Gushel-Mukai varieties with many symmetries and an explicit irrational Gushel-Mukai threefold

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Fano varieties

X threefold $\quad \mathrm{Pic}(X) = \mathbb{Z}K_X \quad (-K_X)^3 = 10$

genus 6

$X = \mathrm{Gr}(2, V_5) \cap \mathbb{P}^7 \cap \text{Quadric} \subseteq \mathbb{P}(\Lambda^2 V_5)$

GM threefold

→ All unirational

→ A general one is irrational : by a degeneration argument, the theta divisor of its int. Jacobian is not singular enough to be Jacobian of curve.

We construct an irrational GM threefold $X / \mathbb{Q}$

Jac(X) has "too many automorphisms"

↳ 10 dim'l ppav

acted on by $G := PSL(2, \mathbb{F}_{11})$ order 660

§2. EPW sextics and GM varieties

2.1 EPW sextics

V_6 , $\Lambda^3 V_6$, $\wedge$ symplectic form

$A \subset \Lambda^3 V_6$ Lagrangian

$$Y_A^{\geq \ell} = \{ [x] \in \mathbb{P}(V_6) \mid \dim (A \cap (x \wedge \Lambda^2 V_6)) \geq \ell \}$$

Lagrangian degeneracy loci

O'Grady

When A is "general" :
quasi-smooth

- $Y_A^{\geq 3} = \emptyset$
- $A \not\ni x \wedge y \wedge z$

- $Y_A := Y_A^{\geq 1}$ sextic hypersurface
- $\mathrm{Sing}(Y_A) = Y_A^{\geq 2}$ smooth surface
- $\tilde{Y}_A \to Y_A$ double cover branched over $Y_A^{\geq 2}$

  smooth HK fourfold of $K3^{[2]}$-type

2.2 (Ordinary) GM varieties

$$X = Gr(2, V_5) \cap \mathbb{P}^{n+4} \cap \text{Quadric} \subseteq \mathbb{P}(\Lambda^2 V_5)$$

$n = \dim(X)$
$n \in \{3, 4, 5\}$
$\mathrm{Pic}(X) \simeq \mathbb{Z}H$
$K_X = -(n-2)H$
coindex 3

There is a nontrivial connection between EPW and GM (Iliev-Manivel, D-Kuznetsov)

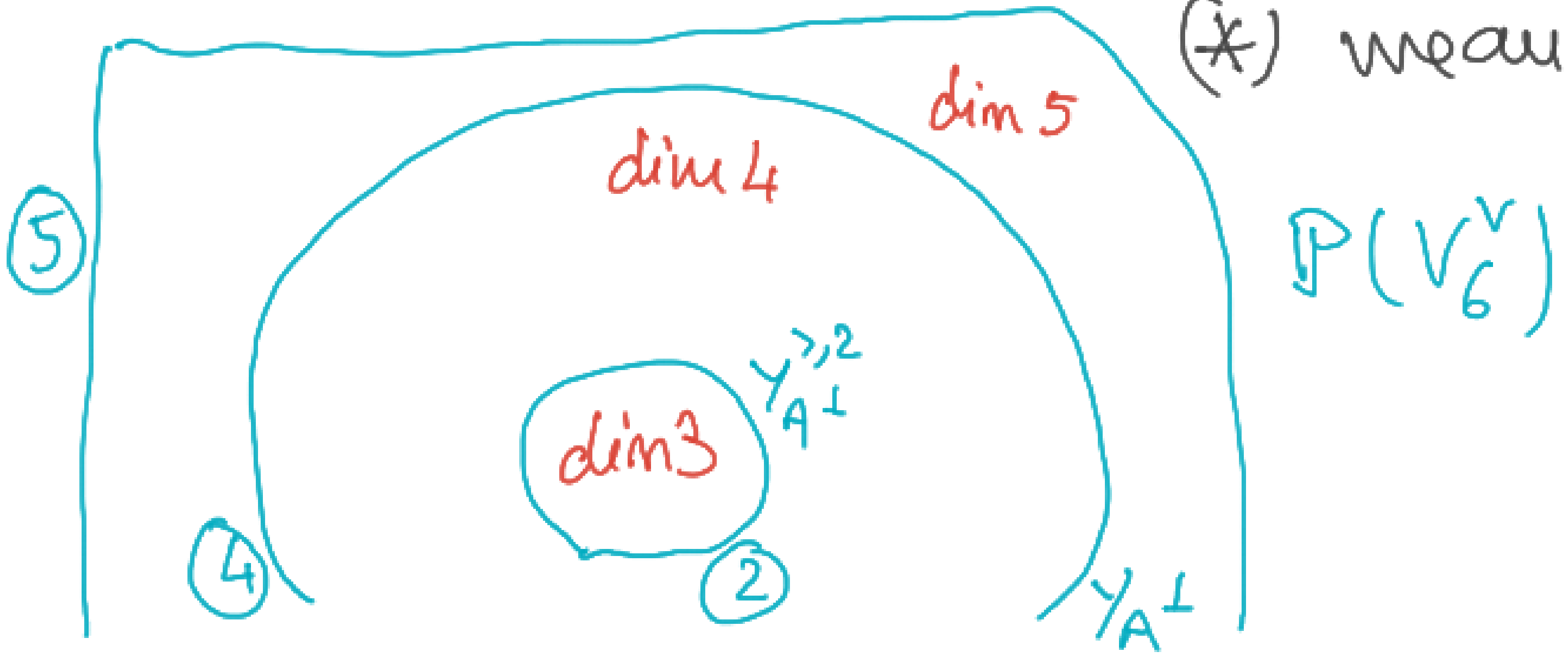
$$\left\{ \begin{array}{c} \text{EPW} \quad Y_A \subseteq \mathbb{P}(V_6) \\ + \quad \text{hyperplane} \quad V_5 \subset V_6 \end{array} \right\} / \text{isom} \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{c} \text{GM varieties} \\ \text{of dim } n \end{array} \right\} / \text{isom.}$$

(*) $n = 5 - \dim(A \cap \Lambda^3 V_5)$

$A \subset \Lambda^3 V_6$ Lagrangian "quasi-smooth" $\longleftrightarrow$ $A^\perp \subset \Lambda^3 V_6^\vee$ dual Lagrangian

$Y_{A^\perp}^{\geq \ell} \subseteq \mathbb{P}(V_6^\vee)$

(*) means $[V_5] \in Y_{A^\perp}^{5-n}$



In terms of moduli spaces

GM_5 (dim 25) $\xrightarrow[\mathbb{P}^5 \setminus Y_{A^\perp}]{\text{fibers}}$ (double) EPW sextics (dim 20)

GM_4 (dim 24) $\xrightarrow[Y_{A^\perp} \setminus Y_{A^\perp}^{\geq 2}]{\text{fibers}}$ (double) EPW sextics

GM_3 (dim 22) $\xrightarrow[Y_{A^\perp}^{\geq 2}]{\text{fibers}}$ (double) EPW sextics

2.3. The Mongardi Lagrangian

Aim: construct A s.t. G acts on Y_A

$G := PSL(2, \mathbb{F}_{11})$

V_5 irr. rep'n of dim 5

$Q \subseteq \mathbb{P}(\Lambda^2 V_5)$ unique G-invariant quadric

- $X_5 := Gr(2, V_5) \cap Q \subseteq \mathbb{P}(\Lambda^2 V_5)$ GM 5-fold with $\mathbb{G}$-action

- $V_6 := \underset{\text{trivial}}{V_1} \oplus V_5$

$$\Lambda^3 V_6 = (V_1 \wedge \Lambda^2 V_5) \oplus \Lambda^3 V_5 \quad \text{isom. irr. } \mathbb{G} \text{ rep'ns.}$$

$$(V_1 \wedge \Lambda^2 V_5) \xrightarrow[v]{\sim} \Lambda^3 V_5$$

Mongardi Lagrangian $A =$ graph of $v = \{ e_0 \wedge x + v(x) \mid x \in \Lambda^2 V_5 \}$ $\quad V_1 = \mathbb{C} e_0$

quasi-smooth $\rightsquigarrow Y_A, \widetilde{Y}_A \circlearrowleft \mathbb{G}$

$\longrightarrow$ X_5 is obtained from A and the hyperplane V_5

$\longrightarrow$ if we choose other hyperplanes W_5 we obtain GM varieties of other dimensions and automorphism groups

$$\{ g \in \mathbb{G} \mid g(W_5) = W_5 \}$$

We get a 3-fold X_3 with aut. group $\mathbb{Z}/11\mathbb{Z}$.

We get as automorphism groups D_{12}, D_{10}, $\mathfrak{S}_4$, abelian gps...

<u>Theorem</u> X_3 is irrational.

$X_3 = Gr(2,V_5) \cap \mathbb{P}^7 \cap Q$

<u>Proof</u> We use CG criterion: sufficient to prove that 10 dim'l ppav $(\,Jac(X_3),\ \theta\,)$ is <u>not</u> Jacobian of curve

$\longrightarrow$ prove that theta divisor is not singular enough

$\longrightarrow$ prove it has "too many" automorphisms (Beauville)

$|Aut(X_3)| = 11$ not enough

D-Kuznetsov

$$G \circlearrowright (\,Jac(X_3),\theta) \simeq (\,Alb(\,\tilde{Y}_A^{\geq 2}),\theta) \circlearrowleft G$$

canonical étale double cover of surface $Y_A^{\geq 2}$.

$$\wr\!\wr$$

$$(\,Jac(X_5),\theta) \not\simeq \quad \circlearrowleft G$$

Hurwitz $\cancel{84\,(10-1)}$

But $g=10$

$|Aut\ C| \leq 432$.

∎

§3. An interesting 10 dim'l ppav

contains no elliptic curves

$$(\mathbb{J}, \Theta) := (\mathrm{Jac}(X_5), \Theta) = (\mathrm{Alb}(\tilde{Y}_A^{\geq 2}), \Theta_{\text{induced}})$$

surface of gen'l type

$\circlearrowright G$

- "Analytic representation" $G \to \mathrm{Aut}(T_{\mathbb{J},0}) = \mathbb{C}^{10}$ is $\Lambda^2 V_5$, irreducible, defined over $\mathbb{Q}$ (over $\mathbb{Z}$)

$\rightsquigarrow$ Ekedahl–Serre Lange: $\mathbb{J} \underset{\text{isog}}{\sim} E^{10}$, E elliptic curve

$G \to GL(10, \mathbb{Z}) \rightsquigarrow G$ acts on F^{10} for any elliptic curve F

Question: What is E?

E should be elliptic curve with endo ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-11})}$.

Analogous situation

Klein cubic $\quad W \subseteq \mathbb{P}^4 = \mathbb{P}(V_5)$ $\quad$ defined over $\mathbb{Q}(\sqrt{-11})$

$\circlearrowleft G$

$\mathrm{Jac}(W)$ $\quad$ 5 dim'l ppav $\quad \circlearrowleft G$

$\overset{12}{\simeq}$ isom

E^5 $\quad$ (Adler)

$\hookrightarrow$ curve F mentioned earlier.

av of dim $\frac{p-1}{2}$ with auto of order p:

Thank you!

Questions

$\mathrm{Aut}(X) \leq G$

X_3 is the only GM 3-fold with an aut. of order ≥ 11

Special GM $\rightsquigarrow$ add $\underbrace{\mathbb{Z}/2\mathbb{Z}}$
acts on Jac by -1

One dim'l family of special GM 3-folds

$\mathbb{Z}/2\mathbb{Z} \times \mathfrak{A}_5$

Cheltsov

Cremona groups

$X_3 \underset{\text{bir'l}}{\not\simeq}$ Cubic 3-fold because $\mathrm{Jac}(X_3)$ indec.