



# Finite groups of birational transformations

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**Abstract.** We survey new results on finite groups of birational transformations of algebraic varieties.

## 1. Introduction

We work over a field  $\mathbb{k}$  of characteristic 0. Typically, unless otherwise mentioned, we assume that  $\mathbb{k}$  is algebraically closed. The *Cremona group*  $\mathrm{Cr}_n(\mathbb{k})$  of rank  $n$  is the group of  $\mathbb{k}$ -automorphisms of the field  $\mathbb{k}(x_1, \dots, x_n)$  of rational functions in  $n$  independent variables. Equivalently,  $\mathrm{Cr}_n(\mathbb{k})$  can be viewed as the group of birational transformations of the projective space  $\mathbb{P}^n$ . It is easy to show that for  $n = 1$ , the group  $\mathrm{Cr}_n(\mathbb{k})$  consists of linear projective transformations:

$$\mathrm{Cr}_1(\mathbb{k}) = \mathrm{PGL}_2(\mathbb{k}).$$

On the other hand, for  $n \geq 2$ , the group  $\mathrm{Cr}_n(\mathbb{k})$  has an extremely complicated structure. In particular, it contains linear algebraic subgroups of arbitrary dimension and has a lot of normal non-algebraic subgroups [18, 24]. We refer to [3, 22, 23, 38, 48, 95] for surveys, historical résumés, and introductions to the subject.

**Examples.** (i) Any matrix  $A = \|a_{i,j}\| \in \mathrm{GL}_n(\mathbb{Z})$  defines an element  $\varphi_A \in \mathrm{Cr}_n(\mathbb{k})$  via the following action on  $\mathbb{k}(x_1, \dots, x_n)$ :

$$\varphi_A : x_i \mapsto x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}.$$

Such Cremona transformations are called *monomial*. For  $n = 2$  and  $A = -\mathrm{id}$ , the transformation  $\varphi_A$  is known as the *standard quadratic involution*

$$(x_1, x_2) \mapsto (x_1^{-1}, x_2^{-1}).$$

(ii) Let  $S$  be an algebraic variety admitting a generically finite rational map

$$\pi : S \dashrightarrow \mathbb{P}^{n-1}$$

of degree 2. In an affine piece and suitable coordinates,  $S$  can be given by the equation  $y^2 = f(x_1, \dots, x_{n-1})$ . One can associate with  $(S, \pi)$  an involution  $\tau \in \text{Cr}_n(\mathbb{k})$  acting on  $\mathbb{k}(x_1, \dots, x_{n-1}, y)$  via

$$\tau : (x_1, \dots, x_{n-1}, y) \mapsto (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}) \cdot y^{-1}).$$

If  $n = 2$  and  $S$  is a hyperelliptic curve, then  $\tau$  is known as the *de Jonquières involution*.

The study of the Cremona group has a very long history. Basically, it was started in earlier works of A. Cayley and L. Cremona, and since then, this group has been the object of many studies. In these notes, we concentrate on the following particular problem.

**Problem 1.1.** Describe the structure of finite subgroups of  $\text{Cr}_n(\mathbb{k})$ .

Note, however, that the projective space is not an exceptional variety from the algebro-geometric point of view. So one can ask a similar question replacing  $\text{Cr}_n(\mathbb{k})$  with the group of birational transformations  $\text{Bir}(X)$  of an arbitrary algebraic variety  $X$ . Hence it is natural to pose the following problem.

**Problem 1.2.** Describe the structure of finite subgroups of  $\text{Bir}(X)$ , where  $X$  is an algebraic variety.

We deal with the most recent results related to these problems. Definitely, our survey is not exhaustive.

## 2. Equivariant minimal model program

In this section, we collect basic facts on the so-called  $G$ -minimal model program (abbreviated as  $G$ -MMP). This program is the main tool in the study of finite groups of birational transformations. For a detailed exposition, we refer to [89].

Let  $G$  be a *finite* group. Following Yu. Manin [68], we say that an algebraic variety  $X$  is a  $G$ -variety if it is equipped with a regular faithful action  $G \curvearrowright X$ , i.e., if there exists an injective homomorphism  $\alpha : G \hookrightarrow \text{Aut}(X)$ . A morphism (resp. rational map)  $f : X \rightarrow Y$  of  $G$ -varieties is a  $G$ -morphism (resp.  $G$ -map) if there exists a group automorphism  $\varphi : G \rightarrow G$  such that, for any  $g \in G$ ,

$$f \circ \alpha(g) = \beta(\varphi(g)) \circ f,$$

where  $\alpha : G \hookrightarrow \text{Aut}(X)$  and  $\beta : G \hookrightarrow \text{Aut}(Y)$  are the embeddings corresponding to the actions  $G \curvearrowright X$  and  $G \curvearrowright Y$ , respectively.

For any  $G$ -variety  $X$ , the action  $G \curvearrowright X$  induces an embedding  $G \hookrightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$  to the automorphism group of the field of rational functions  $\mathbb{k}(X)$ . Conversely, given any finitely generated extension  $\mathbb{K}/\mathbb{k}$  and any finite subgroup  $G \subset \text{Aut}_{\mathbb{k}}(\mathbb{K})$ , there exists a  $G$ -variety  $X$  and an isomorphism  $\mathbb{k}(X) \simeq_{\mathbb{k}} \mathbb{K}$  inducing  $G \subset \text{Aut}_{\mathbb{k}}(\mathbb{K})$ . Thus, we have the following fact.

**Proposition 2.1.** *Let  $\mathbb{K}/\mathbb{k}$  be finitely generated field extension. Then there exists a 1-1 correspondence between finite subgroups  $G \subset \text{Aut}_{\mathbb{k}}(\mathbb{K})$  considered modulo conjugacy and  $G$ -varieties  $X$  such that  $\mathbb{k}(X) \simeq_{\mathbb{k}} \mathbb{K}$  considered modulo  $G$ -birational equivalence.*

Recall that a variety  $X$  is said to be *rational* if it is birationally equivalent to the projective space  $\mathbb{P}^n$  or, equivalently, if the field extension  $\mathbb{k}(X)/\mathbb{k}$  is purely transcendental.

**Corollary.** *There exists a 1-1 correspondence between finite subgroups  $G \subset \text{Cr}_n(\mathbb{k})$  considered modulo conjugacy and rational  $G$ -varieties  $X$  such that  $\mathbb{k}(X) \simeq_{\mathbb{k}} \mathbb{K}$  considered modulo  $G$ -birational equivalence.*

Next, due to the equivariant resolution theorem (see e.g. [1]), it is possible to replace  $X$  with a smooth projective model.

**Proposition 2.2** (see, e.g., [89, Lemma 14.1.1]). *For any  $G$ -variety  $X$ , there exists a smooth projective  $G$ -variety  $Y$  that is  $G$ -birationally equivalent to  $X$ .*

Thus the above considerations allow us to reduce the problem of classification of finite subgroups of  $\text{Bir}(X)$  to the study of subgroups in  $\text{Aut}(Y)$ , where  $Y$  is a smooth projective variety. The main difficulty arising here is that this  $G$ -variety  $Y$  is not unique in its  $G$ -birational equivalence class. So, given  $G$ -birational equivalence class of algebraic  $G$ -varieties, we need to choose some good representative in it. This can be done by means of the  $G$ -MMP. The higher-dimensional MMP forces us to consider varieties with certain very mild, so-called terminal singularities.

**Definition.** A normal variety  $X$  has *terminal singularities* if some multiple  $mK_X$  of the canonical Weil divisor  $K_X$  is Cartier, and for any birational morphism  $f : Y \rightarrow X$ , one can write

$$mK_Y = f^*mK_X + \sum a_i E_i,$$

where  $E_i$  are all the exceptional divisors and  $a_i > 0$  for all  $i$ . The smallest positive  $m$  such that  $mK_X$  is Cartier is called the *Gorenstein index* of  $X$ .

**Definition.** A  $G$ -variety  $X$  has  $G\mathbb{Q}$ -factorial singularities if a multiple of any  $G$ -invariant Weil divisor on  $X$  is Cartier.

It is important to note that terminal singularities lie in codimension  $\geq 3$ . In particular, terminal surface singularities are smooth.

**Example** ([72, 93]). Let the cyclic group  $\mu_r$  act on  $\mathbb{A}^4$  diagonally via

$$(x_1, x_2, x_3, x_4) \mapsto (\zeta x_1, \zeta^{-1} x_2, \zeta^a x_3, x_4), \quad \zeta = \zeta_r = \exp(2\pi i / r), \quad \gcd(a, r) = 1.$$

Then for a polynomial  $f(u, v)$ , the singularity of the quotient

$$\{x_1 x_2 + f(x_3^r, x_4) = 0\} / \mu_r$$

at 0 is terminal whenever it is isolated.

The aim of the  $G$ -MMP is to replace a  $G$ -variety with another one, which is “minimal” in some sense. As we mentioned above, running the  $G$ -MMP we have to consider singular varieties, and the class of terminal  $G\mathbb{Q}$ -factorial singularities is the smallest class that is closed under the  $G$ -MMP.

**Definition** (For simplicity, we assume that  $\mathbb{k}$  is uncountable). A variety  $X$  is *uniruled* if for a general point  $x \in X$ , there exists a rational curve  $C \subset X$  passing through  $x$ . A variety  $X$  is *rationally connected* if two general points  $x_1, x_2 \in X$  can be connected by a rational curve.

Note that a rationally connected surface is rational, and an uniruled surface is birationally equivalent to  $C \times \mathbb{P}^1$ , where  $C$  is a curve.

**Definition.** Let  $Y$  be a  $G$ -variety with only terminal  $G\mathbb{Q}$ -factorial singularities and let  $f : Y \rightarrow Z$  be a  $G$ -equivariant morphism with connected fibers to a lower-dimensional variety  $Z$ , where the action of  $G$  on  $Z$  is not necessarily faithful. Then  $f$  is called  *$G$ -Mori fiber space* (abbreviated as  $G$ -Mfs) if the anti-canonical class  $-K_Y$  is  $f$ -ample and  $\mathrm{rk} \mathrm{Pic}(Y/Z)^G = 1$ . If  $Z$  is a point, then  $-K_Y$  is ample, and  $Y$  is called  *$G\mathbb{Q}$ -Fano variety*. Two-dimensional  $G\mathbb{Q}$ -Fano varieties are traditionally called  *$G$ -del Pezzo surfaces*.

**Definition.** A  $G$ -variety  $Y$  is said to be a  *$G$ -minimal model* if it has only terminal  $G\mathbb{Q}$ -factorial singularities and the canonical class  $K_Y$  is numerically effective (nef).

It is not difficult to show that the concepts of  $G$ -minimal model and  $G$ -Mori fiber space are mutually exclusive. Moreover, if  $f : Y \rightarrow Z$  is a  $G$ -Mfs, then its general fiber is rationally connected; hence  $Y$  is uniruled. On the other hand, a  $G$ -minimal model is never uniruled [70]. The following assertions are usually formulated for varieties without group actions. The corresponding equivariant versions can be easily deduced from non-equivariant ones (see [89]).

**Theorem 2.3** ([14]). *Let  $X$  be an uniruled  $G$ -variety. Then there exists a birational  $G$ -map  $X \dashrightarrow Y$ , where  $Y$  has a structure of  $G$ -Mfs  $f : Y \rightarrow Z$ .*

**Conjecture 2.4.** *Let  $X$  be a non-uniruled  $G$ -variety. Then there exists a birational  $G$ -map  $X \dashrightarrow Y$ , where  $Y$  is a  $G$ -minimal model.*

The conjecture is known to be true in dimension  $\leq 4$  (see [73, 99]), as well as in the case where  $K_X$  is big [14], and in some other cases. In arbitrary dimension, a weaker notion of quasi-minimal models works quite satisfactory [82].

### 3. Cremona group of rank 2

The  $G$ -MMP for surfaces is much more simple than in higher dimensions. It was developed in the works of Yu. Manin and V. Iskovskikh (see [68]). In the two-dimensional case, the  $G$ -MMP works in the category of smooth  $G$ -surfaces, and all the birational transformations are contractions of disjoint unions of  $(-1)$ -curves. For a  $G$ -Mfs  $f : Y \rightarrow Z$ , there are two possibilities:

- (i)  $Z$  is a point and then  $Y$  is a  $G$ -del Pezzo surface,
- (ii)  $Z$  is a curve, any fiber of  $f$  is a reduced plane conic and  $\text{rk Pic}(Y)^G = 2$ . In this case,  $f$  is called  $G$ -conic bundle.

Thus to study finite subgroups of  $\text{Cr}_2(\mathbb{k})$ , one has to consider the above two classes of  $G$ -Mfs's in detail. The classification of del Pezzo surfaces is well known and very short. Hence, to study the case (i) one has to list all finite subgroups  $G \subset \text{Aut}(Y)$  satisfying the condition  $\text{rk Pic}(Y)^G = 1$ . The full list was obtained by Dolgachev and Iskovskikh [40]. In contrast, the class of conic bundles is huge and consists of an infinite number of families. In this case, a reasonable approach is to find an algorithm of enumerating conic bundles  $Y/Z$  together with subgroups  $G \subset \text{Aut}(Y/Z)$  satisfying  $\text{rk Pic}(Y)^G = 2$ . This also was done by Dolgachev and Iskovskikh [40] (see also [103]). However, even using this algorithm, it is very hard to get a complete list of corresponding groups.

As an example, we present a well-known classical result on the classification of subgroups of order 2 in  $\text{Cr}_2(\mathbb{k})$ . It was obtained by E. Bertini [12] in 1877; however, his arguments were incomplete from a modern point of view. A new rigorous proof was given by L. Bayle and A. Beauville [8].

**Theorem 3.1.** *Let  $G = \{1, \tau\} \subset \text{Cr}_2(\mathbb{k})$  be a subgroup of order 2. Then the embedding  $G \subset \text{Cr}_2(\mathbb{k})$  is induced by one of the following actions on a rational surface  $X$ :*

$\tau$	$X$ and $\tau$
$1^o$ Linear involution	$\mathbb{P}^2$
$2^o$ de Jonquières involution of genus $g \geq 1$	$X = \{y_1 y_2 = p(x_1, x_2)\} \subset \mathbb{P}(1, 1, g + 1, g + 1)$ $p$ is a homogeneous form of degree $2g + 2$ , $\tau$ is the deck involution of the projection $X \xrightarrow{2:1} \mathbb{P}(1, 1, g + 1), (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1 + y_2)$
$3^o$ Geiser involution	$X = \{y^2 = p(x_1, x_2, x_3)\} \subset \mathbb{P}(1, 1, 1, 2),$ $p$ is a homogeneous form of degree 4, $\tau$ is the deck involution of the projection $X \xrightarrow{2:1} \mathbb{P}(1, 1, 1) = \mathbb{P}^2$
$4^o$ Bertini involution	$X = \{z^2 = p(x_1, x_2, y)\} \subset \mathbb{P}(1, 1, 2, 3),$ $p$ is a quasihomogeneous form of degree 6, $\tau$ is the deck involution of the projection $X \xrightarrow{2:1} \mathbb{P}(1, 1, 2)$

Here  $\mathbb{P}(w_1, \dots, w_n)$  denotes the weighted projective space with corresponding weights.

In the cases  $1^o$ ,  $3^o$ , and  $4^o$ , the variety  $X$  is a del Pezzo surface of degree 9, 2, and 1, respectively. In the case  $2^o$ , the projection  $X \dashrightarrow \mathbb{P}(1, 1) = \mathbb{P}^1$  becomes a  $G$ -conic bundle after blowing up the indeterminacy points.

The  $G$ -MMP was successfully applied for the classification of various classes of finite subgroups in  $\mathrm{Cr}_2(\mathbb{k})$ : groups of prime order [36],  $p$ -elementary groups [9], abelian groups [15, 16], and finally, arbitrary groups [40]. Here is another example of classification results.

**Theorem 3.2** ([40]). *Let  $G \subset \mathrm{Cr}_2(\mathbb{C})$  be a finite simple group. Then  $G$  is isomorphic to one of the following:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(\mathbf{F}_7),$$

where  $\mathfrak{A}_n$  is the alternating group of degree  $n$  and  $\mathrm{PSL}_n(\mathbf{F}_q)$  is the projective special linear group over the finite field  $\mathbf{F}_q$ .

Moreover, if  $G \not\cong \mathfrak{A}_5$ , then the embedding  $G \subset \mathrm{Cr}_2(\mathbb{k})$  is induced by one of the following actions on a del Pezzo surface  $X$ :

$G$	$ G $	$X$
$\mathfrak{A}_6$	360	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathbf{F}_7)$	168	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathbf{F}_7)$	168	$\{y^2 = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1\} \subset \mathbb{P}(1, 1, 1, 2)$

A complete classification of embeddings  $\mathfrak{A}_5 \hookrightarrow \mathrm{Cr}_2(\mathbb{k})$  can be found in [31].

## 4. Cremona group of rank 3

The MMP in dimension 3 is more complicated than the two-dimensional one, but still, it is developed very well. In particular, terminal threefold singularities are classified up to analytic equivalence [72,93]. The structure of all intermediate steps of the MMP and Mfs's is also studied relatively well (see [89] for a survey).

For a three-dimensional  $G$ -Mori fiber space  $f : Y \rightarrow Z$ , there are three possibilities:

- (i)  $Z$  is a point, then  $Y$  is a (possibly singular)  $G\mathbb{Q}$ -Fano threefold,
- (ii)  $Z$  is a curve, then  $f$  is called a  $G\mathbb{Q}$ -del Pezzo fibration,
- (iii)  $Z$  is a surface, then  $f$  is a  $G\mathbb{Q}$ -conic bundle.

A  $G\mathbb{Q}$ -conic bundle can be birationally transformed into a *standard  $G$ -conic bundle*, i.e.,  $G\mathbb{Q}$ -conic bundle such that both  $X$  and  $Z$  are smooth [6]. For  $G\mathbb{Q}$ -del Pezzo fibrations, there are only some partial results of this type (see [35,66]). Nevertheless, the main difficulty in the application  $G$ -MMP to the classification of finite groups of birational transformations is the lack of a complete classification of Fano threefolds with terminal singularities. At the moment, only some very particular classes of  $G\mathbb{Q}$ -Fano threefolds are studied (see [4, 5, 52, 79, 80, 88] and references therein). Some roundabout methods work in the case of “large” in some sense (in particular, simple) finite groups.

**Theorem 4.1** ([78]). *Let  $G \subset \mathrm{Cr}_3(\mathbb{C})$  be a finite simple subgroup. Then  $G$  is isomorphic to one of the following:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathfrak{A}_7, \quad \mathrm{PSL}_2(\mathbf{F}_7), \quad \mathrm{PSL}_2(\mathbf{F}_8), \quad \mathrm{PSp}_4(\mathbf{F}_3),$$

where  $\mathrm{PSp}_4(\mathbf{F}_3)$  is the projective symplectic group over  $\mathbf{F}_3$ . All the possibilities occur.

This classification is a consequence of the following more general result.

**Theorem 4.2** ([78]). *Let  $Y$  be a rationally connected threefold and let  $G \subset \mathrm{Bir}(Y)$  be a finite simple group. If  $G$  is not embeddable to  $\mathrm{Cr}_2(\mathbb{C})$ , then  $Y$  is  $G$ -birationally equivalent to one of the following  $G\mathbb{Q}$ -Fano threefolds:*

$G$	$X$	Rational?
$1^\circ \mathfrak{A}_7$	$X'_6 = \{\sigma_{1,7} = \sigma_{2,7} = \sigma_{3,7} = 0\} \subset \mathbb{P}^5 \subset \mathbb{P}^6$	no
$2^\circ \mathfrak{A}_7$	$\mathbb{P}^3$	yes
$3^\circ \mathrm{PSp}_4(\mathbf{F}_3)$	$\mathbb{P}^3$	yes
$4^\circ \mathrm{PSp}_4(\mathbf{F}_3)$	Burkhardt quartic $X_4^b = \{\sigma_{1,6} = \sigma_{4,6} = 0\} \subset \mathbb{P}^4 \subset \mathbb{P}^5$	yes
$5^\circ \mathrm{PSL}_2(\mathbf{F}_8)$	Special Fano threefold $X_{12}^m \subset \mathbb{P}^8$ of genus 7	yes
$6^\circ \mathrm{PSL}_2(\mathbf{F}_{11})$	Klein cubic $X_3^k = \{x_1x_2^2 + x_2x_3^2 + \cdots + x_5x_1^2 = 0\} \subset \mathbb{P}^4$	no
$7^\circ \mathrm{PSL}_2(\mathbf{F}_{11})$	Special Fano threefold $X_{14}^a \subset \mathbb{P}^9$ of genus 8	no

Here  $\sigma_{d,k} = \sigma_{d,k}(x_1, \dots, x_k)$  is the elementary symmetric polynomial of degree  $d$  in  $k$  variables.

Below we outline the proof of Theorem 4.2.

Assume that  $G$  is not embeddable to  $\mathrm{Cr}_2(\mathbb{k})$ , i.e., it is not isomorphic to any of the groups listed in Theorem 3.2. First, Proposition 2.2 allows us to assume that the action of  $G$  is regularized on some smooth projective  $G$ -variety  $X$ . By running the equivariant MMP, we may assume that  $X$  has a structure of a  $G$ -Mfs  $f : X \rightarrow Z$  (because  $X$  is rationally connected). Consider the case  $\dim Z > 0$ . Since  $G$  is a simple group, it must act faithfully on the base  $Z$  or on the general fiber  $F$ . Since the varieties  $F$  and  $Z$  are rational, this means that  $G$  is contained in the plane Cremona group  $\mathrm{Cr}_2(\mathbb{k})$ . The contradiction proves Theorem 4.2 in the case  $\dim Z > 0$ .

Hence, we may further assume that  $Z$  is a point and  $X$  is a  $G\mathbb{Q}$ -Fano threefold. Consider the case where  $X$  is not Gorenstein, i.e., the canonical class  $K_X$  is not a Cartier divisor. It turns out that this case does not occur. Let  $P_1, \dots, P_n \in X$  be all non-Gorenstein points and let  $r_1, \dots, r_n$  be the corresponding Gorenstein indices. Arguments based on Bogomolov–Miyaoaka inequality (see [55, 57] and [89, §12]) show that

$$\sum \left( r_i - \frac{1}{r_i} \right) < 24.$$

Hence,  $n \leq 15$ . Then using the classification of transitive actions of simple groups [33] and analyzing the action of stabilizers of  $P_i$ , one obtains the only possibility:

- $n = 11$ ,  $G \simeq \mathrm{PSL}_2(\mathbf{F}_{11})$ ,  $r_1 = \dots = r_n = 2$ .

This case is excluded by a more detailed geometric consideration (see [78, §6]).

Thus, we may assume that  $K_X$  is a Cartier divisor. In this case, according to [74], the variety  $X$  has a smoothing, that is, there exists a one-parameter flat family  $\mathcal{X}/\mathfrak{B} \ni o$  such that the special fiber  $\mathcal{X}_o$  is isomorphic to  $X$ , and a general geometric fiber  $\mathcal{X}_t$  is a smooth Fano threefold. Hence some discrete invariants of  $X$ , such as the Picard lattice  $\mathrm{Pic}(X)$  and the anticanonical degree  $-K_X^3$ , are the same as for smooth Fano threefolds, which are completely classified (see [52]). Recall that the Fano index  $\iota(X)$  of  $X$  is the maximal integer that divides the canonical class  $K_X$  in the lattice  $\mathrm{Pic}(X)$  [52]. By [80], we have  $\mathrm{rk} \mathrm{Pic}(X) \leq 4$ . Since  $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$  and a simple group that is not isomorphic to  $\mathfrak{A}_5$  cannot have a nontrivial integer representation of dimension  $\leq 4$ , we have  $\mathrm{rk} \mathrm{Pic}(X) = 1$ . If  $\iota(X) \geq 4$  (resp.  $\iota(X) = 3$ ), then  $X$  is isomorphic to the projective space  $\mathbb{P}^3$  (resp. a quadric in  $\mathbb{P}^4$ ) [52]. Then from the classification of finite subgroups in  $\mathrm{PSL}_4(\mathbb{k})$  and  $\mathrm{PSL}_5(\mathbb{k})$ , we get cases  $2^\circ$  and  $3^\circ$ . Three-dimensional Fano varieties with  $\iota(X) = 2$  are called del Pezzo threefolds.  $G$ -Fano threefolds of this type were studied in [79]. As a consequence of these results, we get the case of the group  $G = \mathrm{PSL}_2(\mathbf{F}_{11})$  acting on the Klein cubic (case  $6^\circ$ ).



Finally, let  $\text{Pic}(X) = \mathbb{Z} \cdot K_X$ . Recall that in this case, the anticanonical degree is written in the form  $-K_X^3 = 2g(X) - 2$ , where  $g(X) \in \{2, 3, \dots, 10, 12\}$  [52]. For  $g(X) \leq 5$ , the variety  $X$  has a natural embedding to a (weighted) projective space as a complete intersection [52]. Using this and some facts from representation theory, we obtain for the group  $G$  two cases,  $1^\circ$  and  $4^\circ$ . The case  $g(X) = 6$  can be excluded using [37, Corollary 3.11]. For  $g(X) \geq 7$ , the variety  $X$  must be smooth (see [78, Lemma 5.17] and [88]). Further, using some facts about automorphisms of smooth Fano threefolds [63], we obtain for the group  $G$  two possibilities,  $5^\circ$  and  $7^\circ$ . This completes our sketch of the proof of Theorem 4.2. ■

A similar technique was applied to the study of finite  $p$ -subgroups and quasi-simple subgroups in  $\text{Cr}_3(\mathbb{k})$  (see [17, 64, 67, 77, 81, 86]).

Note that Theorem 4.2 does not describe *embeddings* of groups  $\mathfrak{A}_5$ ,  $\mathfrak{A}_6$ , and  $\text{PSL}_2(\mathbf{F}_7)$  to the space Cremona group. It is obvious that such embeddings exist, but their full classification should be significantly more difficult. There are only some partial results in this direction (see e.g. [26–29, 62]).

## 5. Jordan property

The methods and results of [40] show that one cannot expect a reasonable classification of all finite subgroups of Cremona groups of higher rank. Thus it is natural to concentrate on the study of general properties of these subgroups. Recall the following two famous results by C. Jordan and H. Minkowski.

**Theorem 5.1** ([53]). *There exists a function  $j(n)$  such that for any finite subgroup  $G \subset \text{GL}_n(\mathbb{C})$ , there exists a normal abelian subgroup  $A \subset G$  of index at most  $j(n)$ .*

**Theorem 5.2** ([69]). *There exists a function  $b(n)$  such that for every finite subgroup  $G \subset \text{GL}_n(\mathbb{Q})$ , one has  $|G| \leq b(n)$ .*

J.-P. Serre [94, 96] asked if these properties hold for Cremona groups. Complete answers to these questions were given in [82, 83] (see below). The following very convenient definitions were suggested by V. L. Popov [75].

- Definition.** • A group  $\Gamma$  is *Jordan* if there exists a constant  $j(\Gamma)$  such that any finite subgroup  $G \subset \Gamma$  has a normal abelian subgroup  $A$  of index  $[G : A] \leq j(\Gamma)$ .
- A group  $\Gamma$  is *bounded* (or satisfy *bfs* property) if there exists a constant  $b(\Gamma)$  such that for any finite subgroup  $G \subset \Gamma$ , one has  $|G| \leq b(\Gamma)$ .

### Rationally connected varieties

**Theorem 5.3** ([13, 83]). *Let  $X$  be a rationally connected variety. Then  $\text{Bir}(X)$  is Jordan. Moreover,  $\text{Bir}(X)$  is uniformly Jordan; that is, the constant  $j(\text{Bir}(X))$  depends only on  $\dim(X)$ .*

As a consequence, we obtain that the group  $\mathrm{Cr}_n(\mathbb{k})$  is Jordan.

Originally, Theorem 5.3 was proved modulo the so-called BAB conjecture (in a weak form), which is now settled by C. Birkar:

**Theorem 5.4** ([13]). *Fix  $d > 0$ . The set of all Fano varieties  $X$  of dimension at most  $d$  with at worst terminal singularities form a bounded family; i.e., they are parameterized by a scheme of finite type.*

It follows from Theorem 5.3 that there is a constant  $L = L(n)$  such that for any rationally connected variety  $X$  of dimension  $n$  and for any prime  $p > L(n)$ , every finite  $p$ -subgroup of  $\mathrm{Bir}(X)$  is abelian and generated by at most  $n$  elements (see [83]). Recently this result was essentially improved by Jinsong Xu [104]; he showed that  $L(n) = n + 1$ . The proof is based on a result by O. Haution [47]. Thus we have the following theorem.

**Theorem 5.5.** *Let  $X$  be a rationally connected variety of dimension  $n$  and let  $G \subset \mathrm{Bir}(X)$  be a finite  $p$ -subgroup. If  $p > n + 1$ , then  $G$  is abelian and is generated by at most  $n$  elements.*

The results of Theorems 5.3 and 5.5 were applied in the proof of Jordan property of local fundamental groups of log terminal singularities [20, 71].

### Varieties over non-closed fields

**Theorem 5.6** ([13, 82]). *Let  $X$  be a variety over a field  $\mathbb{k}$  of characteristic 0, which is finitely generated over  $\mathbb{Q}$ . Then the group  $\mathrm{Bir}(X)$  is bfs.*

Similar to Theorem 5.3, the proof of this result is based on the BAB conjecture (Theorem 5.4).

In the case  $X = \mathbb{P}^2$ , an explicit bound was obtained in [94] (see also [41]) in terms of cyclotomic invariants of the field  $\mathbb{k}$ . Theorem 5.6 can be reformulated in an algebraic form, which gives the positive answer to a question of J.-P. Serre [96].

**Theorem 5.6a.** *Let  $\mathbb{K}$  be a finitely generated field over  $\mathbb{Q}$ . Then the group  $\mathrm{Aut}(\mathbb{K})$  is bfs.*

**Jordan constants.** Define the *Jordan constant* of a group  $\Gamma$  as the number  $j(\Gamma)$  that appears in the definition of Jordan property. The *weak Jordan constant*  $\bar{j}(\Gamma)$  of  $\Gamma$  is the minimal  $j$  such that for any finite subgroup  $G \subset \Gamma$ , there exists an abelian (not necessarily normal) subgroup  $A \subset G$  such that  $[G : A] \leq j$ . Easy group-theoretic arguments show that

$$\bar{j}(\Gamma) \leq j(\Gamma) \leq \bar{j}(\Gamma)^2.$$

The exact value of the Jordan constant is known only for the Cremona group of rank two:  $j(\text{Cr}_2(\mathbb{k})) = 7200$  (see [105]). On the other hand, weak Jordan constants are easier to compute. It was proved in [84] that

$$\bar{j}(\text{Cr}_2) = 288, \quad \bar{j}(\text{Cr}_3) = 10368.$$

Moreover, the inequality  $\bar{j}(\text{Bir}(X)) \leq 10368$  holds for any rationally connected three-fold  $X$ .

**Jordan property of arbitrary varieties.** It turns out that the group of birational transformations of an algebraic variety is not always Jordan. The first example was discovered by Yu. Zarhin.

**Example** ([106]). Let  $C$  be an elliptic curve and let  $X = C \times \mathbb{P}^1$ . Then the group  $\text{Bir}(X)$  is not Jordan.

On the other hand, the exceptions as above are very rare.

**Theorem 5.7** (V. L. Popov [75]). *Let  $X$  be an algebraic surface. The group  $\text{Bir}(X)$  is not Jordan if and only if  $X$  is birationally equivalent to  $\mathbb{P}^1 \times C$ , where  $C$  is an elliptic curve.*

The proof of this theorem given in [75] essentially uses a result of I. Dolgachev, which in turn is based on the classification of algebraic surfaces. Later, Theorem 5.7 was generalized to higher dimensions with classification independent proofs.

**Theorem 5.8** ([82]). *Let  $X$  be an algebraic variety. Then the following assertions hold.*

- (i) *If  $X$  either is non-uniruled or has irregularity  $q(X) = 0$ , then  $\text{Bir}(X)$  is Jordan.*
- (ii) *If  $X$  is non-uniruled and  $q(X) = 0$ , then  $\text{Bir}(X)$  is bfs.*

Similar to Theorems 5.6 and 5.3, the proof of Theorem 5.8(i) is based on the boundedness of terminal Fano varieties (Theorem 5.4).

In dimension three, there is the following much more precise result.

**Theorem 5.9** ([85]). *Let  $X$  be a three-dimensional algebraic variety. Then  $\text{Bir}(X)$  is not Jordan if and only if either*

- (i)  *$X$  is birationally equivalent to  $C \times \mathbb{P}^2$ , where  $C$  is an elliptic curve, or*
- (ii)  *$X$  is birationally equivalent to  $S \times \mathbb{P}^1$ , where  $S$  is one of the following:*
  - *a surface of Kodaira dimension  $\kappa(S) = 1$  such that the Jacobian fibration of the pluricanonical map  $\phi: S \rightarrow B$  is locally trivial;*
  - *$S$  is either an abelian or bielliptic surface (and  $\kappa(S) = 0$ ).*

Below we explain the main idea of the proof of the necessity. So we assume that  $\text{Bir}(X)$  is not Jordan. By Theorems 5.3 and 5.8, the variety  $X$  is uniruled, but it is not rationally connected. Hence there exists a map  $X \dashrightarrow Z$  with rationally connected fibers (so-called maximal rationally connected fibration) such that  $Z$  is not uniruled and  $\dim(Z) = 1$  or  $2$  (see [56]). We have a natural exact sequence

$$1 \rightarrow \text{Bir}(X_\eta) \rightarrow \text{Bir}(X) \rightarrow \text{Bir}(Z),$$

where  $X_\eta$  is the generic scheme-theoretic fiber. Since  $X_\eta$  is rationally connected and  $Z$  is not uniruled, the groups  $\text{Bir}(X_\eta)$  and  $\text{Bir}(Z)$  must be Jordan. Then group-theoretic arguments show that both groups  $\text{Bir}(X_\eta)$  and  $\text{Bir}(Z)$  are not bfs (see, e.g., [82, Lemma 2.8]). In the case where  $Z$  is a curve, this implies that  $Z$  is elliptic, and applying the following fact with  $\mathbb{K} = \mathbb{k}(Z)$  and  $S := X_\eta$ , we obtain that  $X$  is birationally equivalent to  $Z \times \mathbb{P}^2$ .

**Proposition 5.10** ([85]). *Let  $\mathbb{K}$  be a field containing all roots of 1 and let  $S$  be a surface over  $\mathbb{K}$  such that  $S$  is not  $\mathbb{K}$ -rational,  $S$  is  $\bar{\mathbb{K}}$ -rational, and  $S(\mathbb{K}) \neq \emptyset$ . Then the group  $\text{Bir}(S)$  is bfs.*

Note that the condition of the existence of a  $\mathbb{K}$ -point on  $S$  in the above statement is important. The groups of (birational) automorphisms of geometrically rational surfaces without rational points were studied in the series of papers [100–102].

Now assume that  $Z$  is a surface. According to the main result of [7], the threefold  $X$  is birationally equivalent to  $Z \times \mathbb{P}^1$ . By Theorem 5.8 we have  $q(Z) > 0$ . Thus in the case  $\kappa(Z) = 0$ , the surface  $Z$  must be either abelian or bielliptic. Since the group  $\text{Bir}(Z)$  is not finite in our case,  $Z$  cannot be a surface of general type. Consider the case  $\kappa(Z) = 1$ . Then the pluricanonical map  $\phi : Z \rightarrow B$  is a  $\text{Bir}(Z)$ -equivariant elliptic fibration. Let

$$\text{Jac}(\phi) : E \rightarrow B$$

be the corresponding Jacobian fibration. The automorphism group  $\text{Aut}(Z_\eta)$  of the generic fiber  $Z_\eta$  over  $B$  is embedded to  $\text{Bir}(Z)$  as a normal subgroup. Analyzing singular fibers, one can conclude that  $\text{Aut}(Z_\eta)$  is of finite index in  $\text{Bir}(Z)$ . In turn,  $\text{Aut}(Z_\eta)$  has a subgroup  $\text{Aut}'(Z_\eta)$  of index at most 6 isomorphic to the group of  $\mathbb{k}(B)$ -points of  $E_\eta$ . Assume that the fibration  $\text{Jac}(\phi)$  is not locally trivial. Then by the functional version of Mordell–Weil theorem, known as Lang–Néron theorem (see, e.g., [32]), the group of  $\mathbb{k}(B)$ -points of  $E_\eta$  is finitely generated, and in particular, the torsion subgroup of the group of points of  $E_\eta$  is finite. This implies that  $\text{Aut}'(Z_\eta)$  is finite. ■

## 6. Invariants and rigidity

The most important part of the classification of finite subgroups in  $\text{Bir}(X)$  is to distinguish conjugacy classes.

**Problem 6.1.** Let  $G, G' \subset \text{Bir}(X)$  be finite subgroups such that  $G \simeq G'$ . How can one conclude that  $G$  and  $G'$  are *not* conjugate?

This is equivalent to the following.

**Problem 6.1a.** Let  $X$  and  $X'$  be  $G$ -varieties. How can one conclude that  $X$  and  $X'$  are *not*  $G$ -birational?

Below we describe a few approaches to solve the above problems. Note, however, that there are no universal methods.

**Fixed point locus.** Let  $X$  be a smooth projective  $G$ -variety. By  $\text{Fix}(X, G)$ , we denote the set of  $G$ -fixed points. It is not difficult to show (see [87]) that  $\text{Fix}(X, G)$  has at most one codimension one component that is not uniruled. Denote this component by  $F^{\text{nu}}(X, G)$ . This is a natural birational invariant in the category of smooth projective  $G$ -varieties.

**Proposition 6.2** ([87]). *Let  $X$  and  $X'$  be smooth projective  $G$ -varieties. If  $X$  and  $X'$  are  $G$ -birational, then  $F^{\text{nu}}(X, G_0)$  and  $F^{\text{nu}}(X', G_0)$  are birational for any subgroup  $G_0 \subset G$ .*

If  $G_0 \subset G$  is a normal subgroup, then the set  $F^{\text{nu}}(X, G_0)$  (if it is not empty) has a structure of  $(G/G_0)$ -variety. Clearly, the birational type of this  $(G/G_0)$ -variety is also a birational invariant (cf. [16]).

**Example.** According to Theorem 3.1 for subgroups  $G \subset \text{Cr}_2(\mathbb{k})$  of order 2, we have one of the following possibilities:

	Involution $\tau \in G$	$F^{\text{nu}}(X, G)$
1 <sup>o</sup>	Linear on $\mathbb{P}^2$	$\emptyset$
2 <sup>o</sup>	de Jonquières of genus $g \geq 1$	Hyperelliptic curve of genus $g$
3 <sup>o</sup>	Geiser	Non-hyperelliptic curve of genus 3
4 <sup>o</sup>	Bertini	Special non-hyperelliptic curve of genus 4

Thus the curve  $F^{\text{nu}}(X, G)$  distinguishes conjugacy classes in this case. The same assertion is true for subgroups of prime order [36], but it fails in general [15].

**Cohomological invariants.** It is not difficult to see that for a smooth projective  $G$ -variety  $X$ , the cohomology group

$$H^1(G, \text{Pic}(X))$$

is a  $G$ -birational invariant (see [19]). More generally, we say that  $G$ -varieties  $X$  and  $X'$  are *stably*  $G$ -birationally equivalent if for some  $n$  and  $m$  the products  $X \times \mathbb{P}^n$  and  $X' \times \mathbb{P}^m$  are  $G$ -birationally equivalent, where the action of  $G$  on  $\mathbb{P}^n$  and  $\mathbb{P}^m$  is supposed to be trivial. Then we have the following theorem.

**Theorem 6.3** ([19]). *Let  $X$  and  $X'$  be smooth projective  $G$ -varieties. If  $X$  and  $X'$  are stably  $G$ -birationally equivalent, then*

$$H^1(G, \text{Pic}(X)) \simeq H^1(G, \text{Pic}(X')).$$

Surprisingly, in some cases, the invariant  $H^1(G, \text{Pic}(X))$  can be computed in terms of  $G$ -fixed locus.

**Theorem 6.4** ([19]). *Let  $G$  be a cyclic group of prime order  $p$  and let  $X$  be a smooth projective rational  $G$ -surface. Assume that  $F^{\text{nu}}(X, G)$  is a curve of genus  $g$ . Then*

$$H^1(G, \text{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

This theorem was slightly generalized with a more conceptual proof in [97]. Another cohomological invariant which is called *Amitsur group* was introduced in [17].

As a consequence of Theorem 6.4, one can see that involutions from different families in Theorem 3.1 are not stably conjugate in  $\text{Cr}_2(\mathbb{k})$ . Note, however, that  $H^1(G, \text{Pic}(X))$  is a discrete invariant. For example, stable conjugacy of involutions whose curves  $F^{\text{nu}}(X, G)$  are non-isomorphic but have the same genus is not known.

A natural question that arises here is to find examples of subgroups in  $\text{Cr}_n(\mathbb{k})$  that are stably conjugate but not conjugate. This question is similar to the birational Zariski problem [11].

**Example.** Let  $G = \mathfrak{S}_3 \times \mu_2$ . There are two embeddings of this group into the Cremona group  $\text{Cr}_2(\mathbb{k})$  induced by the following actions:

- (i) action on  $\mathbb{P}^2 = \{x_1 + x_2 + x_3 = 0\} \subset \mathbb{P}^3$  by permutation and reversing signs;
- (ii) action on the sextic del Pezzo surface  $\{y_1 y_2 y_3 = y'_1 y'_2 y'_3\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  by permutation and taking inverses.

It was shown in [65] that these two subgroups in  $\text{Cr}_2(\mathbb{k})$  are stably conjugate; in fact, they are conjugate in  $\text{Cr}_4(\mathbb{k})$ . On the other hand, they are not conjugate [51].

Here is another example of this kind, which was pointed out to us by Yuri Tschinkel.

**Example ([92]).** Let  $V$  and  $W$  be faithful linear representations of  $G$  with  $\dim(V) = \dim(W) = n$ . Assume that the images of  $G$  in  $\mathrm{GL}(V)$  and  $\mathrm{GL}(W)$  do not contain non-identity scalar matrices. Then by a variant of the no-name lemma [39], we have the following  $G$ -birational equivalences of  $G$ -varieties:

$$\mathbb{P}(V) \times \mathbb{k}^{n+1} \underset{\mathrm{bir}}{\sim} V \times W \underset{\mathrm{bir}}{\sim} \mathbb{P}(W) \times \mathbb{k}^{n+1},$$

where  $\mathbb{k}^{n+1}$  is viewed as the trivial representation. Hence  $G$ -varieties  $V$  and  $W$  are stably  $G$ -birationally equivalent. On the other hand, it may happen that they are *not*  $G$ -birationally equivalent.

For example, Reichstein and Youssin [92] showed that the *determinant* of the action in the tangent space at a fixed point of a finite abelian group, up to sign, is a birational invariant of the action. This allowed them to produce nonbirational linear actions, e.g., of groups  $\mu_{p^n}$  on  $\mathbb{P}^n$ , with  $p \geq 5$ . Many new examples of nonbirational linear actions were given in [60, Section 10-11]; these are based on new invariants introduced in [61] (see also [46, 59]). These invariants take into account more refined information about the action on subvarieties with nontrivial abelian stabilizers.

A prime number  $p$  is said to be a *torsion prime* for the group  $\mathrm{Bir}(X)$  if there is a finite abelian  $p$ -subgroup  $G \subset \mathrm{Bir}(X)$  not contained in any algebraic torus of  $\mathrm{Bir}(X)$  [76]. Note that if a group  $G$  is contained in an algebraic torus  $T \subset \mathrm{Bir}(X)$ , then for any smooth projective birational model  $Y$  of  $X$  on which  $T$  acts biregularly, we have  $H^1(G, \mathrm{Pic}(Y)) = 0$ . Then by Theorem 6.3, the inequality  $H^1(G, \mathrm{Pic}(Y)) \neq 0$  for a finite  $p$ -subgroup  $G \subset \mathrm{Aut}(Y)$  implies that a prime number  $p$  is a torsion prime for  $\mathrm{Bir}(Y)$  and for  $\mathrm{Bir}(Y \times \mathbb{P}^n)$  for any  $n$ . Using Theorem 6.4 and the classification [36], one can immediately see that the set of all torsion primes for  $\mathrm{Cr}_2(\mathbb{k})$  is equal to  $\{2, 3, 5\}$ , and the numbers 2, 3, and 5 are torsion primes for  $\mathrm{Cr}_n(\mathbb{k})$  for any  $n \geq 2$ . This fact was proved in [76] by using another argument. In the case  $n \geq 3$ , the collection of all torsion primes for  $\mathrm{Cr}_n(\mathbb{k})$  is unknown.

**Maximal singularities method.** The maximal singularities method is the most powerful tool to study birational maps between Mfs's. It goes back to the works of G. Fano and even earlier works of other Italian geometers. However, the first application of this technique with rigorous proofs appeared much later in the breakthrough paper of Manin and Iskovskikh [49]. For an introduction to the “standard,” non-equivariant maximal singularities method, we refer to the book [90]. Below we outline very briefly an equivariant version of the method.

**Definition ([40, Definition 7.10], [29, Definition 3.1.1]).** A  $G\mathbb{Q}$ -Fano variety  $X$  is said to be  *$G$ -birationally rigid* if given birational  $G$ -map  $\Phi : X \dashrightarrow X^\sharp$  to the total space of another  $G$ -Mfs  $X^\sharp/Z^\sharp$ , there exists a birational  $G$ -selfmap  $\psi : X \dashrightarrow X$

such that the composition  $\Phi \circ \psi : X \dashrightarrow X^\sharp$  is an isomorphism (in particular,  $Z^\sharp$  is a point; i.e.,  $X^\sharp$  is also a  $G\mathbb{Q}$ -Fano variety).

A  $G\mathbb{Q}$ -Fano variety  $X$  is said to be  *$G$ -birationally superrigid* if any birational  $G$ -map  $\Phi : X \dashrightarrow X^\sharp$  to the total space of another  $G$ -Mfs  $X^\sharp/Z^\sharp$  is an isomorphism.

The maximal singularities method allows to check  $G$ -birational (super)rigidity using only internal geometry of the original variety, without considering all other  $G$ -Mfs's. We need the following technical definition which has become common nowadays.

**Definition.** Let  $X$  be a normal variety, let  $\mathcal{M}$  be a linear system of Weil divisors on  $X$  without fixed components, and let  $\lambda$  be a rational number. We say that the pair  $(X, \lambda\mathcal{M})$  is *canonical* if some multiple  $m(K_X + \lambda\mathcal{M})$  is Cartier, where  $M \in \mathcal{M}$ , and for any birational morphism  $f : Y \rightarrow X$ , one can write

$$m(K_Y + \lambda\mathcal{M}_Y) = f^*m(K_X + \lambda\mathcal{M}) + \sum a_i E_i,$$

where  $\mathcal{M}_Y$  is the birational transform of  $\mathcal{M}$ ,  $E_i$  are prime exceptional divisors, and  $a_i \geq 0$  for all  $i$ .

In the surface case, the canonical property is very easy to check: a pair  $(X, \lambda\mathcal{M})$  is canonical if and only if

$$\text{mult}_P(\mathcal{M}) \leq 1/\lambda$$

for any point  $P \in X$ .

Now, suppose that a  $G\mathbb{Q}$ -Fano variety  $X$  is not  $G$ -birationally superrigid. Then the Noether–Fano inequality [34, Theorem 4.2] implies the existence of a  $G$ -invariant linear system  $\mathcal{M}$  on  $X$  without fixed components such that the pair  $(X, \lambda\mathcal{M})$  is not canonical, where  $\lambda \in \mathbb{Q}$  is taken so that  $K_X + \lambda\mathcal{M}$  is numerically trivial. Moreover, any  $\mathcal{M}$  as above defines a birational  $G$ -map  $X \dashrightarrow X^\sharp$  to the total space of a  $G$ -Mfs  $X^\sharp/Z^\sharp$ . To show the existence or non-existence of such  $\mathcal{M}$ , one needs to analyze the geometry of the variety  $X$  carefully.

**Example.** Let  $X$  be a del Pezzo surface of degree 1. Assume that  $X$  is a  $G$ -del Pezzo with respect to some group  $G \subset \text{Aut}(X)$ . This means that  $G$  acts on  $X$  so that  $\text{rk Pic}(X)^G = 1$ . For example, this holds for any subgroup  $G \subset \text{Aut}(X)$  containing the Bertini involution. Let  $\mathcal{M}$  be a  $G$ -invariant linear subsystem without fixed components. Since  $\text{Pic}(X)^G = \mathbb{Z} \cdot K_X$ , we have  $\mathcal{M} \subset |-nK_X|$  for some  $n > 0$ . Suppose that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical. Then  $\text{mult}_P(\mathcal{M}) > n$ . Since  $\mathcal{M}$  has no fixed components,

$$n^2 = (-nK_X)^2 = \mathcal{M}^2 \geq (\text{mult}_P(\mathcal{M}))^2 > n^2.$$

The contradiction shows that  $X$  is  $G$ -birationally superrigid.



Similar arguments show that any  $G$ -del Pezzo surface  $X$  of degree  $\leq 3$  is  $G$ -birationally rigid. Moreover, it is  $G$ -birationally superrigid if and only if  $G$  has no orbits of length  $\leq K_X^2 - 2$  on  $X$ . In particular,  $\mathrm{PSL}_2(\mathbf{F}_7)$ -del Pezzo surface from Theorem 3.2 is  $G$ -birationally superrigid.

**Example.** All the  $G\mathbb{Q}$ -Fano threefolds from Theorem 4.2 are  $G$ -birationally super-rigid [17, 28, 30]. In particular, different embeddings of  $\mathrm{PSP}_4(\mathbf{F}_3)$  and  $\mathrm{PSL}_2(\mathbf{F}_{11})$  are not conjugate in  $\mathrm{Cr}_3(\mathbb{k})$ .

There is another relevant and very important notion called  $G$ -solidity [25]. For Fano varieties without group action, this notion has been introduced earlier by Shokurov [98] (who called solid Fano varieties primitive) and by Ahmadinezhad and Okada [2].

**Definition ([25]).** A  $G$ -Fano variety  $X$  is  $G$ -solid if  $X$  is not  $G$ -birational to a  $G$ -Mfs with a positive dimensional base.

For example, a  $G$ -del Pezzo surface  $X$  of degree 4 is  $G$ -solid if and only if  $G$  has no fixed points on  $X$  [40, §8].

A part of the maximal singularities method is the so-called Sarkisov program [34, 45]. It allows us to decompose any birational map between Mfs's into a composition of elementary ones. Refer to [50] for an explicit description of this program in dimension two and to [31] for examples and applications.

## 7. Application: Essential dimension

The notion of the essential dimension of a finite group  $G$ , denoted by  $\mathrm{ed}(G)$ , was introduced by Buhler and Reichstein [21]. Informally,  $\mathrm{ed}(G)$  is the minimal number of algebraic parameters needed to describe a faithful representation. More precisely, given a faithful linear representation  $V$  of  $G$  viewed as a  $G$ -variety, the *essential dimension*  $\mathrm{ed}(G, V)$  is the minimal value of  $\dim(X)$ , where  $X$  is taken from the set of all  $G$ -varieties admitting dominant rational  $G$ -equivariant map  $V \dashrightarrow X$ . It can be shown that  $\mathrm{ed}(G, V)$  does not depend on  $V$ , so we can omit  $V$  in the notation. It is easy to see that  $\mathrm{ed}(G) = 1$  if and only if  $G$  is cyclic or dihedral of order  $2n$  where  $n$  is odd. Finite groups of essential dimension  $\leq 2$  have been classified [43].

The essential dimension of symmetric groups  $\mathfrak{S}_n$  is important because it is equal to the minimal number of parameters needed to describe the general polynomial of degree  $n$  modulo Tschirnhaus transformations [21]. The values of  $\mathrm{ed}(\mathfrak{S}_n)$ , as well as of  $\mathrm{ed}(\mathfrak{A}_n)$ , are known for  $n \leq 7$ , and bounds exist for any  $n$  as follows.

**Theorem 7.1** ([21, 42]). *If  $n \geq 6$ , then*

$$n - 3 \geq \text{ed}(\mathfrak{S}_n) \geq \lfloor n/2 \rfloor,$$

$$\text{ed}(\mathfrak{S}_n) \geq \text{ed}(\mathfrak{A}_n) \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 2\lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

In many cases, the computations of  $\text{ed}(G)$  use the machinery of  $G$ -varieties. As an example, following Serre [95], we show that  $\text{ed}(\mathfrak{A}_6) = 3$ . Let  $V$  be the standard six-dimensional permutation representation of  $\mathfrak{A}_6$ . There exists an equivariant open embedding  $V \subset (\mathbb{P}^1)^6$ . On the other hand, the group  $\text{PSL}_2(\mathbb{k})$  also acts on  $(\mathbb{P}^1)^6$  so that the two actions commute. Hence we have a dominant rational  $\mathfrak{A}_6$ -map

$$V \hookrightarrow (\mathbb{P}^1)^6 \rightarrow (\mathbb{P}^1)^6 / \text{PSL}_2(\mathbb{k}),$$

where  $(\mathbb{P}^1)^6 / \text{PSL}_2(\mathbb{k})$  is a birational quotient. Since  $\dim((\mathbb{P}^1)^6 / \text{PSL}_2(\mathbb{k})) = 3$ , we have  $\text{ed}(\mathfrak{A}_6) \leq 3$ . Thus it is sufficient to show that  $\text{ed}(\mathfrak{A}_6)$  is not equal to 2. If so, there exists a dominant rational  $G$ -map  $V \dashrightarrow X$  to a surface which must be rational. According to Theorem 3.2, we may assume that  $X = \mathbb{P}^2$ . But in this case, a Sylow 3-subgroup  $S \subset \mathfrak{A}_6$  is abelian and acts without fixed points on  $\mathbb{P}^2$ . On the other hand,  $S$  has a fixed point on  $V$ , and the same should be true for the image of any rational  $S$ -map to a projective variety [58]. Therefore,  $\text{ed}(\mathfrak{A}_6) = 3$  as claimed.

Using similar arguments and the classification of embeddings of  $\mathfrak{A}_7$  to groups of birational transformations of rationally connected threefolds (Theorem 4.2), A. Duncan proved that  $\text{ed}(\mathfrak{A}_7) = \text{ed}(\mathfrak{S}_7) = 4$  [42].

Denote by  $\text{rdim}(G)$  (resp.  $\text{cdim}(G)$ ) the minimal dimension of faithful representations of  $G$  (resp. the smallest  $n$  such that  $G$  is embeddable to  $\text{Cr}_n(\mathbb{k})$ ). It immediately follows from the definition that

$$\text{ed}(G) \leq \text{rdim}(G).$$

If  $G$  is a  $p$ -group, then the equality holds  $\text{ed}(G) = \text{rdim}(G)$  [54]. In general, this equality fails, but there is a bound in terms of Jordan constants.

**Theorem 7.2** ([91]).  $\text{rdim}(G) \leq \text{ed}(G) \cdot j(\text{ed}(G))$ , where  $j(n)$  is the Jordan constant.

I. Dolgachev conjectured that  $\text{ed}(G) \geq \text{cdim}(G)$  (see [44]). It would be interesting to test this conjecture for the group  $G = \text{PSL}_2(\mathbf{F}_{11})$ . In fact, we have

$$3 \leq \text{ed}(\text{PSL}_2(\mathbf{F}_{11})) \leq 4$$

by Theorem 3.2 and because the group  $\text{PSL}_2(\mathbf{F}_{11})$  is simple and has a faithful five-dimensional representation. Assuming Dolgachev's conjecture, by Theorem 4.2 we

would have  $\text{ed}(\text{PSL}_2(\mathbf{F}_{11})) = 4$ . But this is unknown. See [44] for interesting discussions. The computation of the essential dimension of  $\text{PSL}_2(\mathbf{F}_{11})$  should complete Beauville's classification of finite simple groups of essential dimension  $\leq 3$  [10].

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