An abelian fibered Calabi-Yau 3-fold with a relative automorphism of positive entropy

Workshop on birational geometry

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18:30 - Japan Time)

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§0. Introduction /$\mathbb{C}$

Problem (Open minded / Dinh-Sibony)

Find interesting (smooth proj) varieties with interesting automorphisms of positive entropy.

Aim of this talk (One phenomenon)

Main Thm (make more precise in §3)

There is exactly one abelian fibered Calabi-Yau 3-fold $\varphi: X \longrightarrow \mathbb{P}^1$ with $g \in \mathrm{Aut}(\varphi: X \to \mathbb{P}^1)$ of positive entropy up to isomorphisms as fiber spaces.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\varphi"] \\ \mathbb{P}^1 \arrow[r, "g_{\mathbb{P}^1}"'] & \mathbb{P}^1 \end{tikzcd}$$

(but not for K3 fibered Calabi-Yau 3-folds.)

Plan of this talk

§1. entropy and dynamical degree

§2 Surface case

§3 fibered Calabi-Yau 3-fold/$\mathbb{P}^1$ case

§4. C_2-contraction on Calabi-Yau 3-fold

§5. Proof of Main Thm

Def X: Calabi-Yau 3-fold (CY3)

$\Leftrightarrow$ smooth projective 3-fold /$\mathbb{C}$ with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, $\pi_1(X^{an}) = \{1\}$

$\Rightarrow$ fibration over a curve is either

(i) $X \xrightarrow[|F|]{} \mathbb{P}^1$ & gen. fiber F = K3 surface $c_2(X).F = c_2(F) = 24 > 0$

or

(ii) $X \xrightarrow[|F|]{} \mathbb{P}^1$ & gen. fiber F = abelian surface $c_2(X).F = c_2(F) = 0$

(by $0 \to T_F \to T_X|_F \to N_{F/X} = \mathcal{O}_F \to 0$)

§1. Entropy & dynamical degrees

(after Gromov-Yomdin, Dinh-Sibony)

X: smooth proj. var. dim $X = d$

$f: X \dashrightarrow X$ dom. rat. map $f^n = \underbrace{f\circ\cdots\circ f}_{n}$

H: ample class on X.

$$d_p(f) := \lim_{n\to\infty} \left((f^n)^*H^p \cdot H^{d-p}\right)^{\frac{1}{n}} \quad (\geq 1)$$

$$= \lim_{n\to\infty} \| (f^n)^* \circlearrowright N^p(X) \|^{\frac{1}{n}}$$

p-th dynamical degree of f $(0 \leq p \leq d)$

Well-defined & birat inv (Dinh-Sibony)

$$\begin{tikzcd} X \arrow[r, dashed, two heads, "f_X"] \arrow[d, dashed, two heads, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, dashed, two heads, "\varphi"] \\ Y \arrow[r, dashed, two heads, "f_Y"'] & Y \end{tikzcd} \Rightarrow \quad d_p(f_X) = d_p(f_Y)$$

if φ: gen. finite

& Concave: $d_{p+1}(f)\, d_{p-1}(f) \leq d_p(f)^2$.

Case $f: X \twoheadrightarrow X$ morphism

$h_{top}(f)$ (= topological entropy of f) is originally defined as a measure "how fast two general orbits $\{x, f(x), \ldots, f^n(x), \ldots\}$ $\{y, f(y), \ldots, f^n(y), \ldots\}$ spread out?"

Gromov-Yomdin's Thm

$$h_{top}(f) = \max_{0 \leq p \leq d} \log d_p(f) \quad (\geq 0)$$

$(f^n)^* = (f^*)^n$

$$= \max_{0 \leq p \leq d} \log \left(\text{spectral radius of } f^* \circlearrowright N^p(X) \right)$$

$h_{top}(f) > 0 \iff d_1(f) > 1$ $(\Rightarrow \mathrm{ord}\, f = \infty)$

(f: complicated) when f is automorphism

Q (Dinh-Sibony) Find "interesting" (X, f) for $f \in \mathrm{Aut}\, X, \mathrm{Bir}\, X, \mathrm{End}\, X$? with $d_1(f) > 1$

Ref O— ArXiv 1404.2982 (Some aspects...)

T.T. Truong ArXiv 1501.01523 (Relative...)

§2 Surface case

f:automorphism

dim $X = 1$ $\Rightarrow$ $\nexists X$ with $d_1(f) > 1$

dim $X = 2$ (first non-trivial case)

Thm (Cantat around 2000)

X : smooth projective surface

$\exists f \in \mathrm{Aut}(X)$ with $d_1(f) > 1$

$$\Rightarrow X \underset{\text{birat}}{\sim} \begin{cases} \mathbb{P}^2 & (\text{rational surf}) \\ K3 \\ \text{Enriques} \\ \text{Abelian surface} \end{cases}$$

Thm (McMullen around 2000)

Under the situation of Thm

$d_1(f) =$ Salem number

Slightly more precisely

$\Phi_{f^*|N^1(X)}(t)$ (characteristic poly)

$=$ (Salem poly) $\cdot \prod$ (cyclotomic poly)

Here : α : Salem number

$\Leftrightarrow$ α : real. alg. integer s.t.

def $\alpha > 1$ & Galois conj of α are

$\alpha, 1/\alpha, \exists \varepsilon_1, \overline{\varepsilon_1}, \cdots, \varepsilon_\ell, \overline{\varepsilon_\ell}$

$(|\varepsilon_i| = 1 \ \forall i = 1, \cdots, \ell)$

Salem poly $=$ min. poly of Salem $\# / \mathbb{Z}$

($\Rightarrow$ even degree)

cf.

Blanc-Cantat : ArXiv1307.0361 (Dynamical …)

O-Yu: ArXiv1807.09412 (Minimum …)

Bell-Diller-Jonsson : ArXiv1907.00675 (A transcendental …)

Ex Abelian surface T with $P \in \mathrm{Aut}(T)$ with $d_1(P) > 1$

Ex1 $T = E \times E$ E: ell. curve

$$\begin{cases} \tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} P \\ Q \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \\ \tau_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \begin{pmatrix} P \\ Q \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \end{cases}$$

$\Rightarrow$ $\tau_1, \tau_2 \in \mathrm{Aut}(T)$ &

$P := \tau_1 \tau_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

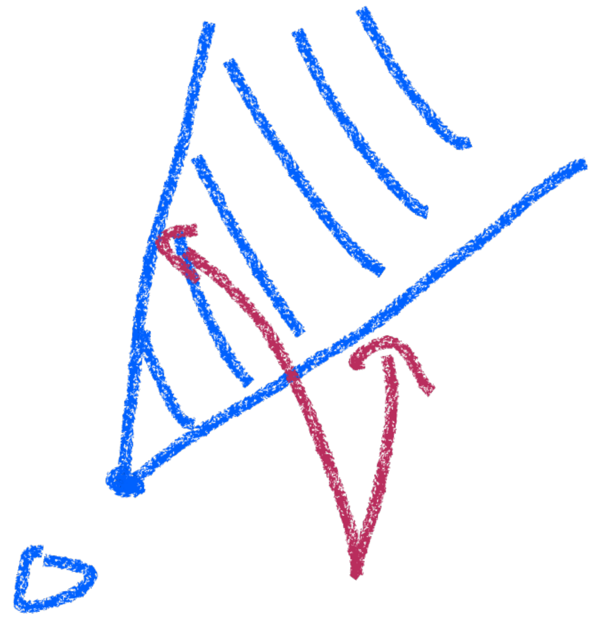
$$\Phi_{\tau_i^*|H^1(T)} = (t-1)^2 (t-1)^2$$

$$\Phi_{P^*|H^1(T)} = \underset{H^{1,0}(T)}{(t^2-3t+1)} \oplus \underset{\overline{H^{1,0}(T)}}{(t^2-3t+1)}$$

$\Rightarrow$ $d_1(\tau_i) = 1^2 = 1$ $(i = 1.2)$

$d_1(P) = \left(\dfrac{3+\sqrt{5}}{2}\right)^2 > 1$

Ex2 $\exists$ abelian surface T with $\rho(T) = 2$ &

$\overline{\mathrm{Amp}}(T) =$  $\subset N^1(T)_{\mathbb{R}} = N^1(T) \otimes \mathbb{R}$

$\mathbb{Z}$

irrational w.r.t. $N^1(T)_{\mathbb{Q}}$

($\forall$ even lattice $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ of sign $(1,1)$ is realized as $N^1(T)$ by the surjectivity of period mapping by Shioda)

$\Rightarrow$ T is simple & $\exists P \in \mathrm{Aut}_{gp}(T)$ s.t. $d_1(P) > 1$

(by cone conjecture for ab. var by Prendergast-Smith)

A lot of examples.

Ex K3 surface S with $f \in \mathrm{Aut}(S)$ with $d_1(f) > 1$

Ex3 (Wehler, Cantat-O —)

$S = (2,2,2) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ generic

$\Rightarrow S : K3$, $\mathrm{Pic}\, S = N^1(S) = \bigoplus_{i=1}^{3} \mathbb{Z} H_i$

$\iota_1, \iota_2, \iota_3$ covering involution of

$$S \xrightarrow[p_{23}]{} \mathbb{P}^1 \times \mathbb{P}^1,\quad S \xrightarrow[p_{13}]{} \mathbb{P}^1 \times \mathbb{P}^1,\quad S \xrightarrow[p_{12}]{} \mathbb{P}^1 \times \mathbb{P}^1$$

$\Rightarrow \iota_1, \iota_2, \iota_3 \in \mathrm{Bir}\, S = \mathrm{Aut}\, S$

$$\iota_1^* = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \iota_2^* = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad \iota_3^* = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P = \iota_3 \circ \iota_2 \circ \iota_1 \Rightarrow P^* = \begin{pmatrix} -1 & -2 & -6 \\ 2 & 3 & 10 \\ 2 & 6 & 15 \end{pmatrix}$$

$$d_1(P) = 9 + 4\sqrt{5} > 1$$

Rem

- In Ex3, $S \in |-K_V|$
  V : Fano 3-fold.
- $X = (2,2,\cdots,2) \subset \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{d+1 \geqq 4}$
  general

$\Rightarrow \mathrm{Aut}(X) = \{1\}$ $\mathrm{Bir}\, X = \langle \iota_1 \rangle * \cdots * \langle \iota_{d+1} \rangle$

(Cantat-O —)

cf. Long Wang ArXiv 2012.00272
(Remarks on nef & movable …)

Ex4 (T, P) in Ex1 or Ex2

$\Longrightarrow P \circ (-1_T) = (-1_T) \circ P$

$\Rightarrow P$ induces

$f \in \mathrm{Aut}(\mathrm{Km}\, T)$

with $d_1(f) = d_1(P) > 1$

$$\begin{tikzcd} & T \arrow[d] \\ \mathrm{Km}\, T \arrow[r] & T/(-1_T) \end{tikzcd}$$

Ex 5 (Nikulin, Allcock-Dolgachev, O-Yu)

S : 2-elementary K3 surface i.e.

$\exists \iota \in \mathrm{Aut}\, S$ s.t. $\begin{cases} \iota^* | N^1(S) = id_{N^1(S)} \\ \iota^*(\omega_S) = -\omega_S \end{cases}$

Assume $S^{\iota} = \underbrace{\mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1}_{\ell \geqq 1}$.

$\Rightarrow$ $f \circ \iota = \iota \circ f$ $\forall f \in \mathrm{Aut}(S)$ &

$\exists f \in \mathrm{Aut}(S)$ with $d_1(f) > 1$

Rem $\ell = 1, 2, \cdots, 10$ ($\forall$ occurs)

Concrete cases

Ex5-1) $S = \mathrm{Km}(\underbrace{E \times F}_{\text{non-isogenous}})$ $\iota = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$S^{\iota} = \underbrace{\mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1}_{8}$

Ex5-2) $\mathbb{P}^2 \supset \bigcup_{1 \leq i \leq 6} L_i$ generic 6 lines L6

$\uparrow$ blow-up 15 pts $\bigcup_{i \neq j} L_i \cap L_j$

$W \supset \bigcup_{1 \leq i \leq 6} \overline{L_i} =: B$ proper transform

$\uparrow$ 2:1 branched along B ($\in |-2K_W|$)

S ι : covering involution

$S^{\iota} = \underbrace{\mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1}_{6}$

Ex5-3) (Coble) $\mathbb{P}^2 \supset (6)$ 10 nodes, generic

$\uparrow$ b-up at 10 nodes

$W \supset \overline{(6)} = B$ proper transform

$\uparrow$ 2:1 : branched along B ($\in |-2K_W|$)

S ι : covering involution $S^{\iota} = \mathbb{P}^1$

Existence of f with $d_1(f) > 1$:

X-Yu ArXiv 2211.07526 (K3 surface entropy...)

Classify $\forall N^1(S)$ of K3 surfaces S of $P(S) \geqq 3$

with $\nexists f \in \mathrm{Aut}\, S$ with $d_1(f) > 1$ (331 cases)

§3 fibered C.Y. 3-folds/$\mathbb{P}^1$

(non-) Ex 6

V: Fano mfd $\dim V = d+1 \geq 4$

$X \in |-K_V|$ smooth

$\Rightarrow X$: CY d-fold (in the strict sense)

But $|\mathrm{Aut}(X)| < \infty$ & $\nexists f$ $d_1(f) > 1$

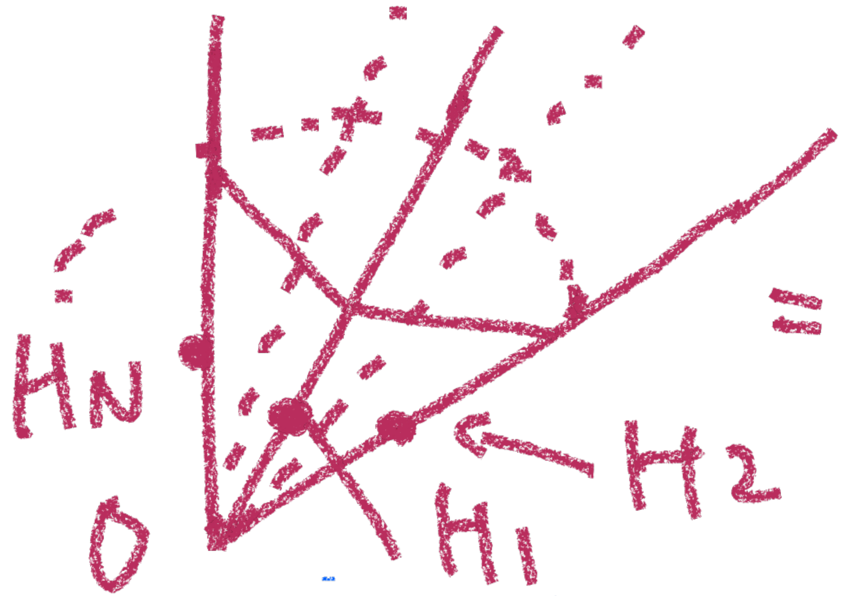
Pf) Kollár Under $\iota : X \hookrightarrow V$

$\iota^* : N^1(V) \xrightarrow{\sim} N^1(X)$ (Lefschetz)

$$\begin{tikzcd} \& \iota^* \quad N^1(V)_{\mathbb{R}} \arrow[r] \& N^1(X)_{\mathbb{R}} \\ \overline{\mathrm{Amp}}(V) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] \& \overline{\mathrm{Amp}}(X) \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

fin. rat poly.

$\Rightarrow H = \sum_{i=1}^{N} H_i$



$H_i (1 \leq i \leq N)$: $\forall$ prim. elements of 1-dim rays of $\partial \overline{\mathrm{Amp}}(X)$

$\Rightarrow H$: ample & inv. under $\mathrm{Aut}(X)$

$\Rightarrow$ (by Fujiki-Lieberman, or by the Hilbert scheme for the graphs of $\mathrm{Aut}(X)$ in $X \times X$ w.r.t $H \boxtimes H$)

$|\mathrm{Aut}(X) / \mathrm{Aut}^0(X)| < \infty$

$\Rightarrow |\mathrm{Aut}(X)| < \infty \Rightarrow \forall f \in \mathrm{Aut}\, X$ $d_1(f) = 1$. $\square$

Rem Also

$$\begin{tikzcd} V \arrow[r, "\pi"] \& \mathbb{P}^1 \\ X \arrow[u, hook] \arrow[ur, "\pi_X"'] \& \end{tikzcd}$$

$\Rightarrow$ general fiber of π_X is strict CY (never ab. var.)

Ex 7 (Borcea-Voisin Type)

(S, ι_S) 2-elem K3 in Ex 5

$f \in \mathrm{Aut}\, S$ $\quad d_1(f) > 1$

$$S^{\iota_S} = \underbrace{\mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1}_{\ell \geq 1}$$

$Z = S \times E$, E : ell. curve

$\iota_Z = \iota_S \times (-1)_E \quad f_Z = f \times id_E$

$\Rightarrow \iota_Z^* \omega_Z = \omega_Z \quad \iota_Z \circ f_Z = f_Z \circ \iota_Z$

$\Rightarrow \exists\, X \xrightarrow{\nu} Z/\langle \iota_Z \rangle$ crep. proj. resol

(b.up along 4ℓ (A_1 sing) $\times \mathbb{P}^1$).

$\Rightarrow \varphi : X \xrightarrow{\nu} Z/\langle \iota_Z \rangle \xrightarrow{p_2} E/\langle -1_E \rangle = \mathbb{P}^1$

is a K3 fibered C.Y. 3-fold.

f_Z induces $f_X \in \mathrm{Aut}(X/\mathbb{P}^1)$

with $d_1(f_X) = d_1(f_Z) = d_1(f_S) > 1$.

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[dr, "\varphi"'] & X \arrow[d, "\varphi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathbb{P}^1 \end{tikzcd}$$

Rem

X : Calabi-Yau 3-folds studied by Borcea, Voisin (in conn. with classical mirror symmetry)

$$S^{\iota} = \underbrace{\mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1}_{\ell}$$

$\Rightarrow \chi_{top}(X) = 12\ell$ so that

X is <u>not</u> deformation equiv. if ℓ is different.

$\Rightarrow$ many K3 fibered C.Y. 3-folds $X \xrightarrow{\varphi} \mathbb{P}^1$ with $f \in \mathrm{Aut}(X/\mathbb{P}^1)$ with $d_1(f) > 1$.

Ex 8 (Schoen Type)

$S_i \xrightarrow[\varphi_i]{} \mathbb{P}^1$: rel. min. rational ell. surface with section

$(i=1,2)$

s.t. $\mathrm{Crit}(\varphi_1) \cap \mathrm{Crit}(\varphi_2) = \phi$.

$$X := S_1 \times_{\mathbb{P}^1} S_2 \xrightarrow{\varphi} \mathbb{P}^1.$$

$\Rightarrow$ X: smooth C.Y. 3-fold

φ: abelian fibration

$X_t = (S_1)_t \times (S_2)_t$ $(\forall t \in \mathbb{P}^1)$

$\chi_{top}(X) = 0$.

$\mathrm{Aut}(S_1/\mathbb{P}^1) \times \mathrm{Aut}(S_2/\mathbb{P}^1) \curvearrowright X$

cf Gachet-Lin-Wang
ArXiv: 2210.02779

Special Case (isotrivial case)

$(S_1)_t \cong (S_2)_t \cong E$ for general $t \in \mathbb{P}^1$

$\Rightarrow$ $\forall$ general $t \in \mathbb{P}^1$, $X_t \cong E \times E$ & has an automorphism $g_t = \begin{pmatrix} 2 & 1 \\ 1 & \underset{1}{0} \end{pmatrix}$ with $d_1(g_t) > 1$. (Ex 1)

Rem If $\{g_t\}$ would extend to ,

$g \in \mathrm{Bir}(X/\mathbb{P}^1)$, then

$g \in \mathrm{Aut}(X/\mathbb{P}^1)$ $\left(\begin{array}{c} \nexists \text{ rigid } \mathbb{P}^1 \\ \text{in fibers of} \\ X \to \mathbb{P}^1 \end{array}\right)$

& $d_1(g) = d_1(g_t) > 1$.

But it will turn out that this is not the case.

Ex9 (Calabi-Beauville Type)

$E_\omega := \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\omega \qquad \omega = \frac{-1+\sqrt{-3}}{2}$

$\Rightarrow X_3 := \left(\begin{matrix}\text{blow up at}\\ \text{27 sing pts of}\\ \text{type } \frac{1}{3}(1,1,1)\end{matrix}\right) \xrightarrow{\varphi_3} \overline{X}_3 := \dfrac{E_\omega^3}{\left\langle\begin{pmatrix}\omega & & 0\\ & \omega & \\ 0 & & \omega\end{pmatrix}\right\rangle}$

Calabi-Yau 3-fold ($\rho(X_3)=36$, $h^{2,1}(X_3)=0$)
rigid

$$\begin{tikzcd} X_3 \arrow[r, "\varphi_3"] \arrow[rr, bend right, "p_3"'] & \overline{X}_3 \arrow[r, "pr_3"] & \mathbb{P}^1 = \frac{E_\omega}{\langle\omega\rangle} \end{tikzcd}$$

abelian fibration &

$g = \begin{pmatrix}2 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in \mathrm{Aut}(E_\omega^3)$ induces

$\overline{g} \in \mathrm{Aut}(X_3/\mathbb{P}^1)$ with Ex1

$d_1(\overline{g}) = d_1(g) = d_1\left(\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\right) > 1.$

Main Thm (precise form / in preparation) [10

Let $\varphi : X \to \mathbb{P}^1$ be an abelian fibered CY 3-fold.

Assume $\exists\, g \in \mathrm{Aut}(X \xrightarrow{\varphi} \mathbb{P}^1)$ s.t. $d_1(g) > 1$. Then:

$(X \xrightarrow{\varphi} \mathbb{P}^1) \simeq (X_3 \xrightarrow{p_3} \mathbb{P}^1)$ in Ex9.

Rem $\chi_{top}(X \text{ in Ex8}) = 0 \neq \chi_{top}(X_3)$

So Rem in Ex8 follows.

Rem X_3 is also an example of Calabi-Yau 3-fold with a primitive automorphism of positive entropy (O- & Truong)

§4 (Maximal) c_2-contraction

(after [OS]: O– & Sakurai: ArXiv 9909175)

Def X : Calabi-Yau 3-fold / $\mathbb{C}$

$\varphi : X \twoheadrightarrow W$, W : normal proj
φ : conn. fiber

is c_2-contraction if

$\varphi^* H . c_2(X) = 0$ for $\exists H$ ample on W

(hence $\varphi^* N^1(W) . c_2(X) = 0$ & $\varphi = \Phi_{|m\varphi^* H|}$ $\exists m > 0$)

c_2-contraction $\varphi_0 : X \to W_0$ is maximal c_2-contraction

$\overset{\text{def}}{\iff}$ $\forall \varphi : X \to W$ c_2-contraction

$$\begin{tikzcd}
X \arrow[r, "\varphi_0"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & W_0 \arrow[dl, "\exists \pi"] \\
W &
\end{tikzcd}$$

Rem

(1) $c_2(X) > 0$ on $\mathrm{Amp}(X) \setminus \{0\}$

(hence) $c_2(X) \geq 0$ on $\overline{\mathrm{Amp}}(X)$.

[Miyaoka-Yau]

(2) $\forall X$ $\exists$ max c_2-contr $\varphi : X \to W_0$ (possibly $W_0 = pt$) & $\varphi : X \to W_0$ is unique in the sense that if $\varphi' : X \to W_0'$ is also max c_2-contraction, then

$$\begin{tikzcd}
& X \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] \arrow[d, phantom, "\circlearrowleft"] & \\
W_0 \arrow[rr, "\simeq", "\exists \tau"'] & & W_0'
\end{tikzcd}$$

hence

$\forall f \in \mathrm{Aut}(X)$ $\exists f_{W_0} \in \mathrm{Aut}(W_0)$ s.t.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\varphi"] \\
W_0 \arrow[r, "f_W"] & W_0
\end{tikzcd}$$

.

$\boxed{\text{Thm A}}$ ([OS] & references therein)

X : Calabi-Yau 3-fold. Then:

(1) c_2-contraction $X \to \overline{X}$ with $\dim \overline{X} = 3$ is either one of

(1-1) $\varphi_3 : X_3 \to \overline{X}_3$ (in Ex9)

(1-2) $\varphi_7 : X_7 \to \overline{X}_7$ unique proj. crepant resolution of $\overline{X}_7 = A_7/\langle g_7 \rangle$:

$$\begin{cases} A_7 : \mathrm{Jac}(C := (XY^3 + YZ^3 + ZX^3 = 0)) \\ g_7 \in \mathrm{Aut}\,A_7 \text{ induced from } \begin{pmatrix} \zeta_7 & & 0 \\ & \zeta_7^2 & \\ 0 & & \zeta_7^4 \end{pmatrix} \in \mathrm{Aut}\,C \end{cases}$$

(Hence both are necessarily maximal)

Rem ([OS]) (i) X_7 has no abelian fib.

(ii) ∀ abelian fibration on X_3 is isomorphic to $X_3 \xrightarrow{p_3} \mathbb{P}^1$ in Ex9.

(2) c_2-contraction $X \to W$ with $\dim W = 2$ is of the following form (cf. Ex7):

$$\begin{cases} {}^\exists S : \text{K3 or ab. surf} \ \& \ {}^\exists E : \text{ell. curve} \\ {}^\exists G < \mathrm{Aut}(S \times E) \text{ finite, Gorenstein, diagonal} \\ \exists\, S \to S' \text{ birat contraction} \end{cases}$$

s.t. $G \subset \mathrm{Aut}(S')$ & $W = S'/G$ is a rat. surf. with RDP sing & $K_W \equiv 0$ and $X \to W$ is a flop $/W$ of a crepant projective resolution $Y \to S \times E/G$:

$$\begin{tikzcd} X \arrow[rr] & & W \arrow[d, equal] \\ Y \arrow[u, dashed, "\text{flops}/W"] \arrow[urr] \arrow[r] & S \times E/G \arrow[r] & S'/G \end{tikzcd}$$ □

Rem We may choose by Bridgeland-King-Reid:

$Y = G\text{-Hilb}(S \times E)$

$:= \{ Z \in \mathrm{Hilb}^{|G|}(S \times E) \mid GZ = Z \ \& \ G \curvearrowright H^0(\mathcal{O}_Z) \text{ is a regular representation} \}$

Proof of Main Thm

K_X nef $\Rightarrow$ $\varphi : X \to \mathbb{P}^1$ has $\exists 3$ sing. fibers (Viehweg-Zuo)

$\Rightarrow$ replacing g by g^n ($\exists n>0$), we may assume that $g \in \mathrm{Aut}(X/\mathbb{P}^1)$

For $d_1(g) =: \delta_1 > 1$,
$\exists D_1 \in \mathrm{Nef}(X) \setminus \{0\}$ s.t. $g^* D_1 = \delta D_1$
by Birkhoff-Perron-Frobenius
applied for $\overline{\mathrm{Amp}}(X) \subset N^1(X)_{\mathbb{R}}$:

V : real vect. sp. $\dim V < \infty$ &
$C \subset V$ closed strictly convex cone
s.t. $\mathrm{Int}(C) \neq \phi$.

Let $\Phi \in GL(V)$ s.t. $\Phi(C) = C$.
$\Rightarrow \exists v \in C \setminus \{0\}$ s.t. $\Phi(v) = \underbrace{\rho(\Phi)}_{\text{spectral radius of } \Phi} v$.

Let A be a general fiber of φ
$\Rightarrow g(A) = A$ (also as a variety)

Claim1 $D_1|_A \neq 0$ in $N^1(A)_{\mathbb{R}}$ & $\delta_1 =$ Salem #.

pf of Claim1 H : ample smooth div. on X.

If $D_1|_A = 0$, then $(A|_H . D_1|_H) = (D_1|_A . H|_A) = 0$.
$A|_H$ & $D_1|_H$ nef $\Rightarrow A|_H \backsim D_1|_H$ Hodge index Thm on H
$\Rightarrow A \backsim D_1$ Lefschetz Thm
But $g^*A = A$, $g^*D_1 = \delta_1 D_1$, $\delta_1 \neq 1$ a contradiction.

Then $g_A \in \mathrm{Aut}\, A$ &
$g_A^*(D_1|_A) = \delta_1 D_1|_A$ ($D_1|_A \neq 0$, $\delta_1 > 1$)
Hence δ_1 is a Salem # by
McMullen's Thm. $\square$

Then $\delta_2 := d_1(g^{-1}) > 1$ as well (as the Salem polynomial is reciprocal)

Hence, applying Claim 1 for g^{-1}, we find:

$\exists D_2 \in \overline{\mathrm{Nef}}(X) \setminus \{0\}$, $(g^{-1})^* D_2 = \delta_2 D_2$

$\delta_2 > 1$, $D_2|A \not\equiv 0$.

We have now:

$g^*A = A$, $g^*D_1 = \delta_1 D_1$, $g^*D_2 = \delta_2^{-1} D_2$

$0 < \delta_2^{-1} < 1 < \delta_1$.

& all A, D_1, D_2 are nef & linearly independent in $N^1(X)_{\mathbb{R}}$.

Claim 2

(1) $(A . D_1 . D_2) > 0$.

(2) X has a nef and big divisor E such that $(E . c_2(X)) = 0$.

Claim 2 $\Rightarrow$ Main Thm

Pf) Claim 2 + (base point freeness for $E = K_X + E$.)

$\Rightarrow \Phi_{|mE|} : X \to \overline{X}$ is a c_2-contraction with $\dim \overline{X} = 3$. & X has $\exists$ ab. fib

$\overset{\text{Thm A}}{\Rightarrow} (\Phi_{|mE|} : X \to \overline{X}) \cong (\varphi_3 : X_3 \to \overline{X_3})$ & $X \cong X_3$. (+ Rem (ii)) □

Pf of Claim 2 $D_1|A \neq 0$, $D_2|A \neq 0$

If $(A.D_1.D_2)=0$, then

$(D_1|A . D_2|A)=0$ & $D_1|A$, $D_2|A$ nef

Hence $D_1|A \sim D_2|A$. ✗ to

$$\left\{ \begin{array}{l} g_A^*(D_1|A) = \delta_1(D_1|A) \quad g_A^*(D_2|A) = \delta_2^{-1}(D_2|A) \\ \delta_1 \neq \delta_2^{-1}. \end{array} \right.$$

Hence $(AD_1D_2)>0$ (as $\forall$nef).

Note : by $\delta_1>1$, $0<\delta_2^{-1}<1$,

$(D_1^3)=(D_2^3)=0$, $(D_1^2A)=(D_2^2A)=0$, $A^2=0$

$(A.c_2(X))=(D_1.c_2(X))=(D_2.c_2(X))=0$

(eg. $(D_1 c_2(X)) = (g^*D_1 . g^*c_2(X)) = (\delta_1 D_1 . c_2(X))$
& $\delta_1 \neq 1 \Rightarrow (D_1 . c_2(X)) = 0$ etc)

Hence $(xA+yD_1+zD_2)^3 = 6xyzAD_1D_2$

& $E'' := x_0A + y_0D_1 + z_0D_2$
$(x_0>0, y_0>0, z_0>0)$ is a nef & big $\mathbb{R}$-divisor. (by $(AD_1D_2)>0$).

By Kawamata, $\overline{\mathrm{Amp}}(X)$ is locally rat. polyhedral in $\mathrm{Big}(X)$.

Hence so is near $E'' \in \overline{\mathrm{Amp}}(X)$.

Moreover $(E''.c_2(X))=0$ &
$c_2(X)^\perp = ((*.c_2(X))=0)$ is a rational hyperplane bounding $\overline{\mathrm{Amp}}(X)$.

Hence $\overline{\mathrm{Amp}}(X) \cap c_2(X)^\perp$ is also locally rat. polyhedral cone near E'' in $\mathrm{Big}(X) \cap c_2(X)^\perp$.

Thus, $\exists E'$ near E'' which is nef & big $\mathbb{Q}$-divisor class with $(E'.c_2(X))=0$. Then multiply E'. □

Remarks

My original motivation of Main Thm: finiteness problem of real forms of an abelian fibered CY 3-fold

cf Dinh-Gachét-Lin-O_ -Wang-Yu (ArXiv 2210.04760)

[Cone Conjecture ⇒ finiteness of real forms]

Q1 $X \not\equiv X_3$.

Is $\mathrm{Aut}(X/\mathbb{P}^1)$ virtually abelian group for an abelian fibered CY 3-fold $\varphi : X \to \mathbb{P}^1$?

cf. Dinh-Lin-O_ -Zhang (ArXiv 1810.04827)

Q2 $\varphi : X \to \mathbb{P}^1$ K3 fibered CY 3-fold with $g \in \mathrm{Aut}(X/\mathbb{P}^1)$ with $d_1(g) > 1$.

Is X "close to" CY 3-fold with C_2-contr. to a surface?

Thank you very much for your attension!