exists f, g$

$$\begin{tikzcd} X \arrow[rr, dashed, "id"] \arrow[dr, dashed, "f"'] & & X \\ & \mathbb{P}^N \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

$(N \geq \dim X)$

Ex. If $X \times Y$ rat'l $\Rightarrow$ X is retract rat'l

Rmk.

$$\text{RAT'L} \underset{\not\Leftarrow}{\Rightarrow} \text{ST. RAT'L} \underset{?}{\Rightarrow} \text{RETR RAT'L} \underset{\not\Leftarrow}{\Rightarrow} \text{UNI RAT'L}$$

$X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of deg d, very general

$d=4, n=3$: Iskovskikh–Manin, CT Pirutka not retract rat'l

$d=4, n=4$, Totaro '16 not retract rat'l

$d \geq \log_2 n + 2$, $n \geq 3$ $\rightsquigarrow$ X is not retr rat'l [Sch]

$n=5$: $d \geq 5$

<u>Thm</u> (Nicaise - Ottem) $X \subset \mathbb{P}^6_{\mathbb{C}}$ is vg quartic
$\Rightarrow$ X is not stably rat'l.

<u>Rmk.</u> Does not work for retr. rat'l
or in char > 0.

<u>NO used:</u>

<u>Thm</u> (Nicaise Shinder, Kontsevich-Tschinkel)
$\mathcal{X} \to \operatorname{Spec} \mathbb{C}[[t]]$ is str. ss with special
fibre $Y = \bigcup_{i \in I} Y_i$, then
$X_{\bar{\eta}}$ stably rat'l $\Rightarrow \sum_{J \subset I} (-1)^{|J|-1} [Y_J] = [pt.]$
in $\mathbb{Z}\{$st. bir'l equiv classes$\}$
where $Y_J = \bigcap_{j \in J} Y_j$

<u>Proof of NO-thm.</u>
$$\mathcal{X} = \{ wt = z_5 z_6,\ w^2 = F \} \subseteq \mathbb{P}_{\mathbb{C}[[t]]}(\underbrace{1}_{z_0,\dots,z_6}, \underbrace{2}_{w})$$
$F \in \mathbb{C}[z_0, \dots, z_6]$ quartic v.g. subject to inv

under $z_5 \leftrightarrow z_6$.

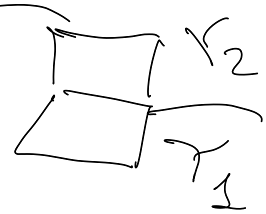
special fibre $Y = \{z_5 \cdot z_6 = 0,\ w^2 = F\} = Y_1 \cup Y_2$

$$Y_i \xrightarrow{2:1} \mathbb{P}^5 \text{ branched at } F|_{z_6=0}$$

$Z = Y_1 \cap Y_2 = \{z_5 = z_6,\ w^2 = F\}$

$$Z \xrightarrow{2:1} \mathbb{P}^4 \text{ branched at } F|_{z_5 = z_6 = 0}.$$

generic fibre: $w = t^{-1} z_5 z_6 \rightsquigarrow \{t^{-2} z_5^2 z_6^2 = F\} \subseteq \mathbb{P}^6$ is a quartic

$Y =$ 

$Z \xrightarrow{2:1} \mathbb{P}^4 \qquad y^2 = f$

need to check

$$\underbrace{[Y_2] + [Y_1]}_{=2[Y_1]} - \underset{\substack{\uparrow \\ \neq [\mathrm{pt.}] \text{ by } [\mathrm{L\text{-}P}]}}{[Z]} \overset{?}{=} [\mathrm{pt.}] \in \mathbb{Z}\left[\begin{matrix}\text{st. bir'l} \\ \text{equiv.}\end{matrix}\right]$$

↯

$\Rightarrow$ NO

Alternatively, we can make Y_1 & Y_2 singular (triple pt) $\rightsquigarrow$ rat'l

Q1 $X_4 \subset \mathbb{P}^6$ is it retr. rat'l?

Q2 How to replace motivic obstruction of NS - KT in the above argument.

2. Obstruct Retr. rat'l

$F : \{\text{Sm Proj}_k\} \to Ab$ functorial for rat'l maps

Lem. X retr rat'l & $F(\mathbb{P}^N) = 0 \ \forall N$
$\Rightarrow F(X) = 0$.

pf.
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, dashed] & & X \\ & \mathbb{P}^N \arrow[ur, dashed] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(X) \arrow[rr, "id"] \arrow[dr] & & F(X) \\ & F(\mathbb{P}^N) = 0 \arrow[ur] & \end{tikzcd} \qquad \square$$

Ex1 $F(X) = H^0(X, \Omega^i_X)$, $i \geq 1$ contravariant

Ex2 $F(X) = \ker\left(CH_0(X) \xrightarrow{\deg} \mathbb{Z}\right)$ covariant

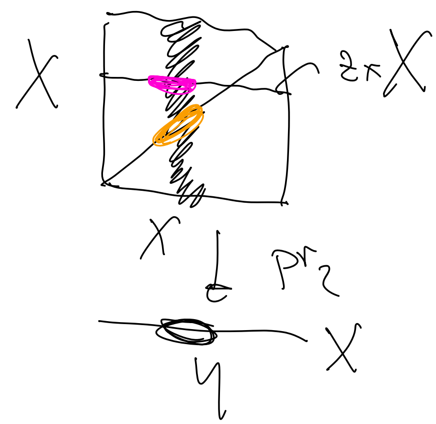
Ex3 $\forall L/k \rightsquigarrow F_L(X) = \ker(CH_0(X_L) \to \mathbb{Z})$

Def. X is univ CH_0-triv if $CH_0(X_L) \xrightarrow[\simeq]{\deg} \mathbb{Z}$ is an iso $\forall L/k$.

Def. $\exists$ dec of Δ of X if $\exists z \in CH_0(X)$ with

$$\delta_X = z_{k(X)} \in CH_0(X_{k(X)})$$

class of Δ_X



Lem. X sm proj. TFAE:

(1) X is univ CH_0-triv

(2) $\exists$ dec of Δ.

UPSHOT: If X is not univ CH_0-trivial

$\Rightarrow$ X is not retract rat'l

Back to NO degeneration:

$\mathcal{X} \to \operatorname{Spec} \mathbb{C}[[t]]$, $Y = Y_1 \cup Y_2$, $Z = Y_1 \cap Y_2$

For simplicity assume Y_1 & Y_2 are rat'l

Z is not CH_0-univ trivial (OK by [HPT])

$\Rightarrow$ Y is CH_0-univ trivial ~~and~~ ;)

<u>1st idea</u> 2:1 base change + blow-up Z

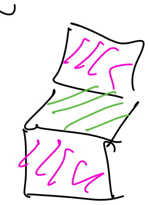
get special fibre $Y_1 \cup P_Z \cup Y_2$

rat'l

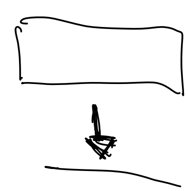
$\mathbb{P}^1$-bdl/Z

is not univ CH_0 trivial

rat'l

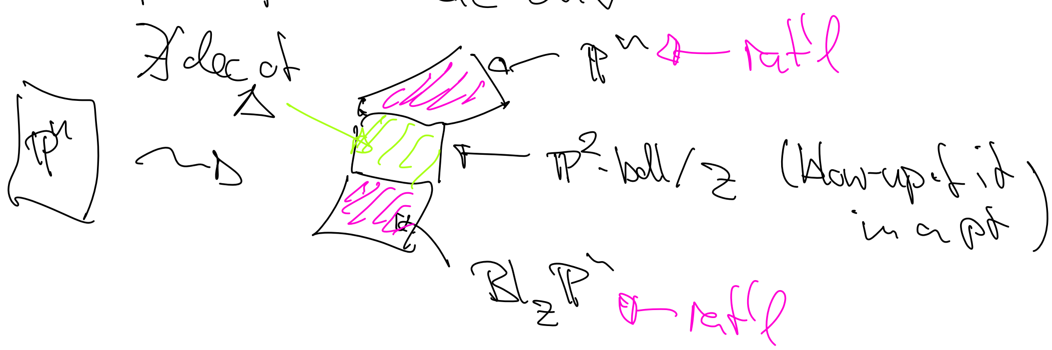


<u>Ex.</u> • start with $\mathbb{P}^n_{\mathbb{C}((t))}$



• blow-up $Z \subset \mathbb{P}^n_{\mathbb{C}}$ gen codim 2 subvariety in the special fibre

• blow-up of pt on exc div



Obstruction

$\mathcal{X} \to \operatorname{Spec} R$ str. ss, $R = \text{dvr}$, $R/m = k = \text{alg closed}$

$$\Phi_{\mathcal{X}} : CH_1(Y) \to \ker\Big(\deg : \bigoplus_{i\in I} CH_0(Y_i) \to \mathbb{Z}\Big)$$

$$\gamma \longmapsto \big(\underbrace{i_*\gamma|_{Y_i}}\big)_{i\in I}$$

$Y = \text{special fibre} = \bigcup_{i\in I} Y_i$

$Y \overset{i}{\hookrightarrow} \mathcal{X}$

if γ has sup on Y_j then $i_*\gamma|_{Y_i} = \begin{cases} \gamma \cap [Y_j \cap Y_i] & i \neq j \\ -\sum_{k\neq i} \gamma \cap [Y_k] & i = j \end{cases}$

Thm (Pavic–Sch. '21)

(1) Assume X_η is univ CH_0-trivial

$\Rightarrow$ $\forall$ unramified ext A/R of dvr (ex $L/\mathbb{C}$, $L[[t]]/\mathbb{C}[[t]]$)

$\Phi_{\mathcal{X}_A}$ is surjective.

(2) Assume $X_{\bar\eta}$ is univ CH_0-trivial & Y is a chain of components:

$\Rightarrow$ $\forall A/R$ unramified ext of dvr's $\Phi_{\mathcal{X}_A}$ is surj modulo 2.

We apply ② of the thm to a deg similar to NO to get the following:

<u>Thm</u> (Pavic-Sch '21) char $k \neq 2$.

$X \subseteq \mathbb{P}^6_k$ v.g. quartic $\Rightarrow$ X has no dec of Δ

$\Rightarrow$ X is not retr rat'l.

<u>Rmk.</u> char >0, rat'l was open.