AUTOMORPHISMS OF COBLE SURFACES

IGOR V. DOLGACHEV

Let Σ be a finite set of N points (including infinitely near points) in the projective plane $\mathbb{P}^2$ over an algebraically closed field k. One asks about possible structure of the group of biregular automorphisms of the blow-up B_Σ of this set. It was claimed by Coble (II,(7)) and later rigorously proven by Gizatullin and Hirschowitz that for $N \geq 9$ the group is trivial if Σ is general enough. In the case when $N \leq 8$, the blow-up B_Σ is a del Pezzo surface if Σ is general and possible automorphism groups were essentially known since the 19th century in the case when $k = \mathbb{C}$. We still do not know the answer in case $N = 7, 8$ when k is of positive characteristic.

Apparently Arthur Coble was the first to ask in 1917 about special sets Σ of $N \geq 9$ points for which the group $\mathrm{Aut}(B_\Sigma)$ could be an infinite discrete group. He had never used the language of algebraic surfaces and the question was stated in terms of the group of Cremona transformation. He introduced the notion of Cremona congruent ordered sets of points and asked for the existence of ordered set of points such that its congruence equivalence class consists of finitely many projective equivalence classes. One can also state this in terms of the Coble representation of the Weil group $W(E_N)$ in the group of birational automorphisms of $(\mathbb{P}^2)^n/\mathrm{PGL}(3)$ and ask a question about possible periodic orbits of this action. We are not going to say it more precisely.

Coble was also the first to give examples of special sets Σ. The first example is an Halphen set p 9 base points of an Halphen pencil of curves of degree $3m$ with m-multiple points in Σ. By Bertini's Theorem, each pencil whose general member is a curve of geometric genus 1 can be reduced to such a set by a Cremona transformation. The surface B_Σ is a relatively minimal rational elliptic surface and its group of automorphisms (for a general such Σ) contains a subgroup of finite index in the infinite Coxeter group $W(E_9)$, the affine Weil group of $W(E_8)$.

The second series of examples of special sets σ given by Coble is the subject of my talk. Here Σ is a set of 10 double points of a rational plane curve of degree 6. The surface $X = B_\Sigma$ has the property that $|-K_X| = \emptyset$ but $|-2K_X| \neq \emptyset$. This property was taken for a definition of a *Coble surface* in an unpublished work of Miyanishi in 1980. There exists a classification of all projective algebraic surfaces with $|-K_X| = \emptyset$ and $|-mK_X| \neq \emptyset$ for some $m > 1$, the surfaces which we will be interested and referred to as Coble surface are of special kind: they are rational and $|-2K_X|$ consists of an isolated smooth curve (irreducible in the original case considered by Coble). For a general Coble set of points Σ, $\mathrm{Aut}(B_\Sigma)$ is of finite index in $W(E_{10})$. It was proven by Cantat and myself that Halphen or Coble sets are the only ones in zero characteristic for which $\mathrm{Aut}(B_\Sigma)$ could be of finite index in $W(E_N)$ (in characteristic p we can also blow up any general set of $N \geq 10$ points on a cuspidal cubic, the blow-ups are called Harbourne surfaces).

Let $|-2K_X| = \{C\}$ which we assume to be a smooth and hence consists of m connected components $C_1, \ldots, C_n$. The adjunction formula gives that each $C_i \cong \mathbb{P}^1$ and $C_i^2 = -4$. In particular, $K_X^2 = -n$. So, $\#\Sigma = 9 + n$. If the characteristic p is equal to 0, the double cover of X ramified over C is a K3 surface. The covering involution has n smooth rational curves as its set of fixed points and hence, a generic such surface leads to surfaces with a 2-elementary Picard lattice classified by Nikulin. It follows from this classification that there are 10 different irreducible families of dimension $10 - n$

n	N	K_{V}^2	(r,l,δ)	2-elementary lattice M	$N = M^{\perp}$
1	10	-1	$(11,11,1)$	$\mathsf{E}_{10}(2) \oplus \mathsf{A}_1$	$\mathsf{I}^{2,9}(2)$
2	11	-2	$(12,10,1)$	$\mathsf{E}_8(2) \oplus \mathsf{U} \oplus \mathsf{A}^{\oplus 2}$	$\mathsf{I}^{2,8}(2)$
3	12	-3	$(13,9,1)$	$\mathsf{D}_4^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 3} \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,7}(2)$
4	13	-4	$(14,8,1)$	$\mathsf{D}_4^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 4} \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,6}(2)$
5	14	-5	$(15,7,1)$	$\mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 5} \oplus \mathsf{U}$	$\mathsf{I}^{2,5}(2)$
6	15	-6	$(16,6,1)$	$\mathsf{E}_{10} \oplus \mathsf{A}_1^{\oplus 6}$	$\mathsf{I}^{2,4}(2)$
7	16	-7	$(17,5,1)$	$\mathsf{E}_8 \oplus \mathsf{D}_6 \oplus \mathsf{A}_1 \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,3}(2)$
8	17	-8	$(18,4,0)$	$\mathsf{E}_8 \oplus \mathsf{D}_8 \oplus \mathsf{U}(2)$	$\mathsf{U}(2)^{\oplus 2}$
8	17	-8	$(18,4,1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_6 \oplus \mathsf{A}_1^{\oplus 2}$	$\mathsf{I}^{2,2}(2)$
9	18	-9	$(19,3,1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_8 \oplus \mathsf{A}_1$	$\mathsf{I}^{2,1}(2)$
10	19	-10	$(20,2,1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_{10}$	$\langle 2\rangle^{\oplus 2}$

As you see there are 10 families of such surfaces whose general member has the given 2-elementary Picard lattice of the K3 cover. The number of 'boundary components $C_1,\ldots,C_n$ varies from 1 to 10 and the cardinalities of the Coble sets Σ vary from 10 to 19, respectively.

Over $\mathbb{C}$, it follows from the theory of periods of lattice polarized K3 surfaces that Coble surfaces lie in the boundary of the moduli space of Enriques surfaces that complete this moduli space to a quotient of a symmetric Hermitian domain of type IV by an arithmetic discrete group. In any characteristic one can also obtain an Enriques surface as a deformation of $\mathbb{Q}$-Gorenstein surface obtained by blowing down the boundary components $C_1,\ldots,C_n$.

So, being close relatives of Enriques surfaces it is natural extend the wide study of automorphisms of Enriques surface to Coble surfaces.

As is well-known, a useful tool for study the automorphism group of any algebraic surface X is to consider its natural representation

$$\rho : \mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Num}(X))$$

in the orthogonal group of its numerical lattice $\mathrm{Pic}(X)/\mathrm{num}$. For example, for an Enrique surface $\mathrm{Num}(X) \cong E_{10}$ is the unique (up to isometry) unimodular even lattice of signature $(1,9)$. In the case of Coble surfaces $\mathrm{Num}(X) \cong I^{1,9+n}$, where I is a unique odd unimodular lattice of signature

$(1, 9+n)$. In the representation ρ, the image preserves the canonical class and the nef cone $\mathrm{Nef}(X)$. In our case, it is also preserves the numerical classes $\delta_i = [C_i]$ of boundary components. It was suggested by Mukai that a natural target for the representation ρ is the lattice (which I call the Coble-Mukai lattice) defined as follows.

$$\mathrm{CM}(X) = \{x \in \mathrm{Num}(S)+\mathbb{Z}[\frac{1}{2}\delta_1]+\cdots+\mathbb{Z}[\frac{1}{2}\delta_n] \subset \mathrm{Num}(X)_{\mathbb{Q}} : x\cdot\delta_i = 0, i = 1, \ldots, n\}.$$

The lattice $\mathrm{CM}(X)$ contains the classes of smooth rational curves ((-2)-curves) on X but does not contain classes of (-1)-curves. This makes it possible to describe explicitly the intersection $\mathrm{Nef}_{CM}(X)$ of $\mathrm{Nef}(X)$ with $\mathrm{CM}(X)_{\mathbb{R}}$. For any two boundary components C_i, C_j and a (-1)-curve E with $E \cdot C_i = E \cdot C_j = 1$, the class $\alpha = \frac{1}{2}\delta_i + \frac{1}{2}\delta_j + 2E$ belongs to $\mathrm{CM}(X)$. It has self-intersection equal to -2 and intersects K_X with zero. So, it is an analogue of a (-2)-curve. Together with classes of (-2)-curves we call them irreducible effective roots. Now one can describe $\mathrm{Nef}(X)$ as

$$\mathrm{Nef}_{\mathrm{CM}}(X) = \{x \in \mathrm{CM}_{\mathbb{R}} : x^2 \geq 0, x\cdot\alpha \geq 0, \text{ for all irreducible effective roots}\}.$$

Of course, if $n = 1$, we have $\mathrm{CM}(X) = K_X^\perp \cong E_{10}$, so it coincides with the Enriques lattice. It is not obvious, and the proof is not easy that

$$\mathrm{CM}(X) \cong E_{10}$$

always!

Let me start to give examples where the computation of $\mathrm{Aut}(X)$ is known.

Example 0.1. Suppose $n = 1$, so we are dealing with original Coble surfaces. In 1919 Coble himself have shown that the the image of the group of automorphisms of a general Coble surface in $W(E_{10}) \cong \mathrm{O}(E_{10}/(\pm 1)$ is a normal subgroup with quotient isomorphic to the orthogonal group $\mathrm{O}^+(10, \mathbb{F}_2)$ of the even quadratic space of dimension 10 over $\mathbb{F}_2$. It took 60 years to find the same answer for the group of automorphisms of a complex Enriques surface (Barth-Peters and Nikulin). The same result is true for a general Coble surface in any characteristic and a geometric proof applies to both Coble and Enriques surfaces using a lattice theoretical result (claimed by Coble without a rigorous proof) that any isometry of the form $\mathrm{id}_U + (-\mathrm{id}_{E_8})$ defined by an orthogonal sum decomposition $E_{10} = U \perp E_8$ are conjugate in $W(E_{10})$ and generate a normal subgroup $W(E_{10})(2)$ with quotient $\mathrm{O}^+(10, \mathbb{F}_2)$. These involutions can be realized geometrically as the deck transformations of a double plane model of an Enriques or a Coble surface. A general Coble surface is *unnodal*, i.e does not contain irreducible effective roots.

Example 0.2. Let $V = V(F_3)$ be a cubic surface in $\mathbb{P}^3$ over a field k of characteristic $p \neq 2, 3$. Let $H(V)$ be its Hessian quartic surface given by the determinant of the Hessian matrix of the cubic polynomial F_3. We assume that F_3 is Sylvester non-degenerate, i.e. admits an expression as a sum of 5 linear forms. This linear polynomials embed S into $\mathbb{P}^4$ so that equation of the image can be given in the form

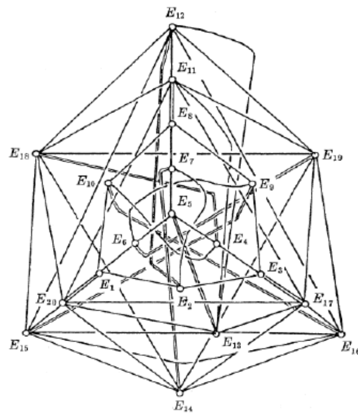
$$\sum_{i=0}^{4} a_i x_i^3 = \sum_{i=0}^{4} x_i = 0.$$

In these coordinates, the equation of the Hessian surface is

$$\sum_{i=0}^{4} \frac{1}{a_i x_i} = \sum_{i=0}^{4} x_i = 0.$$

The Cremona transformation $T : (x_0, \ldots, x_4) \mapsto (1/a_0 x_1, \ldots, 1/a_4 x_4)$ is a birational involution that leaves the Hessian surface invariant. The Hessian surface has at least 10 ordinary nodes P_{ijk} given by the vertices of the Sylvester pentahedron, the union of planes $x_i = x_0 + \cdots + x_4 = 0$. It has 10 lines ℓ_{ij} given by its edges The transformation T extends to a minimal resolution $\widetilde{H(V)}$ of $H(V)$ which is a K3 surface, it exchanges the exceptional curves E_{ijk} over the points P_{ijk} with the proper transforms L_{mn} the edges ℓ_{km} where $\{i, j, k\} \cup \{l.m\} = \{0, \ldots, 4\}$.

If V is a nonsingular cubic surface and $p \neq 5$, T has no fixed points on X' and the quotient by the involution T is an Enriques surface. If V has k nodes (it is known that $k \in \{1, 2, 3, 4\}$) , then these point lie on $H(S)$ and are isolated fixed points of T. The quotient of $\widetilde{H(V)}$ becomes a Coble surface with k boundary components. The nef cone $\mathrm{Nef}_{\mathrm{CM}}(S)$ is a convex cone over the the convex polytope in the Lobachevsky space associated with $\mathrm{CM}(S)_{\mathbb{R}} \cong \mathbb{R}^{1,9}$ with orthonormal vectors of inner product (-2) with incidence matrix defined by the following diagram



U_{01}	U_{02}	U_{03}	U_{04}	U_{12}	U_{13}	U_{14}	U_{23}	U_{24}	U_{34}
E_1	E_8	E_4	E_7	E_5	E_9	E_{10}	E_2	E_3	E_6
α_{01}	α_{02}	α_{03}	α_{04}	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}
E_{20}	E_{11}	E_{18}	E_{14}	E_{13}	E_{15}	E_{16}	E_{12}	E_{17}	E_{19}

The graph looks complicated but if you caefully you recognize that that it is union of two graphs

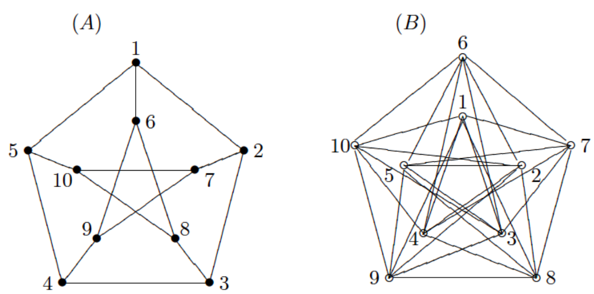


FIGURE 5. Type VI

One is the Petersen graph and another the anti-Petersen graph (the complement of the Petersen graph in the complete graph $K(10)$.

Here the vertices U_{ab} correspond to orbits $\{E_{ijk}, L_{lm}\}$ of T and among E_{ab} there are s effective irreducible roots, where s is the number of Eckardt points on V. For example, if S is a Clebsch diagonal cubic surface with 10 Eckardt point, then all vertices in the diagram are the classes of (-2)-curves, and this implies that $\mathrm{Aut}(S)$ is finite, in fact, isomorphic to $\mathfrak{S}_5 = \mathrm{Aut}(V)$. This is one of the 7 possible types of Enriques surface with finite automorphism group. Another example is when V has 6 Eckardt points when we have a one-dimensional family of such surfaces

$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{t}{x_4} = 0,$$

where $t = 1/16, 1/4$ give Coble surfaces with $4, 1$ boundaries, respectively, and $t = 1$ gives the Enriques-Clebsch surface with automorphism group $\mathfrak{S}_5$. We prove with Daniel Allcock that the automorphism group is the same for all $t \neq 1$ and isomorphic to the group

$$\mathrm{UC}(4) \rtimes \mathfrak{S}_4,$$

where $\mathrm{UC}(4)$ is the free product of 4 groups of order 2.

Since the set of boundary components is invariant with respect to $\mathrm{Aut}(S)$ of a Coble surface, thee are natural questions: Is $\mathrm{Aut}(S) \to \mathrm{Aut}(C)$ injective and what is its image. We assume that $n = 1$, so that $\mathrm{Aut}(C) = \mathrm{PGL}(2)$. Coble conjectured that this homomorphism is injective for a general Coble surface. It is still unknown whether this is true or not.

In the second example of Coble surfaces with $n = 1$ coming from one-nodal cubic surfaces with 6 Eckardt points, we prove that the restriction map is indeed injective and the image is the subgroup of $\mathrm{PGL}(2)$ naturally isomorphic to the group of isometries of 3-dimensional Euclidean space generated by symmetries of a regular tetrahedron and the reflections across its faces. It is not a discrete group in the Lie group of isometries of $\mathbb{R}^3$, however it is isomorphic to a 3-adic lattice in $\mathrm{PGL}_2(\mathbb{Q}_3)$.

Finally let us discuss the problem of classification of Coble surfaces with finite automorphism group. The similar problem was solved for complex surfaces by Kondo and Nikulin in early eighties. Recently, for $p \neq 2$, it was solved by Gebhard Martin, and for $p = 2$ by Katsura-Kondo-Martin. Some of the surfaces from the Kondo-Nikulin list are not realized in other cases, and there are new types in characteristic 2. Recently, yet unpublished, the classification of Coble surfaces with finite automorphisms in characteristic $p \neq 2$ was established by Kondo.

Type	p	k	Aut	R-invariant	Moduli
I	any	12	D_8	$(E_8 \oplus A_1, \{0\})$	$\mathbf{A}^1 \setminus \{0, -2^{10}\}$
II	any	12	$\mathfrak{S}_4$	$(D_9, \{0\})$	$\mathbf{A}^1 \setminus \{0, -2^8\}$
III	any	20	$(\mathbf{Z}/4\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^2) \rtimes D_8$	$(D_8 \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$	unique
IV	any	20	$(\mathbf{Z}/2\mathbf{Z})^4 \rtimes (\mathbf{Z}/5\mathbf{Z} \rtimes \mathbf{Z}/4\mathbf{Z})$	$(D_5^{\oplus 2}, \mathbf{Z}/2\mathbf{Z})$	unique
V	$\neq 3$	20	$\mathfrak{S}_4 \times \mathbf{Z}/2\mathbf{Z}$	$(E_7 \oplus A_2 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$	unique
VI	$\neq 3, 5$	20	$\mathfrak{S}_5$	$(E_6 \oplus A_4, \{0\})$	unique
VII	$\neq 5$	20	$\mathfrak{S}_5$	$(A_9 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$	unique

TABLE 1. Enriques surfaces with finite automorphism group ($p \neq 2$)

In the case $p = 0$, there are three classes of such surfaces, all lie on the boundary of two 1-dimensional families of type I (two surfaces with $n = 1, 2$ and of type II (one with $n = 1$). Some other surfaces occur when we reduce mod p an Enriques surface of type in

Type	p	n	k	Aut	R-invariant
I	any	1	12	D_8	$(E_8 \oplus A_1, \{0\})$
I	any	2	12	$(\mathbf{Z}/2\mathbf{Z})^2$	$(E_8 \oplus A_1^{\oplus 2}, \mathbf{Z}/2\mathbf{Z})$
II	any	1	12	$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	$(D_9, \{0\})$
V	3	2	20	$\mathfrak{S}_3 \times \mathbf{Z}/2\mathbf{Z}$	$(E_7 \oplus A_2 \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$
VI	5	1	20	$\mathfrak{S}_5$	$(E_6 \oplus A_4, \{0\})$
VI	3	5	20	$\mathfrak{S}_5$	$(E_6 \oplus D_5, \mathbf{Z}/2\mathbf{Z})$
VII	5	1	20	$\mathfrak{S}_5$	$(A_9 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$
MI	3	2	40	$\mathrm{Aut}(\mathfrak{S}_6)$	$(A_5^{\oplus 2} \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^3)$
MII	3	8	40	$(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes \mathbf{Z}/2\mathbf{Z}$	$(D_8 \oplus A_2^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$

TABLE 2. Coble surfaces with finite automorphism group ($p \neq 2$)

Example 0.3. Consider the example of an Enriques surface arising from the Hessian surface of Clebsch diagonal cubic surface. When $p = 3$, the standard Cremona involution has 5 fixed points on $H(V)$ with coordinates $(1, 2, 1, 1, 1)$, etc.. the quotient is a Coble surface from the fourth row of the Table. The nef cone has 10 nodes originated by (-2)-curves that form a subdiagram isomorphic to the Petersen graph and ten other nodes originated by other irreducible effective roots that form a subdiagram isomorphic to the anti-Petersen graph (the complement of the Petersen graph in the complete graph $K(10)$).

If $p = 5$, we get one fixed point on $H(V)$ with coordinates $(1,1,1,1,1)$. The quotient of $\widetilde{H(V)}$ by the involution is a Coble surface with one boundary component. It is in Row 5 in the Table.

Here is a new example (Type MI) that is not related to an Enriques surface.

Example 0.4. Let X be the Fermat quartic surface

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0. \tag{0.1}$$

The equation is defined by a Hermitian form over $\mathbb{F}_9$ and hence the unitary group $\mathrm{PGU}(4, \mathbb{F}_9)$ acts on X as projective automorphisms. It is known that X contains 112 lines. Let ℓ, ℓ' be two skew lines on X. Let $p \in X$ not lying on $\ell \cup \ell'$. Then there exists a unique line ℓ'' in $\mathbb{P}^3$ containing p and meeting ℓ, ℓ'. Let $q \in X$ satisfying $\ell'' \cap X = \{p, q, \ell'' \cap \ell, \ell'' \cap \ell'\}$. By associating q with p, we have a birational involution $s_{\ell,\ell'}$ of X which can be extended to a regular automorphism of X by the minimality of K3 surfaces. The fixed point set of $s_{\ell,\ell'}$ is the union of ℓ and ℓ', and the quotient surface V of X by $s_{\ell,\ell'}$ is a Coble surface with two boundary components.

Finally let us give an example in characteristic 2 that does not come as a reduction mod 2 of an Enriques surface.

Example 0.5. let X be a Vinberg most algebraic K3 surface of the first type. Recall that this surface is obtained as the double cover of the blow-up S of 15 intersection points of ten lines on a quintic del Pezzo surface branched along the proper transforms of the lines. The surface S is a Coble surface with 10 boundary components.

The K3 cover X has 25 (-2)-curves with the intersection diagram

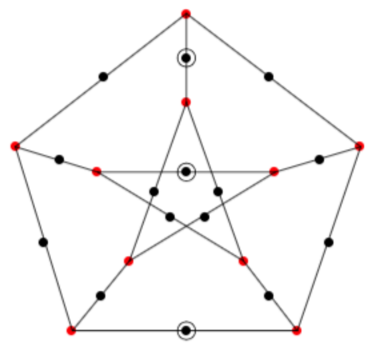


Figure 9.3: Twenty five (-2)-curves on the Vinberg's most algebraic K3 surface

It has also 5 non-effective classes $c_1, \ldots, c_5$ of square-norm -4. The reflections with respect to the classes of the 25 (-2)-curves, 5 classes c_i and the deck transformations σ of $X \to S$ generate $\mathrm{Aut}(X)$ which is isomorphic to a central extension of the group

$$\mathrm{UC}(5) \rtimes \mathfrak{S}_5.$$

by (σ). So the above group is the group of automorphisms of the Coble surface S. The boundary of its nef cone are hyperplanes orthogonal to 25 classes of (-2)-curves from the following diagram (the pre-images of the exceptional curves of the blow-up and the ramification curves of the double cover). There are also 5 hyperplanes orthogonal to some non-effective classes c_i of norm-square -2. In characteristic 2, these classes become effective, and the subgroup $\mathrm{UC}(5)$ disappears and $\mathrm{Aut}(S)$ becomes isomorphic to $\mathfrak{S}_5$.