Markov numbers in arithmetic and geometry

Andrey Andreevich Markov (1856–1922) was an outstanding Russian mathematician. His works on probability theory and mathematical analysis are widely known and generally recognized. The developed by him theory of an extensive class of stochastic processes with discrete and continuous time components, named after him, has countless applications in modern theoretical and applied research, its influence is difficult to overestimate. A. A. Markov made a huge contribution to the theory of continued fractions and the calculus of finite differences. In the theory of pattern recognition and artificial intelligence tasks most of the algorithms use the concept of a hidden Markov model, which originates in Markov's works.

However, A. A. Markov is no less well-known as a specialist in number theory. He received the first significant result in his master thesis *On binary quadratic forms of a positive determinant* [11, 13] (see also [10, 12]). One of the central objects of the dissertation is one Diophantine equation which subsequently arose in many areas of mathematics, quite far from the original problem of minimizing of quadratic forms. In this note we will discuss this aspect of A. A. Markov's extensive mathematical heritage.

The Markov equation

The *Markov equation* is a Diophantine equation of the form

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3. \tag{3.6}$$

Solutions of this equation are now known as *Markov triples*. The *Markov numbers* are all natural numbers appearing in these triples. Let (x_1, x_2, x_3) be a Markov triple. Consider the following three transformations

$$\begin{array}{ccccc} & & (x_1, x_2', x_3) & & \\ & & \uparrow t_2 & & \\ & & (x_1, x_2, x_3) & & \\ & \swarrow t_1 & & t_3 \searrow & \\ (x_1', x_2, x_3) & & & & (x_1, x_2, x_3') \end{array} \tag{3.7}$$

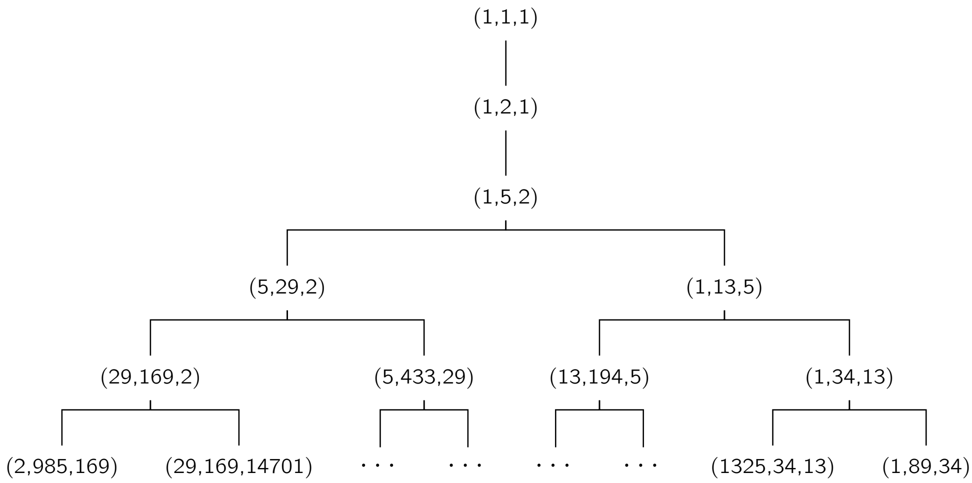
where

$$x_i' := \frac{3x_1x_2x_3}{x_i} - x_i.$$

According to Vieta's formulas, new triples

$$(x_1', x_2, x_3), \quad (x_1, x_2', x_3), \quad (x_1, x_2, x_3') \tag{3.8}$$

are also solutions of the Markov equation, moreover $x_i' \neq x_i$. Such a procedure t_i is called an *elementary transformation* or *mutation* in the element x_i, and the corresponding triples are called *neighboring*. It can be shown that if all three entries x_1, x_2, x_3 are different, then all triples (3.8) are also different. Moreover, a mutation in the maximal element of the triple reduces this element. For example, if $x_1 = \max(x_1, x_2, x_3)$, then $x_1' < \max(x_2, x_3) < x_1$. It follows that any solution of the Markov equation is obtained from $(1, 1, 1)$ by successive application of mutations. All the Markov triples can be written as a graph in which the neighboring ones are connected by an edge. The graph has the form of an infinite trivalent tree:



It is easy to see that any Markov number is maximal in some triple. In 1913, Frobenius proposed the following conjecture.

Conjecture (uniqueness conjecture). A Markov triple is uniquely determined by its maximal element.

Despite numerous attempts, the conjecture has not yet been proven, see [1] for a very good introduction and historical overview.

The geometry of the Markov surface

Consider the surface X defined in the affine space $\mathbb{A}^3$ by the equation (3.6). Its projective closure $\bar{X} \subset \mathbb{P}^3$ is a nodal cubic with a unique singular point so that the boundary divisor is the union of three lines forming a "triangle".

The maps t_i are automorphisms of the surface X as an affine variety. One can check that they generate a subgroup $\Gamma_0 \subset \mathrm{Aut}(X)$ isomorphic to the free product

$$(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}).$$

The complete automorphism group $\mathrm{Aut}(X)$ is generated by Γ_0, permutations, and sign changes of pairs of coordinates [3]. In this presentation, $\mathrm{Aut}(X)$ acts transitively on the set of integer points of the surface X and its subgroup of index 4, isomorphic to $\mathrm{PGL}_2(\mathbb{Z})$, acts transitively on the set of Markov triples.

The projection

$$\Psi : X \dashrightarrow \mathbb{P}^2$$

from the origin is a birational map, i.e. it is one-to-one on nonempty Zariski-open subsets of $U \subset X$ and $V \subset \mathbb{P}^2$. Moreover, Ψ induces an embedding of

$\mathrm{Aut}(X)$ into the group of birational transformations of the plane so that all the elements preserve, up to sign, the symplectic form

$$\frac{\mathrm{d}u \wedge \mathrm{d}v}{uv}.$$

Thus, the subgroup of the index 2 in $\mathrm{Aut}(X)$ can be embedded to *symplectic Cremona group* [15].

Markov numbers in approximation theory and quadratic form theory

In Markov's original work, the equation (3.6) arose in connection with the problem of finding the arithmetic minimum of binary quadratic forms.

Consider a binary quadratic form

$$f(x, y) = \alpha x^2 + \beta xy + \gamma y^2, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

We assume that the form is *indefinite*, i.e. its discriminant

$$D := \beta^2 - 4\alpha\gamma$$

is positive. The *Markov constant of the form f* is the number

$$\mu(f) := \frac{\sqrt{D}}{\min'(f)},$$

where $\min'(f)$ is the arithmetic minimum:

$$\min'(f) := \min\left\{ |f(x, y)| \;\middle|\; x, y \in \mathbb{Z}, \quad (x, y) \neq (0, 0) \right\}.$$

The *Markov spectrum* is the set of all Markov constants:

$$\mathbb{M} := \left\{\mu(f) \mid f \text{ is a binary quadratic form with } D > 0\right\}.$$

The forms f and f' are called *equivalent* if they are obtained from each other by integer coordinate changes. It is clear that the equivalent forms have the same minimum.

It turns out that the problem of computing the arithmetic minimum of quadratic forms is closely related to the theory of Diophantine approximations. The well-known theorem of A. Hurwitz states that for any irrational number θ there are infinitely many rational fractions $\frac{p}{q} \in \mathbb{Q}$ satisfying the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

and the constant $\sqrt{5}$ in the denominator cannot be increased. In this regard, the following natural definition arises: the *Lagrange number* for $\theta \in \mathbb{R}$ is the supremum $\lambda(\theta)$ of the set of all real numbers λ such that the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\lambda q^2}. \tag{3.9}$$

holds for infinitely many rational fractions $\frac{p}{q} \in \mathbb{Q}$. Thus, by Hurwitz's theorem for the irrational θ we have $\lambda(\theta) \geqslant \frac{1}{\sqrt{5}}$. The *Lagrange spectrum* is the set

$$\mathbb{L} := \{\lambda(\theta) \mid \theta \in \mathbb{R}\}$$

of all possible values of Lagrange numbers. The numbers θ, $\theta' \in \mathbb{R}$ are called *equivalent* if they are contained in the same orbit of the action group $GL_2(\mathbb{Z})$ on $\mathbb{R}$ by Mobius transformations. It is clear that the Lagrange numbers of equivalent real numbers are equal.

Note that the exponent 2 for q on the right side of the inequality (3.9) cannot be increased: as was shown by K. Roth (1955), for any irrational *algebraic* number and for any $\epsilon > 0$ inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only a finite number of solutions for coprime p and q.

The results of Markov

Let $m_1 = 1$, $m_2 = 2$, $m_3 = 5, \ldots$ be an ordered sequence of all Markov numbers. Denote

$$\lambda_m = \sqrt{9 - 4/m}.$$

Also, to each ordered Markov triple (m, m', m''), $m > m' > m''$ one can associate, by a certain explicit rule, an indefinite quadratic form

$$F_{m,m',m''}(x, y)$$

which is called the *Markov form*. Assuming the Frobenius conjecture we can think that $F_{m,m',m''}$ depends only on the maximal element: $F_{m,m',m''} = F_m$.

Theorem (Markov)**.** *For an indefinite binary quadratic form $f(x, y)$ the inequality $\mu(f) < 3$ is satisfied if and only if f is equivalent to a multiple of the form F_m for some Markov number m.*

Hurwitz noticed that the methods of of the proof of this theorem allows to obtain immediately a similar result for Diophantine approximations.

Theorem. *For an irrational real number* θ*, the inequality* $\lambda(\theta) < 3$ *holds if and only if* $\lambda(\theta) = \lambda_m$*, where* m *is a Markov number. In this case, the number* θ *is equivalent to a root of the equation* $F_m(x, 1) = 0$.

In particular, it follows that on the interval $[0, 3)$ the Lagrange and Markov spectra are discrete and coincide:

$$\mathbb{L} \cap [0, 3) = \mathbb{M} \cap [0, 3) = \{\lambda_n\}.$$

On the contrary, in the right hand side of the real line, these spectra are continuous: G. A. Freiman in 1975 proved that the Lagrange and Markov spectra contain the interval $[\lambda_{\mathrm{F}}, +\infty]$ (Hall ray), where

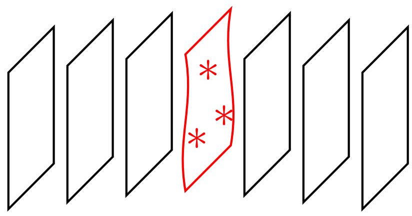
$$\lambda_{\mathrm{F}} := \frac{2221564096 + 283748\sqrt{462}}{491993569} \approx 4.52.$$

On the other hand, the behavior of Lagrange and Markov spectra on the interval $[3, \lambda_{\mathrm{F}}]$ is quite complicated and still not fully understood.

Markov numbers in geometry

Degenerations of the projective plane

Consider an analytic family $\{S_t\}_{t \in \Delta}$ of compact complex surfaces over a disk $\Delta \subset \mathbb{C}$ such that for $t \neq 0$ the fiber S_t is isomorphic the projective plane $\mathbb{P}^2$. In this situation, the central fiber of S_0 is called *degeneration* of $\mathbb{P}^2$.



In general, the structure of degenerations of $\mathbb{P}^2$ can be quite complicated. M. Manetti [9] posed a problem classifications of degenerations of $\mathbb{P}^2$ admitting only *quotient singularities*, i.e. those degenerations whose singularities are analytically equivalent to quotients $\mathbb{C}^2/G$, where $G \subset \mathrm{GL}_2(\mathbb{C})$. This problem is interesting, important, and motivated by its applications in the theory of modules of curves and surfaces, as well as in the Minimal Model Program.

Recall that the *weighted projective plane* $\mathbb{P}(d_1, d_2, d_3)$ is the set of triples of numbers $(x_1, x_2, x_3) \neq (0, 0, 0)$ with identification:

$$(x_1,\ x_2,\ x_3) = (t^{d_1}x_1,\ t^{d_2}x_2,\ t^{d_3}x_3), \qquad t \in \mathbb{C}^*.$$

Here d_1, d_2, d_3 are natural numbers called *weights*. We will assume that the weights are pairwise coprime. For $d_1 = d_2 = d_3 = 1$ we get the usual projective plane. Otherwise, $\mathbb{P}(d_1, d_2, d_3)$ has quotient singularities.

Theorem ([7]). *If the weighted projective plane is a degeneration of* $\mathbb{P}^2$*, then it has the form*

$$\mathbb{P}(m_1^2, m_2^2, m_3^2),$$

where (m_1, m_2, m_3) *is a Markov triple. Conversely, each weighted projective plane* $\mathbb{P}(m_1^2, m_2^2, m_3^2)$ *is a degeneration of* $\mathbb{P}^2$*.*

A complete classification of degenerations of $\mathbb{P}^2$ was obtained in [7], as well as similar results for degenerations of the two-dimensional quadric and other del Pezzo surfaces.

Exceptional vector bundles on $\mathbb{P}^2$

A vector bundle $\mathscr{E}$ on a nonsingular complex projective algebraic variety X is called *exceptional* if

$$\mathrm{Hom}(\mathscr{E}, \mathscr{E}) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^q(\mathscr{E}, \mathscr{E}) = 0 \quad \text{when } q > 0.$$

An ordered collection of vector bundles $\mathscr{E}_1, \ldots, \mathscr{E}_n$ is called *exceptional* if all $\mathscr{E}_i$ are exceptional and

$$\mathrm{Ext}^q(\mathscr{E}_i, \mathscr{E}_j) = 0 \qquad \text{for } i > j \text{ and } q \geqslant 0.$$

An exceptional collection is said to be *complete* if it generates a bounded derived category $\mathcal{D}^{\mathrm{b}}(X)$ of coherent sheaves on X. The presence of a complete exceptional collections imposes very strong restrictions on the variety X. We will consider only the case of the projective plane $X = \mathbb{P}^2$. In this case, any line bundle is exceptional and the triple

$$\big(\mathscr{O}_{\mathbb{P}^2},\ \mathscr{O}_{\mathbb{P}^2}(1),\ \mathscr{O}_{\mathbb{P}^2}(2)\big)$$

is a complete exceptional collection. Moreover, an exceptional collection on $\mathbb{P}^2$ is complete if and only if it consists of three elements.

In the works of A. N. Rudakov [14] and A. L. Gorodentsev and Rudakov [4] a surprising fact was established: one can define certain transformations (mutations) of the complete exceptional collections of vector bundles on $\mathbb{P}^2$, similar to the mutations of Markov triples (3.7). In particular, the ranks of bundles in complete exceptional collections are exactly Markov triples. These results have generalizations to arbitrary del Pezzo surfaces [8].

Markov numbers in Lobachevsky geometry

The classical Fricke identity states that for any matrices A, B, $C = AB \in \mathrm{SL}_2(\mathbb{R})$ the following equality holds

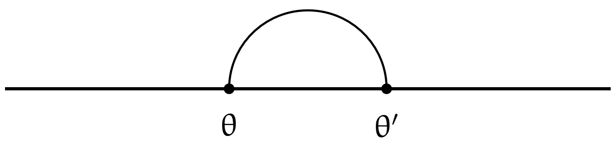
$$\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(C)^2 = \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(C) + \mathrm{tr}(ABA^{-1}B^{-1}) + 2.$$

If the matrices are integer and the commutator $ABA^{-1}B^{-1}$ is a parabolic matrix, then $\mathrm{tr}(ABA^{-1}B^{-1}) = -2$ and the numbers

$$\mathrm{tr}(A)/3, \quad \mathrm{tr}(B)/3, \quad \mathrm{tr}(C)/3$$

form a Markov triple. This observation allows us to reformulate many questions about Markov numbers in terms of the action of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and its congruence subgroup $\Gamma(3)$ on the Lobachevsky plane.

Consider the Poincare model $\mathbb{H}$ (the upper half-plane in $\mathbb{C}$) of the Lobachevsky plane. The action of a hyperbolic transformation $A \in \Gamma(3)$ on the closure $\bar{\mathbb{H}}$ has two real fixed points θ and θ'. The circle passing through these points and perpendicular to the real axis is a straight line in the Lobachevsky geometry,



and its image on the quotient $\mathbb{H}/\Gamma(3)$ is a geodesic γ_A. It turns out that γ_A has no self-intersections if and only if $\lambda(\theta), \lambda(\theta') < 3$ and its length can be expressed in terms of Markov numbers. The uniqueness conjecture also has an interpretation in these terms [1]. This approach, using Lobachevsky geometry was applied by D. S. Gorshkov [5], [6] in order to reprove Markov's results in purely geometric methods.

Markov numbers in symplectic geometry

One of the interesting and important problems in symplectic geometry is the question of the classification of Lagrangian tori in the complex projective plane with a symplectic form equal to the Kähler form of the standard Fubini-Study metric. In the recent works of R. Viano [16], significant progress has been made in this direction. In particular, an infinite family of nonequivalent Lagrangian tori parametrized by Markov triples was constructed.

In conclusion, we note that our brief overview is not complete. Unexpected applications of Markov triples continue to appear in various parts of mathematics. We hope that there will be many more other appearances, as well as interesting connections between them will be found. Here is what the outstanding Soviet mathematician B. N. Delone wrote about the master's thesis of A. A. Markov [2]:

> *"This work, highly appreciated by Chebyshev, is one of the most insightful achievements of the St. Petersburg school of number theory and, perhaps, of all Russian mathematics."*

Yuri Prokhorov

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