1 Elementary proof of Borsuk-Ulam Theorem

Borsuk-Ulam theorem states:

Theorem 1. *Every continuous mapping of n-dimensional sphere S^n into n-dimensional Euclidean space R^n identifies a pair of antipodes.*

This theorem is widely applicable in combinatorics and geometry. And there are sufficiently many nontopologists, who are interested to know the proof of the theorem. But the standard proof of this theorem involves the ring structure of cohomologies of projective space — the subject hardly understanding by nontopologsts. The goal of this article is to present an elementary proof of the theorem. More precisely we present an elementary reduction of high-dimensional Borsuk-Ulam theorem to 2-dimensional, which is known to have an elementary proof. Moreover we reduce this theorem almost to 1-dimensional case: it is enough to proof that the suspension of an odd mapping of the circle is not homotopic to constant.

By the product we get a new elementary proofs for Brouwer Fixed Point theorem, which one apples in the Dimension theory to prove Lebesgue theorem on coverings of Euclidean Space.

Odd mappings of spheres. Let us say that a mapping of one sphere into another sphere is *odd* if it takes any pair of antipodes into antipodes. That is $-f(x) = f(-x)$ for all x — the usual equality to define an odd function . The following reduction of the theorem **1** was made already in the original Borsuk's paper in 1933

Lemma 2. *If there exists a continuous mapping of $f\colon S^n \to R^n$, which does not identify any pair of antipodes, then there exists an odd continuous mapping $g\colon S^n \to S^{n-1}$.*

Proof. Here is the explicit formula for such mapping $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ □

The main new idea is presented by the following Lemma

Lemma 3 (key lemma). *If $n > 3$ and there is an odd PL mapping of S^n to S^{n-1}, then there is a continuous odd mapping $S^{n-1} \to S^{n-2}$*

Closed subset of the sphere S^n is called *simple section* between the poles $P, -P$ if every geodesic arc joining these poles intersects the subset just in one point. It is easy to see that any two simple sections are homeomorphic to each other. Hence all simple sections are homeomorphic to S^{n-1} — the equatorial section of the sphere. Every simple section S separates the sphere in two parts, the connectivity components of $S^n \setminus S$, containing P and $-P$ respectively. And we will say that a simple section separates a pair of the sets if these sets belongs to different components of $S^n \setminus S$.

Lemma 4. *Let X be a closed subset of the sphere S^n, such that every geodesic arc joining the poles $P, -P$ meeting X does not meet the antipode $-X = \{-x \mid x \in X\}$. Then there exists a centrally symmetric simple section separating X and $-X$.*

Proof. Denote by $B, -B$ the poles of the sphere and denote by $S(X)$ and $S(-X)$ respectively the projections (from poles) of X and $-X$ respectively onto the equator of the sphere. Let us choose ε so small that ε-neighborhood of B does not intersect $X \cup -X$. Denote by $d(x)$ the distance from x to $S(X)$ and denote by $d^-(x)$ the distance from x to $S(-X)$. Consider the function $\psi(x)$, defined on S by the formula

$$\frac{d(x)(\pi - \varepsilon) + d^-(x)\varepsilon}{d(x) + d^-(x)} \tag{1}$$

Denote by $B(x)$ the length of geodesic, connecting B with x. Now the simple section S' we are looking for, may be introduced by the following way: the intersection of S' with geodesic between B and $-B$ passing through $x \in S$ is the point, located on the (geodesic) distance $\psi(x)$ from B. $\square$

Proof of Key Lemma. Without loss of generality one can suppose f to be PL with respect to some centrally symmetric triangulation. Denote by A some point of the sphere S^n, which is the center of gravity of some n-dimensional simplex of the triangulation. Then preimages $f^{-1}(A)$ and $f^{-1}(-A)$ are graphes. Indeed, for any $n+1$-dimensional simplex of S^{n+1} its intersection with these preimages represents either interval or emptyset. Let us consider the union of geodesics joining pairs of point from $f^{-1}(A) \cup f^{-1}(-A)$ which are not antipodes to each other. The dimension of this set does not exceed 3, because it is finite union of joins of geodesic arcs.

Let us pick up a point $B \in S^{n+1}$ from the complement of the union. Such point exists due to inequality $n > 3$. In this case the central projections from the poles (B and $-B$) onto the equator of $f^{-1}(A)$ and $f^{-1}(-A)$ do not intersect each other. Hence by virtue the Lemma **4** there is a symmetric simple section S separating $f^{-1}(A)$ from $f^{-1}(-A)$. The restriction of f on this sphere has the image in $S^n \setminus (\{P\} \cup \{-P\})$, which is naturally retracted onto S^{n-1}. As result one gets an odd mapping $S^n \to S^{n-1}$. $\square$

Lemma 5. *If there exists an odd mapping $g\colon S^3 \to S^2$, then there exists a mapping $\psi\colon D^3 \to S^2$ with odd restriction onto the boundary such that the image of upper semisphere of ∂D^3 is upper semisphere of S^2.*

Proof. Let $pr\colon S^3 \to D^3$ denotes the restriction onto the unit sphere S^3 of orthogonal projection of R^4 onto 3-dimensional horizontal hyperplane passing through the origin . Denote by pr^{-1} the inverse mapping of D^3 into upper semisphere of S^3. Consider the composition $f = gpr^{-1}$. Fix poles $A, -A \in S^2$. f-preimages of the poles are finite. Choose points B and $-B$ from which these preimages are projected injectively onto the equator. By the help of the Lemma **4** one constructs a symmetric simple section S of S^2 separating $f^{-1}A$ and $f^{-1}(-A)$. Let $h\colon S^2 \to S^2$ be an odd homeomorphism translating upper semisphere into the component of $S^2 \setminus S$, containing $f^{-1}A$.

Now we define an odd mapping $\phi\colon S^2 \to S^2$ in such a way: $\phi(x)$ for x from the upper semisphere is defined as point, with projection onto equator equal to $fh(x)$ with height over equator equal to the distance from x to S, if this distance is less than 1 and less than the distance from x to $f^{-1}A$. And the image coincides with the pole A in other cases. For x from the lower semisphere $\phi(x)$ is defined as $-\phi(-x)$.

Constructed ϕ transforms upper semisphere into upper semisphere and for any $x \in S^2$ points $\phi(x)$ and $f(x)$ are not opposite. Now the mapping $\psi \colon D^3 \to S^2$ we are looking for may be defined in polar coordinates as follows

$$\psi(r,x) = \begin{cases} (2r-1)\phi(x) + (2-2r)f(1,x), & 1 \geq r \geq \frac{1}{2}; \\ f(2r,x), & r \leq \frac{1}{2}. \end{cases}$$

□

Brouwer Fixed Point Theorem. Let $f \colon D^n \to D^n$ be a continuous mapping without fixed point. Then the mapping $F(x) = (1-|x|)f(x)+|x|x$ is identity on the boundary ($|x| = 1$) and has not fixed points in the interior of D^n. Indeed, the equality $x = (1-|x|)f(x)+|x|x$ implies $(1-|x|)f(x) = (1-|x|)x$, which may be true only for $|x| = 1$. And the following formula for $(t,x) \in R^1 \times R^n$

$$g(t,x) = \begin{cases} F(x), & t > 0; \\ x, & t \leq 0. \end{cases}$$

defines an continuous mapping $S^n \to R^n$, which does not identifies any pair of antipodes.