Rationally connected rational double covers of primitive Fano varieties

Aleksandr V. Pukhlikov

arXiv:1910.08975

University of Liverpool

22 December 2020

[We are working over the field $\mathbb{C}$]

The **unirationality problem:** for a given rationally connected variety X of dimension M, is there a rational dominant map

$$\mathbb{P}^M \overset{d:1}{\dashrightarrow} X?$$

The problem is very old and absolutely open: there are no known examples of non-unirational rationally connected varieties.

Some remarks:

- G.Fano in

  Fano G., Sulle varietà algebriche a tre dimensioni aventi tutti i generi nulli, Atti del Congresso Internazionale dei Matematici, Bologna, 3-10 Settembre 1928, Zanichelli Bologna (1931) 115–121,

  insisted that a typical conic bundle over a rational surface is non-unirational.

- V.A.Iskovskikh in

  Iskovskikh V.A., Birational automorphisms of three-dimensional algebraic varieties, J. Soviet Math. **13** (1980), 815–868,

  mentioned the unirationality problem several times:

— in the Introduction he mentions that the problem is open for *general* three-dimensional non-singular quartics and *any non-singular* sextic double solids;

— in Chapter III, §2 he gives the explicit constructions of unirationality of some special non-singular quartics $V_4 \subset \mathbb{P}^4$ (due to Segre), of double quadrics of index 1 (due to Roth), of complete intersections $V_{2\cdot 3} \subset \mathbb{P}^5$ (due to Enriques);

— he completes §3 of Chapter III (and the chapter) by the following conjecture for standard conic bundles $V \to \mathbb{P}^2$: if the degree of the discriminant curve $\subset \mathbb{P}^2$ is sufficiently high, then V is non-unirational.

- There are plenty of explicit constructions of unirationality; e.g. a non-singular (in fact, with the singular locus of a sufficiently high codimension) hypersurface of a fixed degree m in $\mathbb{P}^N$ is unirational for N sufficiently large, see, for instance,

  Conte, A., Murre, J. P. On a theorem of Morin on the unirationality of the quartic fivefold, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **132** (1998), 49–59.

  Harris, J., Mazur, B., Pandharipande, R., Hypersurfaces of low degree, Duke Math. J. **95** (1998), no. 1, 125–160.

Consider non-singular hypersurfaces of index 1, $V = V_{M+1} \subset \mathbb{P}^{M+1}$:

- the expected answer is NO, at least for a Zariski general variety;
- in

  Kollár, J., Low degree polynomial equations: arithmetic, geometry and topology, European Congress of Mathematics, Vol. I (Budapest, 1996), 255–288, Progr. Math., **168**, Birkhäuser, Basel, 1998.

  János Kollár suggested that on V there are no rational surfaces through a general point $p \in V$, which would imply non-unirationality;

- which we can re-formulate as follows: the space of irreducible rational curves of a given degree on V through the general point p does not contain a rational curve; this conjecture motivated a deep study of spaces of rational curves on Fano varieties, e.g.

  Beheshti, R., Starr, J. M., Rational surfaces in index-one Fano hypersurfaces. J. Algebraic Geom. **17** (2008), no. 2, 255-274,

  Beheshti, R., Kumar, N. M., Spaces of rational curves on complete intersections, Compos. Math. **149** (2013), no. 6, 1041-1060.

  and other papers authored/co-authored by R. Beheshti;

- but the unirationality problem is still open.

Main Theorem (of this talk). *For a Zariski general non-singular hypersurface* $V = V_{M+1} \subset \mathbb{P}^{M+1}$ *of degree* $M+1$*, where* $M \geqslant 5$*, there are no non-trivial rational Galois covers* $X \overset{d:1}{\dashrightarrow} V$ *with an abelian Galois group of order* $d \geqslant 2$*, where* X *is a rationally connected variety; in particular, there are no rational maps* $X \dashrightarrow V$ *of degree 2 with* X *rationally connected.*

(Here "Galois cover" means that $\mathbb{C}(V) \subset \mathbb{C}(X)$ is a Galois extension.) The theorem motivates

Conjecture (on absolute rigidity of hypersurfaces). *For a Zariski general non-singular hypersurface* $V = V_{M+1} \subset \mathbb{P}^{M+1}$ *of degree* $M+1$*, where* $M \geqslant 5$*, every rational dominant map* $X \dashrightarrow V$*, where* X *is a rationally connected variety of dimension* $\dim V$*, is a birational map.*

"Zariski general" has a very precise meaning in the Main Theorem above: let V be a projective factorial variety with at most terminal singularities, such that $\mathrm{Pic}\, V = \mathbb{Z}K_V$ and the anticanonical class $(-K_V)$ is ample (that is, a primitive Fano variety). Then the Fano variety V is *divisorially canonical* if for every effective divisor $D \sim -nK_V$, $n \geqslant 1$, the pair $(V, \frac{1}{n}D)$ is canonical; that is to say, for every exceptional prime divisor E over V the inequality

$$\mathrm{ord}_E\, D \leqslant n \cdot a(E),$$

where $a(E)$ is the discrepancy of E with respect to V, holds. (That is to say, the global canonical threshold of V is $\geqslant 1$.)

Theorem. (P., Izvestiya: Mathematics, 2005.) *A Zariski general non-singular hypersurface* $V = V_{M+1} \subset \mathbb{P}^{M+1}$ *is divisorially canonical for* $M \geqslant 5$.

In the Main Theorem above “Zariski general” means “divisorially canonical”.

The Main Theorem can be generalized as follows: the claim is true for any divisorially canonical primitive Fano variety satisfying some additional technical condition (which is very easy to check for hypersurfaces).

Similarly, the conjecture on absolute rigidity of hypersurfaces generalizes to

Conjecture (on absolute rigidity). *If* V *is a divisorially canonical Fano variety, then every rational dominant map* $X \dashrightarrow V$*, where* X *is a rationally connected variety of dimension* $\dim V$*, is a birational map.*

Proof. Since the image of a rationally connected variety is rationally connected, we may assume that the Galois group of the original extension $\mathbb{C}(V) \subset \mathbb{C}(X)$ is a cyclic group of a prime order $p \geqslant 2$. We will consider the case $p = 2$.

Assume the converse: there is a rational map

$$X \overset{2:1}{\dashrightarrow} V$$

with X RC. Desingularizing, we may assume that X is non-singular and

$$\sigma \colon X \to V$$

is a morphism.

A toy example: why there are no *double covers* (in the usual sense) $\sigma: X \to V$, branched over a non-singular divisor $W \subset V$ with X RC?

Because the anticanonical class of V is too small: say if $W \sim nH$, where H is the hyperplane section of V, then we get

$$K_X = \sigma^* K_V + \frac{1}{2}\sigma^* W = \left(-1 + \frac{n}{2}\right) \sigma^* H.$$

As X must be RC, we get $n \leqslant 1$ so σ is either unramified or branched over a hyperplane section; both cases clearly impossible.

Note: for hypersurfaces of higher index this construction works.

We will keep this toy example in mind.

We say that a family $\mathcal{L}$ of irreducible projective curves on a quasi-projective variety is *free*, if they sweep out a dense subset of that variety and for every subvariety Y of codimension $\geqslant 2$ the subset

$$\{L \in \mathcal{L} \,|\, L \cap Y \neq \emptyset\}$$

is a proper closed subfamily of the family $\mathcal{L}$ (that is to say, a curve $L \in \mathcal{L}$ of general position does not intersect Y). Let us fix a free family $\mathcal{C}_X$ of non-singular rational curves on X.

Our reference for free families of curves (existence and properties) is

Kollár J., Rational curves on algebraic varieties. Springer-Verlag, Berlin, 1996, Sections II.3 and IV.3.

More precisely, we have

$$f\colon \mathbb{P}^1 \to C_X \subset X$$

an isomorphism onto the image with f^*T_X ample $\cong \oplus\, \mathcal{O}_{\mathbb{P}^1}(\alpha_i)$ with all $\alpha_i \geqslant 1$.

So deformations of f are unobstructed and we can deform C_X

- at any point $p \in C_X$ in any direction,
- at any $p \neq q$ on C_X in independent directions,

so away from any $Y \subset X$ of codimension 2 or higher.

Using this principle, we get that we can assume in addition, that

- for every prime divisor $\Delta \subset X$, such that $\sigma_* \colon T_p X \to T_{\sigma(p)} V$ is not an isomorphism for a point of general position $p \in \Delta$ (this is true, in particular, if $\operatorname{codim}(\sigma(\Delta) \subset V) \geqslant 2$), a general curve $C_X \in \mathcal{C}_X$ meets Δ transversally at points of general position,
- and for a general curve $C_X \in \mathcal{C}_X$ the morphism

$$\sigma|_{C_X} \colon C_X \to \sigma(C_X)$$

  is birational.

If necessary, we can shrink the family $\mathcal{C}_X$ removing any proper closed subsets.

Now let $\mathcal{C}_V = \sigma_* \mathcal{C}_X$ be the image of that family on V. The family $\mathcal{C}_V$ is, generally speaking, not free: if the σ-image of a prime divisor $\Delta \subset X$ is of codimension $\geqslant 2$, then the general curve $C_V \in \mathcal{C}_V$ meets $\sigma(\Delta)$.

Technical fact 1. *There is a birational morphism $\varphi\colon V^+ \to V$, where V^+ is a non-singular projective variety, such that the strict transform $\mathcal{C}_V^+$ of the family $\mathcal{C}_V$ on V^+ is a free family of curves.*

Proof. We blow up subvarieties of codimension $\geqslant 2$ intersecting all curves in the family: first points, then curves, etc. $\square$

Technical fact 2. *There is a non-singular quasi-projective variety U_X, a birational map $\varphi_X\colon U_X \dashrightarrow X$ and a Zariski open subset $U \subset V^+$, such that:*

(i) *the rational map*

$$\sigma_* = \varphi^{-1} \circ \sigma \circ \varphi_X\colon U_X \dashrightarrow V^+$$

extends to a morphism $\sigma_U\colon U_X \to V^+$, the image of which is U,

(ii) *the inequality*

$$\operatorname{codim}\left((V^+ \setminus U) \subset V^+\right) \geqslant 2$$

holds,

(iii) *the map $\sigma_U\colon U_X \to U$ is a double cover of U, branched over a non-singular hypersurface $W \subset U$.*

Essentially we remove some subsets of codimension $\geqslant 2$ from V^+ so that over the complement we get a double cover in the usual sense, see the following commutative diagram:

$$\begin{array}{ccccc} U_X & & \overset{\varphi_X}{\dashrightarrow} & & X \\ \sigma_U \downarrow & & & & \downarrow \sigma \\ U & \subset & V^+ & \overset{\varphi}{\to} & V. \end{array}$$

Proof. The field extension $\mathbb{C}(V) \subset \mathbb{C}(X)$ is generated by some element $\xi \in \mathbb{C}(X)$, satisfying the equation

$$\xi^2 - q = 0$$

for some rational function $q \in \mathbb{C}(V) = \mathbb{C}(V^+)$. Using this fact of elementary algebra, we can (birationally) realize X as a hypersurface in $V^+ \times \mathbb{P}^1$, covering V^+, given by an equation, quadratic in the coordinates on $\mathbb{P}^1$.

Then we construct a locally trivial $\mathbb{P}^1$-bundle $\mathcal{X}$ over an open set $U \subset V^+$ such that $\operatorname{codim}(V^+ \setminus U) \geqslant 2$ and X is birational to a non-singular hypersurface $U_X \subset \mathcal{X}$ which is a double cover of U branched over a non-singular (possibly reducible) hypersurface in U. $\square$

(We can remove any closed subset of codimension $\geqslant 2$ in V^+!)

Now we can complete the proof of Main Theorem. A general curve $C \in \mathcal{C}_V^+$ does not meet the closed set $V^+ \setminus U$ and for that reason is contained entirely in U. Therefore,

$$\sigma_U^{-1}(C) = C_1 \cup C_2$$

is a union of 2 distinct (projective!) rational curves on the quasi-projective variety U_X. The curves C_1 and C_2 are permuted by the Galois involution and move in families of irreducible rational curves sweeping out U_X, so that

$$(C_i \cdot K_U) < 0,$$

$i = 1, 2$, where K_U is the canonical class of the variety U_X.

Now

$$K_U = \sigma_U^* \left(K_{V^+} + \frac{1}{2} W \right),$$

where $W \subset U$ is a non-singular hypersurface, over which the double cover σ_U is branched. For the canonical class of V^+ we have the presentation

$$K_{V^+} = -\varphi^* H + \sum_{i \in I} a_i E_i$$

where $E_i \subset V^+$ are all the prime φ-exceptional divisors and $a_i > 0$ are their discrepancies with respect to V (and $H = -K_V$ is the hyperplane section, generating $\operatorname{Pic} V$).

Let us look at the **branch divisor** W: collecting separately the components of the hypersurface W, which are divisorial on V and φ-exceptional, write

$$W = W_{\mathrm{div}} + W_{\mathrm{exc}},$$

where $W_{\mathrm{div}} = nH - \sum_{i\in I} b_i E_i$ with $n \geqslant 1$ and $b_i \in \mathbb{Z}_+$ and $W_{\mathrm{exc}} = \sum_{i\in I} c_i E_i$ with $c_i \in \{0, 1\}$. We get the inequality

$$(C_i \cdot K_U) = (C \cdot K_{V^+}) + \frac{1}{2}(C \cdot W) < 0.$$

(Recall the toy example!)

Assume that $n \geqslant 2$: adding $[\frac{1}{2}n-1](C\cdot K_{V^+})\leqslant 0$, we get

$$n(C\cdot K_V^+)+(C\cdot W)=\left(C\cdot\sum_{i\in I}(na_i-b_i+c_i)E_i\right)<0,$$

so that for some $i\in I$ we have

$$b_i>na_i+c_i\geqslant n\cdot a_i$$

and the pair $(V,\frac{1}{n}\varphi_*W_{\mathrm{div}})$ with $\varphi_*W_{\mathrm{div}}\sim nH$ is not canonical, which contradicts the divisorial canonicity of the variety V.

Therefore, $n = 1$ or 0. Both cases are easy to exclude: if $n = 1$, then $\varphi_* W_{\mathrm{div}}$ is a hyperplane section of V, and it is easy to find a non-singular curve $N \subset V$ of odd degree such that N meets this section transversally outside the closed set

$$\varphi(V^+ \setminus U) \cup \varphi\left(\bigcup_{i \in I} E_i\right)$$

of codimension $\geqslant 2$. Then

$$\sigma^{-1}(N) \to N$$

is a double cover of a non-singular curve, branched over an odd number of points, impossible.

If $n=0$, we can find a non-singular rational curve on V that does not meet the same closed set

$$\varphi(V^+\setminus U)\cup\varphi\left(\bigcup_{i\in I}E_i\right)$$

of codimension $\geqslant 2$, and we get a non-ramified double cover of a non-singular rational curve, also impossible.

The proof of Main Theorem is complete.

Divisorial canonicity has been shown for general members of many families of Fano varieties, including mildly singular.

There are other important applications of the divisorial canonicity.

One example is given in (P., Izvestiya: Mathematics, 2005): if $F_1, \dots, F_k$ are divisorially canonical primitive Fano varieties, then every structure of a RC fibre space on the direct product

$$F_1 \times F_2 \times \dots \times F_k$$

is a projection onto a direct factor (a product of some F_i's).

The other application is for Fano-Mori fibre spaces $\pi\colon V \to S$ over a non-singular RC base S such that

— *every* fibre is a divisorially canonical Fano hypersurface with at most quadratic singularities of rank $\geqslant 8$, satisfying a few additional conditions of general position and such that

— V/S satisfies a global numerial condition similar to Sarkisov condition for conic bundles or the K^2-condition for fibrations over $\mathbb{P}^1$, see [P., Izvestiya: Mathematics, 2015].

Then every birational map $\chi\colon V \dashrightarrow V'$ onto the total space V' of a RC fibre space $\pi'\colon V' \to S'$ (both S' and a general fibre are RC) is fibre-wise (compatible with the projections, that is to say, the fibres of π are mapped into the fibres of π').

Example. $S = \mathbb{P}^m$ with $m \leqslant \frac{1}{2}(M-6)(M-5)-6$ and V is a Zariski general hypersurface of bidegree $(M+1, l)$ in $\mathbb{P}^{M+1} \times \mathbb{P}^m$ with

$$l \geqslant \frac{M+1}{M}(m+1),$$

then $V/\mathbb{P}^m$ satisfies all assumptions above so every structure of a RC fibre space on V factors through the projection $V \to \mathbb{P}^m$. (Note that if $l \leqslant m$, then $V \to \mathbb{P}^{M+1}$ is a structure of a Fano-Mori fibre space not compatible with π, so that the condition for l is close to an optimal one.)

Thank you!