Openness of projectivity

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Main theorem I

Theorem

$g : X \to \mathbb{D}$ proper, flat morphism of complex analytic spaces. Assume that

1. *X_0 is projective,*
2. *X_s has rational singularities for $s \neq 0$, and*
3. *g is bimeromorphic to projective $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$.*

Then g is projective over a smaller punctured disc $\mathbb{D}^{\circ}_{\epsilon} \subset \mathbb{D}$.

— X_0 may have bad singularities,
— K3 surfaces show: (3) is needed,
— (2) is also needed (see later)
— puncture is needed (see later).

Previously known

Kodaira-Spencer, 1958:
X_0 smooth $\Rightarrow$ X_0 Kähler $\Rightarrow$ X_s Kähler

Moishezon, 1966
Kähler$+$ bimeromorphic to projective $\Rightarrow$ projective.

Kollár-Mori, 1992
proof for X_0 terminal, $\mathbb{Q}$-factorial, 3-fold

Villalobos-Paz, 2021
X_0 terminal, $\mathbb{Q}$-factorial, n-fold

Example

$g : X \to \mathbb{D}$ flat, proper morphism, such that,

1. X_0 projective, 1 quotient sing. and ample K_{X_0},
2. X_s is smooth, non-algebraic, $\kappa(X_0) = 0$ for $s \neq 0$.

– Start: K3 surface $Y_0 \subset \mathbb{P}^3$ with a hyperplane section
  $C_0 \subset Y_0$: rational curve with 3 nodes.

– Blow up the nodes and contract C_0' to get X_0.
  singular point: $\mathbb{C}^2/\frac{1}{8}(1,1)$.

– Check: $K_{X_0} \sim E_1 + E_2 + E_3$ (exceptional curves) is ample.

– Deform Y_0 and the 3 points:
  $X_s = Y_s$ is (birationally) a K3. Can be non-algebraic.

Example (Atiyah, 1958)

$g : X \to \mathbb{C}$ smooth, proper morphism, such that,

1. all fibers projective surfaces
2. g is not projective.

- $S_0 := (g = 0) \subset \mathbb{P}^3_x$ and $S_1 := (f = 0) \subset \mathbb{P}^3_x$ same degree.
- S_0 has only ordinary nodes, not in S_1
- S_1 is smooth, $\mathrm{Pic}(S_1) = \mathbb{Z}$
- Consider $X_m := (g - t^m f = 0) \subset \mathbb{P}^3_x \times \mathbb{C}^1_t$.
- Singularties: locally analytically: $xy + z^2 - t^m = 0$.

Claims:

1. X_m is bimeromorphic to a proper, smooth family of projective surfaces iff m is even, but
2. X_m is **not** bimeromorphic to a smooth, projective family of surfaces.

Example

$g : X \to S$ flat, proper

1. fibers normal surfaces, trivial K
2. projective fibers: $\cup_{i=1}^{\infty} H_i \subset S$.

– Start: $E \subset \mathbb{P}^2$ smooth cubic.
– Blow up $m \geq 10$ points $p_i \in E$ and contract E'.
– $S = E^m \setminus \text{(diagonals)}$.
– projective $\Leftrightarrow \exists\ n_i > 0$ such that $\sum_i n_i[p_i] \sim \mathcal{O}(n)|_E$

Note: For $m = 12$: singularities = cones over plane cubics.

Main theorem II

Theorem

$g : X \to S$ proper morphism of complex analytic spaces
$S^* \subset S$ dense, Zariski open, g flat over it. Assume:

1. *X_0 is projective for some $0 \in S$,*
2. *X_s has rational singularities for $s \in S^*$, and*
3. *g is bimeromorphic to a projective morphism.*

Then there are

4. *Zariski open $0 \in U \subset S$,*
5. *locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$*

such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \quad \text{is projective.}$$

Note: g need not be flat.

Open questions

Question

$g : X \to \mathbb{D}$ proper morphism of complex analytic spaces.
What if every fiber is bimeromorphic to a projective variety?

Question

What happens in positive characteristic?

Plan of proof of Main Theorem I

- shrink $\mathbb{D}$: X retracts to X_0.
- X_0 projective, L ample: lift $c_1(L)$ to $\Theta \in H^2(X, \mathbb{Q})$
- Warning: not the Chern class of a holomorphic line bundle.

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Step 1. Fix s very general. For every $p_s \in C_s \subset X_s$
$\exists\ p_0 \in C_0 \subset X_0$ such that $\mathrm{mult}_{p_0} C_0 \geq \mathrm{mult}_{p_s} C_s$.

$\Rightarrow \Theta \cap [C_s] = \Theta \cap [C_0] \geq \epsilon \cdot \mathrm{mult}_{p_0} C_0 \geq \epsilon \cdot \mathrm{mult}_{p_s} C_s.$

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Step 2. Seshadri's criterion for cohomology classes:
$\Rightarrow X_s$ is projective.

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$\Rightarrow X_s$ is projective.

Step 3. Baire category argument: go from
very general fibers to smaller disc.

Step 2 – Seshadri's criterion for cohomology classes

Theorem

X proper algebraic space over $\mathbb{C}$, rational singularities.
X is projective $\Leftrightarrow \exists\, \Theta \in H^2(X, \mathbb{Q})$ and $\epsilon > 0$ such that

$$\Theta \cap [C] \geq \epsilon \cdot \operatorname{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$.

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Proof:

- $\exists$ injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X, \mathbb{Q})$ (proof on next slide)
- $\Theta \cap \in N_1(X, \mathbb{Q})^\vee = N^1(X, \mathbb{Q})$.
- There is L such that $c_1(L) \cap = \Theta \cap$ (in $N_1(X, \mathbb{Q})^\vee$ only!)
- L ample by Seshadri.

$N_1(X,\mathbb{Q}) \hookrightarrow H_2(X,\mathbb{Q})$ smooth, projective case

- natural surjection: $H_2^{\mathrm{alg}}(X,\mathbb{Q}) \twoheadrightarrow N_1(X,\mathbb{Q})$

  Need: $Z \overset{\text{num}}{=} 0$ then $Z \overset{\text{hom}}{=} 0$.
- X smooth, projective: Lefschetz $(1,1)$:

  $\operatorname{rank} N_1(X,\mathbb{Q}) = \operatorname{rank} N^1(X,\mathbb{Q}) = \operatorname{rank} H^2_{\mathrm{alg}}(X,\mathbb{Q})$.
- Hard Lefschetz: $\operatorname{rank} H^2_{\mathrm{alg}}(X,\mathbb{Q}) = \operatorname{rank} H_2^{\mathrm{alg}}(X,\mathbb{Q})$.

$N_1(X,\mathbb{Q}) \hookrightarrow H_2(X,\mathbb{Q})$ singular case

Need: $Z \overset{\text{num}}{=} 0$ then $Z \overset{\text{hom}}{=} 0$.

- $\pi : Y \to X$ projective resolution. Lift Z to Z_Y ($\mathbb{Q}$-coeffs).

  If $Z_Y \in N_1(Y/X,\mathbb{Q})$ then $Z \overset{\text{hom}}{=} 0$.
  Otherwise: $\exists\ L$ that is 0 on $N_1(Y/X,\mathbb{Q})$ but $(L \cdot Z_Y) \neq 0$.

- **Lemma.** If $R^1\pi_*\mathcal{O}_Y = 0$ then L^m descends to X.

Proof: Euclidean local on X:

assume $\pi : V \to U \ni 0$ contractible.

exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_V \to \mathcal{O}_V^\times \to 1$

push forward: $R^1\pi_*\mathcal{O}_V \to \mathrm{Pic}(V) \to R^2\pi_*\mathbb{Z}_V$

$\mathrm{Pic}(V) \hookrightarrow H^2(V_0,\mathbb{Z})$.

Step 1 – Chow variety

$Chow_1^m(X/S)$: $\{p \in Z \subset X : \mathrm{mult}_p Z = m\}$.

Lemma

$g : X \to S$ proper morphism, bimeromorphic to a projective. There are countably many diagrams

$$\begin{array}{ccc} C_i & \to & W_i \times_S X \\ w_i \downarrow\uparrow \sigma_i & & \\ W_i & & \end{array}$$

2. *the w_i proper, rel. dim. 1, flat over $W_i^\circ \subset W_i$,*
3. *the fiber C_p over $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,*
4. *W_i are irreducible, $\pi_i : W_i \to S$ projective, and*
5. *informally: $\cup_i W_i^\circ \hookrightarrow Chow_1^m(X/S)$.*

$H_i := \pi_i(W_i) \subset S$: nowhere dense images.

Claim. X_s is projective for
$s \in S \setminus \cup_{i \in I} H_i$: G_δ-set (= second category)

Proof. Pick $p_s \in C_s \subset X_s$. It is part of a

$$\begin{array}{ccc} C_i & \to & W_i \times_S X \\ w_i \downarrow\uparrow \sigma_i & & \\ W_i & & \end{array}$$

where $W_i \to S$ is dominant. So

$$(p_s \in C_s \subset X_s) \stackrel{\text{specializes}}{\longrightarrow} (p_0 \in C_0 \subset X_0) \text{ and}$$

$\text{mult}_{p_0} C_0 \geq \text{mult}_{p_s} C_s$. □

Step 3 – projective fibers I

Lemma

$g : X \to S$ proper morphism of analytic spaces. Then

1. *either X is locally projective over a Z-dense, open S°,*
2. *or $\{s \in S : X_s \text{ projective}\} \subset F_\sigma$ (= first category).*

Note: F_σ: **locally** a countable union of Z-closed.

Example. image of line $L \subset \mathbb{C}^2$ in $\mathbb{C}^2/\mathbb{Z}^4$.

Step 3 – projective fibers II

- Zariski open + universal cover:
  $R^2 g_* \mathcal{O}_X$ is locally free and $R^2 g_* \mathbb{Z}_X$ is constant.
- exponential sequence $\Rightarrow$
  $\mathrm{Pic}(X)/\mathrm{Pic}^\circ(X) = \ker\big[\partial : R^2 g_* \mathbb{Z}_X \to R^2 g_* \mathcal{O}_X\big]$ and
  $\mathrm{Pic}(X_s)/\mathrm{Pic}^\circ(X_s) = \ker\big[\partial : R^2 g_* \mathbb{Z}_X|_s \to R^2 g_* \mathcal{O}_X|_s\big]$.

Case 1. Some $L \in \mathrm{Pic}(X)$ ample on some X_s: $\Rightarrow$ (1).

Case 2. Ampleness comes from points where $\partial\Theta = 0$ for some $\Theta \in \Gamma(S, R^2 g_* \mathbb{Z}_X)$: $\Rightarrow$ (2).

Baire category

If $\{H_i : i \in I\}$ and $\{G_j : j \in J\}$ are countably many nowhere dense subset of S then

$$S \setminus \cup_{i \in I} H_i \not\subset \cup_{j \in J} G_j.$$