Questions posed after the first part of the "Subgroups of Cremona" workshop in University of Edinburgh, UK

March 30, 2010

1 Questions posed by I.Cheltsov

Question 1.1 *Describe the conjugacy classes of finite simple non-abelian subgroups in* $Cr_3(\mathbb{C})$*. There are only the following possibilities for such groups:*

1. A_5, A_6, $PSL_2(7)$ *(this is the hard case)*
2. A_7, $SL_2(8)$, $PSp_4(3)$ *(the answer is known for the last two)*

For A_7*, have*

$$\left\{\sum x_i = 0,\ \sum x_i^2 = 0,\ \sum x_i^3 = 0\right\} \subset \mathbb{P}^6$$

Is it rational? If not, A_7 *would have a unique conjugacy class in* $Cr_3(\mathbb{C})$.

Question 1.2 *What are the normalizers and centralizers of groups from question 1.1 in* $Cr_3(\mathbb{C})$*?*

Question 1.3 *What are the conjugacy classes of "small" groups in the list above (question 1.1,* (1)*)?*

Question 1.4 *Describe embeddings of "small" groups in the list above (question 1.1,* (1)*) into* $Bir(X)$*, where* X *is a rationally connected 3-fold.*

Question 1.5 *Let* $f \to X;\ S \mapsto$ *be a standard* G*-conic bundle,* X *a rationally-connected 3-fold. (*X *is smooth,* $rk\,Pic(X/S)^G = 1$*, etc. — very good conic bundle).* ***True or False:*** X *is always rational for* G *from (question 1.1,* (1)*)?*
Subquestion: *same question, but assuming also that* $rk\,Pic(X/S) = 1$.

Question 1.6 $\mathbb{P}^3/G$ *— study rationality for* $G \in PGL_4(\mathbb{C})$ *finite. At least,* ***True or False:*** $\mathbb{P}^3/G$ *rational.*

Question 1.7 $g \in Cr_n(\mathbb{C})$, $g \notin PGL_n(\mathbb{C})$*. What is the prime order of* g*? If* $ord(g) \gg 0$*, is* g *linearisable?*

2 Questions posed by J.P.Serre

Question 2.1 ***True or False:*** *Let K/k be an extension of fields of finite type, k small field (as defined in the lecture). Then the finite tame subgroups (of order prime to $char(k)$) of $Aut_k(K)$ have bounded order. Especially for $k = \mathbb{Q}$.*

This is now known to be false in general. The question is open if k is assumed to be a number field.

Question 2.2 *Same as question 2.1, assuming $char(k) = 0$ and $tr.deg(K) = 2$.*

Question 2.3 *Is it possible to introduce such topology on $Cr_2(\mathbb{C})$ that is compatible in $PGL_3(\mathbb{C})$ with $PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$?*

The conjectured answer is "No".

Question 2.4 *If $char(k) = p > 0$, then $Cr_n(k) \not\ni g$ with $ord(g) = p^{n+1}$.*

3 Questions posed by S.Lamy

Question 3.1 *In $Cr_2(\mathbb{C})$, is there a criterion to decide that g is not conjugate to g^{-1}? (or g^n not conjugate to g^m, for $n, m \in \mathbb{Z}$) Does there exist $g \in Cr_2(\mathbb{C})$, hyperbolic, such that g is not conjugate to g^{-1}.*

Question 3.2 *Does there exist a smooth cubic surface S in $\mathbb{P}^3$ and an element $g \in Cr_3(\mathbb{C}) \setminus PGL_4(\mathbb{C})$, such that $g \in Aut(\mathbb{P}^3 \setminus S)$? (This question is due to Gizatulin)*

4 Questions posed by S.Galkin

Question 4.1 *Take $Symp := SCr_2$.* ***True or False:*** *$Symp$ is generated by $(k^*)^2$, $SL_2(\mathbb{Z})$ and $P : (x,y) \mapsto \left(y, \frac{1+y}{x}\right)$. (Usnich conjecture)*

Question 4.2 *Take $H := <P, SL_2(\mathbb{Z})>$, where*

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with $I^4 = C^3 = [I^2, C] = 1$, $P^5 = 1$, $PCP = I$ (take $k = \mathbb{C}$). Is H presented by these?

Question 4.3 *What are the finite subgroups (normal subgroups, etc.) of H (from question 4.2)? Is there an element of order 7?*

Question 4.4 *Is H hyperbolic?*

Question 4.5 *Let w be a Laurent polynomial. Assume we know that*

$$J_w(t) = \int \frac{1}{1-tw}\frac{dx}{x}\wedge\frac{dy}{y} = J_{w_0}(t)$$

for $w_0 = x + y + \frac{1}{xy}$. Is it true that $\mathcal{N}(w)$ is an affine transformation of $\Delta\left(\mathbb{P}\left(x^2, y^2, z^2\right)\right)$ for Markov triple?

Question 4.6 *What are the hierarchies of special birational transform (???) of*

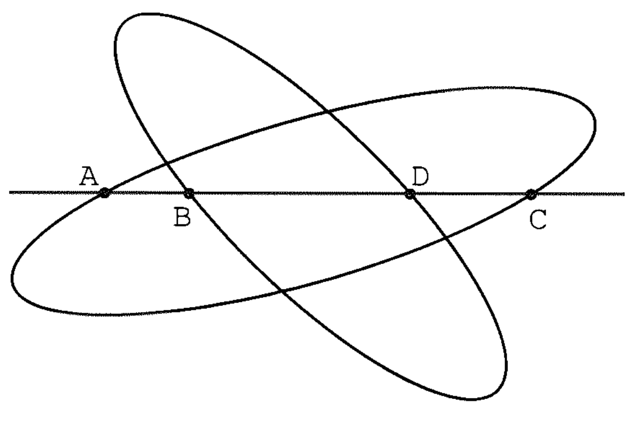
1. *$\mathbb{P}^3$ ($w = x + y + z + \frac{1}{xyz}$)*
2. *del Pezzo surface S of rk Pic$(S) > 1$*

5 Questions posed by A.Veselov

Question 5.1 *Describe $f \in Cr_2(\mathbb{C})$ with non-trivial symmetry $g \in Cr_2(\mathbb{C})$, i.e. such that $g \circ f = f \circ g$ and $g^m \neq f^n$ ($\sim$ description of $\mathbb{Z} \oplus \mathbb{Z}$ in $Cr_2(\mathbb{C})$)*

Question 5.2 *Describe Yang–Baxter maps (birational) $R : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ satisfying $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where R_{ij} is the map R from the (i, j)-th factor of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ to itself and identity on the third factor.*

The expected answer is: take



and let R map (A, B) to (C, D). Is this the only way to obtain R?

6 Questions posed by G.Brown

Question 6.1 *Does $PSL_2(13)$ act on a rational Fano 4-fold? OR: Is it embeddable in Cr_4?*

7 Questions posed by I.Karzhemarov

Question 7.1 *What is* $Aut(\mathbb{H}(x,y))$*?*

Question 7.2 ***True or False:*** $Aut(\mathbb{H}(x)) = PGL_2(\mathbb{H})$

8 Questions posed by Yu.Prokhorov

Question 8.1 *Work in* Cr_2*. How many hyperbolic elements does it contain? For example, study embeddings of* $\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{r}$ *in* Cr_2*. Is there a bound on* r*? Is it contained (up to conjugation) in the deJonquieres group for large* r*?*

Question 8.2 *Are there* $n \neq m$ *with* $Cr_n(k) \cong Cr_m(k)$*?*

Question 8.3 *Is* $Cr_n(\mathbb{C})$ $(n \geq 3)$ *Jordan? (This question is due to Popov, Serre).*

This question relates to the Borisov–Alexeev conjecture.

Question 8.4 (Stable conjugacy) $Cr_2(k) \hookrightarrow Cr_n(k)$ $(n \geq 3)$*.* G, G' *(finite) subgroups, not conjugate in* $Cr_2(k)$*. Are they conjugate in* $Cr_n(k)$*? Classify finite subgroups in Cremona group (e.g.* Cr_2*,* Cr_3*) up to this "stable conjugacy".*

9 Questions posed by J.Blanc

Question 9.1 *Assume* k *is an algebraically closed field. Is* $Cr_2(k)$ *an algebraic group of infinite dimension in the Shafarevich's sence? In other words, is it possible to write*

$$Cr_2(k) = \bigcup_{i=1}^{\infty} U_i$$

where U_i *are algebraic varieties,* U_i *is closed in* U_{i+1} $\forall i$*, and the structure is compatible with the group action.*

Prof. Serre suggests thct the answer is "No"

Question 9.2 *Is* $\{g \in Cr_2 : \deg g \leq 2\}$ *an irreducible algebraic variety?*

Prof. Serre suggests that it is *not*.

Question 9.3 *Are the two natural structures of* $\{f \in Cr_2(k) : \deg f = 2\}$ *and* $PGL_3(\mathbb{C})$ *compatible?*

Question 9.4 *Is* $Cr_n(k)$ *simple for* $n \geq 3$*? (even without Zariski topology)*

Question 9.5 *Give conjugacy classes of elements of finite order (e.g. 2, large prime, etc.) of* $Cr_3(\mathbb{C})$ *(over* $\mathbb{Q}$*, etc).*

Question 9.6 *What are generators of $Cr_3(\mathbb{C})$. Find some other reasonable formulation for this question. For $n \geq 3$, is $Cr_n(\mathbb{C})$ generated bv $Aut(\mathbb{P}^n_{\mathbb{C}})$ and $Cr_{n-1}(\mathbb{C}) \subset Bir(\mathbb{P}^1 \times \mathbb{P}^{n-1}, \pi_2)$ (in a natural way)?*

The conjectured answer to the last part is "No".

Problems for the Edinburgh workshop on Cremona groups, March 2010

Problem 1. Prove (or disprove) that there does not exist a topology on $G = \mathrm{Cr}_2(\mathbf{C})$ with the following properties :

a) It is compatible with the group structure of G ; in particular, the multiplication map $G \times G \to G$ is continuous.

b) Its restriction to $\mathbf{PGL}_3(\mathbf{C})$ (resp. to $\mathbf{PGL}_2(\mathbf{C}) \times \mathbf{PGL}_2(\mathbf{C})$) is the usual topology of that group.

Problem 2. Does there exist a non trivial central extension of $\mathrm{Cr}_2(\mathbf{C})$, for instance with a center of order 2 ?

Problem 3. Let us say that a group G has property (BFS) (" bounded finite subgroups ") if the finite subgroups of G have bounded order.

Let K be a field of characteristic 0 which is finitely generated over $\mathbf{Q}$. Prove (or disprove) that the group $\mathrm{Aut}\, K$ has property (BFS).

Remarks on Problem 3.

1. Let k be a field which is a finitely generated extension of $\mathbf{Q}$ and let V be a projective smooth k-scheme. The group $\mathrm{Aut}_k(V)$ has property (BFS).
[By Néron and Weil, the group $\mathrm{Pic}\, V$ is a finitely generated $\mathbf{Z}$-module, hence its group of automorphisms has property (BFS). A finite subgroup of $\mathrm{Aut}_k(V)$ which acts trivially on $\mathrm{Pic}\, V$ is isomorphic to a subgroup of some $\mathbf{PGL}_n(k)$ for some integer n depending only on V ; the group $\mathbf{PGL}_n(k)$ has property (BFS).]

2. Problem 3 has a positive answer when the transcendence degree of K over $\mathbf{Q}$ is $\leqslant 2$.
[Use the birational classification of surfaces : the case where the Kodaira dimension is > 0 is easy ; the case where there is a unique minimal model follows from Remark 1 ; the case of a rational field is standard.]

J-P.Serre, April 3, 2010

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