Mori fibred Calabi-Yau pairs

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Mori fibred CY pairs

(+ C. Araujo, A. Massarenti) / $\mathbb{C}$

<u>CY pair</u> (X, D)

D reduced divisor $\sim -K_X$.

[also fix ω rational differential s.t. $D + \mathrm{div}_X \omega = 0$]

<u>singularities</u> X terminal & (X, D) log canonical

<u>Mori fibred</u> $f: X \to Z$ Mori fibration ($-K_X$ f-ample $\rho(X/Z) = 1$ &c.)

(X, D) (X', D')

$t: X \dashrightarrow X'$ is <u>volume preserving</u>

if $t^*(\omega') = \omega$

<u>Why, Alessio?</u> It makes sense. (relevant for mirror symmetry)

Work with C. Araujo & A. Massarenti

<u>Theorem A</u> (X, D)

X $\mathbb{Q}$-Fano $(X = \mathbb{P}^n)$

(X, D) terminal

$\mathrm{Cl}\, D = \mathbb{Z} A$ $\left(\exists f \in \mathbb{N}_+ \; -K_X = fA\right)$

$\Rightarrow \mathrm{Bir}(X; D) = \mathrm{Aut}(X; D)$

[more strongly: $(X/_{\operatorname{Spec}\mathbb{C}}, D)$ is birationally super-rigid]

COMMENTS

(X, D) terminal:

$\forall E$ $\operatorname{codim}_Z X \leq 2 \Rightarrow (X, D, Z)$ smooth

$\geq 3 \quad = a(E, K_X + D) > 0.$

Ex. X smooth 4-fold ($\mathbb{P}^4$) & D has nodes $xy+zt=0$

Ex. $\mathbb{P}^3$, $D_4 \subset \mathbb{P}^3$ smooth

non generic $\rho(D_4) \geq 2$
then Thm does not hold
(Oguiso after Gizatullin ...)

$\mathbb{P}^4$, D_5 containing a $\mathbb{P}^2$
then does not hold

Birationally rigid

$$\mathcal{P}((X/Z, D)) = \left\{ \begin{matrix} \text{Mf CY pairs} \\ (Y/T, D^*) \end{matrix} \;\middle|\; \begin{matrix} \exists \text{ volume preserving} \\ (X, D) \dashrightarrow Y, D^* \end{matrix} \right\} \Big/ \text{Sq}$$

in the theorem A

$$\mathcal{P}(X/Z, D) = \{(X/_{\operatorname{Spec}\mathbb{C}}, D)\}$$

[the proof is not difficult
in the appropriate sense (X, D)

is a "minimal model" & therefore it is unique.]

<u>Theorem B</u> $(\mathbb{P}^3, D_4)$ Assume:

- $\exists_1$ A_1 - singularity $z \in D$
- $Cl(D) = \mathbb{Z}[\mathcal{O}_D(1)]$.

Then: $Bir(\mathbb{P}^3, D) = G \rtimes \mu_2$
where G is the form of $\mathbb{G}_m / \mathbb{C}(x,y)$

D_η D is a 2-to-1 cover of $\mathbb{P}^2$
$\downarrow$ $\rightsquigarrow$ form of $\mathbb{G}_m$ to
$\eta \in \mathbb{P}^2$ degree 2 field extension
$\mathbb{C}(x,y) \subset k(D)$.

[we can <u>also</u> say that
$$\mathcal{P}(\mathbb{P}^3, D) = \left\{ (\mathbb{P}^3, D) \; ; \; \begin{matrix} (\tilde{\mathbb{P}}^3, \tilde{D}) \\ \downarrow \\ \mathbb{P}^2 \end{matrix} \right.$$
]

<u>Remark</u>

$$\left\{ \begin{matrix} \text{volume preserving} \\ \varphi : (\mathbb{P}^3, D) \dashrightarrow (\mathbb{P}^3, D) \end{matrix} \right\}$$

$$\boxed{D \subset \mathbb{P}^1_{k(x,y)}} \quad = \left\{ t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \;\middle|\; t(D) \subseteq D \right\}$$

$Aut(\mathbb{P}^1_{k(x,y)}, D_k$

$k(x,y)$

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* The proof is not too difficult but note this:

* there are several ways of embedding $\tilde{D}$ = min resolution of D_4 in a Fano 3-fold. $\rho(\tilde{D})=2$

$$\mathrm{NE}\,\tilde{D} = \langle E, E' \rangle_+ \qquad \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}$$

$$E^2 = E'^2 = -2 \quad E\cdot E' = 6$$

$$(E+E')^2 = -4+12 = 8$$

$\tilde{D} \subset X(2,2,2)$ complete inters.

the Theorem states in particular that $(X_{2,2,2}, \tilde{D})$ is not birational to $(\mathbb{P}^3, D)$

$(2E+E')$

<u>Theorem C</u> $(\mathbb{P}^3, D)$ Assume:

- $\exists_1$ A_2-singularity $z \in D$

$\mathcal{Cl}\, D = \mathbb{Z}\, \mathcal{O}_D(1)$

<u>THEN</u>:

- Description of $\mathrm{Bir}(\mathbb{P}^3, D)$ same as for Theorem B

• $\mathbb{P}(\mathbb{P}^3/\mathrm{Spec}\,\mathbb{C}, D)$ is given in the following Table:

thmc

Table 1: Mf CY pairs $(X^\dagger, D^\dagger)$ birational to $(\mathbb{P}^3, D_4)$ where $D_4 \subset \mathbb{P}^3$ is a quartic surface with one A_2-singularity.

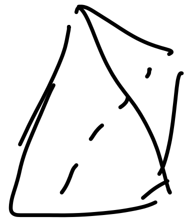
Object	Ambient	Ambient coords. & wts.	Equation of $X^\dagger$	Equation of $D^\dagger$
1	$\mathbb{P}^3$	$\begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ \hline 1 & 1 & 1 & 1 \end{array}$	0	$x_0x_1x_3^2 + Bx_3 + C$
2	$\mathbb{F}_1^3$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x \\ \hline 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}$	0	$x_0x_1x_3^2 + Bx_3x + Cx^2$
$2^\sharp$	$\mathbb{F}_2^3$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x \\ \hline 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}$	0	$x_0x_3^2 + Bx_3x + x_1Cx^2$
$2^\flat$	$\mathbb{F}_2^3$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x \\ \hline 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}$	0	$x_1x_3^2 + Bx_3x + x_0Cx^2$
3^a	$\mathbb{P}(1^3, 2)$	$\begin{array}{cccc} x_0 & x_1 & x_2 & y \\ \hline 1 & 1 & 1 & 2 \end{array}$	0	$x_0y^2 + By + x_1C$
3^b	$\mathbb{P}(1^3, 2)$	$\begin{array}{cccc} x_0 & x_1 & x_2 & y \\ \hline 1 & 1 & 1 & 2 \end{array}$	0	$x_1y^2 + By + x_0C$
4	$X_4 \subset \mathbb{P}(1^3, 2^2)$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & y_0 & y_1 \\ \hline 1 & 1 & 1 & 2 & 2 \end{array}$	$y_0y_1 + C - L(x_0y_1 - x_1y_0 - B)$	$x_0y_1 - x_1y_0 - B$
5	$X_4 \subset \mathbb{P}(1^4, 2)$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & y \\ \hline 1 & 1 & 1 & 1 & 2 \end{array}$	$y(y+Q) - C + x_3\big((x_0+x_1)y + x_1Q + B\big)$	$y + x_1x_3$
$5^\flat$	$X_4 \subset \mathbb{P}(1^4, 2)$	$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & y \\ \hline 1 & 1 & 1 & 1 & 2 \end{array}$	$y(y+Q) - C + x_3\big((x_0+x_1)y + x_0Q + B\big)$	$y + x_0x_3$

this was quite a bit of work to prove.

[illegible]

Where does this end :

$\mathbb{P}^3, (x_0 x_1 x_2 x_3) = 0$



<u>Conjecture</u> :

$Bir(\mathbb{P}^3, D) = Bir\left(\mathbb{T}^3_{x_1, x_2, x_3} ; \Omega \right.$ $\left(\frac{1}{2\pi i}\right)^3 \frac{dx_1}{x_1} \frac{d}{-}$

is generated by $SL_3(\mathbb{Z})$

& $\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \dashrightarrow \begin{matrix} x_1 \\ x_2 \\ x_3 A(x_1, x_2) \end{matrix}$

$\hookrightarrow k[x_1^{\pm 1}, x_2$

& I this maybe doable.

For del Pezzo surfaces

<u>conjecture</u>

$\left\{ \begin{matrix} \text{log} \\ \text{del Pezzo} \\ \text{surfaces} \end{matrix} \right\}$ s.t. $\exists$ toric degeneration / deformation $= \left\{ \begin{matrix} \text{lattice} \\ \text{plane } \\ \text{polygons} \end{matrix} \right.$ /

$\left\{ \begin{matrix} \text{deformations of} \\ X \\ \text{spanning fan}(P) \end{matrix} \right\} \longleftrightarrow \{(P, *$