Lax operator algebras: unexpected outcome, and a new tool of the theory of integrable systems

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Southeast Lie Theory Workshop
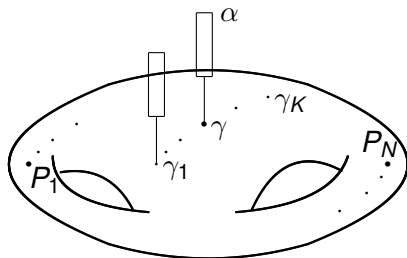*College of Charleston*
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Outline

SECTION 1. LAX OPERATOR ALGEBRAS

Geometrical data

- Riemann surface Σ
- Classical Lie alg. $\mathfrak{g}$ over $\mathbb{C}$
- Marked points of two kinds: $P_1, \ldots, P_N \in \Sigma$ and $\gamma_1, \ldots, \gamma_K \in \Sigma$
- Vectors $\alpha_1, \ldots, \alpha_K \in \mathbb{C}^n$ associated with γ's.

Pairs γ, α are referred to as <u>Tyurin parameters</u>, due to the

THEOREM (*A.N.Tyurin):*** Let $g = genus \, \Sigma$, $n \in \mathbb{Z}_+$. Then there is a 1–1 correspondence between the pairs of sets
$$\gamma_1, \ldots, \gamma_{ng} \in \Sigma; \quad \alpha_1, \ldots, \alpha_{ng} \in \mathbb{C}P^{n-1},$$
and the equivalence classes of the semi-stable holomorphic rank n vector bundles on Σ.

Lax operator with the spectral parameter on a Riemann surface

— it is a meromorphic function L on Σ with arbitrary poles at P_i's, simple or double poles at the γ's, holomorphic at the other points, and having the expansion at a γ of the form

$$L(z) = \frac{L_{-2}}{(z - z_\gamma)^2} + \frac{L_{-1}}{(z - z_\gamma)} + L_0 + L_1(z - z_\gamma) + O((z - z_\gamma)^2)$$

where z is a local coordinate at γ, $\quad z_\gamma = z(\gamma)$, and

$$L_{-1} = \begin{cases} \alpha\beta^t, & \mathfrak{g} = \mathfrak{gl}(n) \\ \alpha\beta^t - \beta\alpha^t, & \mathfrak{g} = \mathfrak{so}(n) \\ (\alpha\beta^t + \beta\alpha^t)\sigma, & \mathfrak{g} = \mathfrak{sp}(2n) \end{cases}$$

$$\boxed{\begin{array}{l} \beta^t\sigma\alpha = 0 \\ L_0\alpha = k\alpha \end{array}}$$

$$\boxed{L_{-2} = \nu\alpha\alpha^t\sigma, \ \mathfrak{g} = \mathfrak{sp}(2n)}$$

$\alpha, \beta \in \mathbb{C}^n$, α is associated with γ, β arbitrarily, σ — is the matrix of the quadratic form

Lax operator algebras

THEOREM: For given $\mathfrak{g}$, (α, γ, P_i), the Lax operators form a Lie algebra with respect to the point-wise matrix commutator $[L, L'](P) = [L(P), L'(P)]$ $(P \in \Sigma)$ (Krichever-Sh., 2007).

This algebra is referred to as Lax operator algebra, and is denoted by $\mathcal{L}$, or by $\overline{\mathfrak{g}}$.
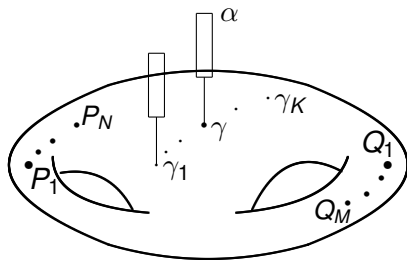
Examples:
1) If $g = 0$, $P_1 = 0$, $P_2 = \infty$, $\{\gamma\} = \varnothing$ then $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ — *a loop algebra*.

2) If $g > 0$, $\{\gamma\} = \varnothing$ then $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}$ where $\mathcal{A}$ consists of meromorphic functions holomorphic outside $\{P\}$ — *a Krichever–Novikov current algebra*.

Almost graded structure

Lax operator algebras are by no means graded. Define $\overline{\mathfrak{g}}_m \subset \overline{\mathfrak{g}}$ by

$$(\overline{\mathfrak{g}}_m)_I = -m \sum_{i=1}^N P_i$$
$$(\overline{\mathfrak{g}}_m)_O = \sum_{j=1}^M (a_j m + b_{j,m}) Q_j,$$
where $\sum_{j=1}^M a_j = N$,
$$\sum_{j=1}^M b_{j,m} = N + g - 1,$$
$$|b_{j,m}| \leq B \; \forall \, j, m \; (N \geq M)$$

THEOREM: (1) $\dim \overline{\mathfrak{g}}_m = N \dim \mathfrak{g}$;   (2) $\overline{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \overline{\mathfrak{g}}_m$

(3) $\exists$ a constant integer $R > 0$ such that

$$[\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_k] \subseteq \bigoplus_{h=m+k}^{m+k+R} \overline{\mathfrak{g}}_h. \tag{1}$$

Central extensions

A skew-symmetric bilinear form γ on $\mathcal{L}$ is called a *2-cocycle* if

$$\gamma([X, Y], Z) + \gamma([Z, X], Y) + \gamma([Y, Z], X) = 0$$

for all $X, Y, Z \in \mathcal{L}$.

Given a Lie algebra $\mathcal{L}$, and a cocycle γ, we can construct a new Lie algebra $\widehat{\mathcal{L}}$ called central extension of $\mathcal{L}$. As a linear space

$$\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} \cdot t$$

and the bracket is given by

$$[X + c_1 t, Y + c_2 t] = [X, Y]_{\mathcal{L}} + \gamma(X, Y)t, \quad [X, t] = 0 \; \forall X \in \mathcal{L}.$$

Example. $\mathcal{L} = $ loop algebra, $\gamma(X, Y) = \mathrm{res}_0 \, \mathrm{tr}(X dY)$.

Equivalence of central extensions

Equivalently, a central extension of $\mathcal{L}$ is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \xrightarrow{\ i\ } \widehat{\mathcal{L}} \xrightarrow{\ p\ } \mathcal{L} \longrightarrow 0 \qquad (2)$$

where $\mathrm{Im}(i) = \ker(p)$ is the center of $\widehat{\mathcal{L}}$. Then $\widehat{\mathcal{L}}$ is a central extension of $\mathcal{L}$ in the above defined sense.

Two central extensions $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}'$ are called equivalent if there exists an isomorphism e (*equivalence*) such that the following diagram is commutative

(3)

Classification of almost graded central extensions

Geometric cocycle:
$\gamma(L, L') = \sum_{i=1}^{N} \text{res}_{P_i} \text{tr}(L\nabla^{(\theta)}L') = \sum_{j=1}^{M} \text{res}_{Q_i} \text{tr}(L\nabla^{(\theta)}L')$ where θ is a $\mathfrak{g}$-valued 1-form, $\nabla^{(\theta)} = d + \text{ad}\,\theta$.

THEOREM: 1) If θ possesses the same expansion as L at any γ-point, with $\beta^t\sigma\alpha = 1$, then γ gives an almost graded central extension.

2) If $\mathfrak{g}$ is simple, the only central extension of $\overline{\mathfrak{g}}$, up to equivalence and normalization of the central generator, is given by the above cocycle γ.

3) If $\mathfrak{g} = \mathfrak{gl}(n)$ then $\overline{\mathfrak{g}}$ has only one more central extension given by $\gamma' = \sum_{i=1}^{N} \text{res}_{P_i} \text{tr}(L)\text{tr}(\nabla^{(\theta)}L')$.

Example. For a loop algebra, or a KN current algebra $\theta = 0$, and $\gamma(X, Y) = \text{tr}(XdY)$.

Canonical representation of $\mathcal{L}$

$\mathcal{F}$ — the space of meromorphic functions on Σ with the same singularities as L. Expansion at a γ is of the form

$$\varphi = const \cdot \frac{\alpha}{z - z_\gamma} + \varphi_0 + O(z - z_\gamma).$$

$\mathcal{L}$ has a natural representation in $\mathcal{F}$. This representation is almost graded.

Almost grading in $\mathcal{F}$ is given by means a certain base (the Krichever–Novikov base).

Example. For two points P, Q the degree is essentially given by the order at P.

The $\mathcal{L}$-module $\mathcal{F}$ by no means is of vacuum type.

Vacuum representation

$\mathcal{F}^{\infty/2}$ — space of semi-infinite forms on $\mathcal{F}$, $\mathcal{F}^{\infty/2} = \oplus \mathcal{F}_k^{\infty/2}$,

$$\mathcal{F}_k^{\infty/2} = \{f_{i_1} \wedge \ldots \wedge f_{i_k} \wedge f_{i_k+1} \wedge f_{i_k+2} \ldots\}$$

where f_j is the Krichever–Novikov base in $\mathcal{F}$.

$$\boxed{\mathcal{F}_k^{\infty/2} \text{ is a vacuum representation of } \mathcal{L}}$$

where the vacuum is given by the semi-infinite monomial
$f_{k+1} \wedge f_{k+2} \wedge \ldots$, and the action by Leibniz rule.

Canonical commutative subalgebra

Given an *L*-operator, its spectral curve Σ_L is defined by means the equation $\det(L(z) - \varkappa) = 0$. It is an *n*-fold branch covering of Σ.

Spectrum *K* of *L* is a diagonal matrix defined by the relation $\Psi L = K \Psi$. The spectrum defines the function $\varkappa$ on Σ_L.

LEMMA: $\varkappa$ is meromorphic and holomorphic outside the pre-images of *P*-*Q*-points (in particular, at pre-images of γ's).

Vise versa, the direct image of such a function is a spectrum of the operator $L(\varkappa) = \Psi^{-1} K(\varkappa) \Psi$.

THEOREM: The maximal subalgebra in $\mathcal{L}$ commuting with an $L \in \mathcal{L}$ is isomorphic to the algebra of meromorphic functions on Σ_L, holomorphic outside the pre-images of *P*-*Q*-points.

SECTION 2. LAX INTEGRABLE SYSTEMS

Lax equations

Lax equation: $\qquad \dot{L} = [L, M]$ $\qquad$ where

$$L = L(z, \alpha, \beta, \gamma, \varkappa, \ldots), M = M(z, \alpha, \mu, \gamma, \ldots).$$

It is regarded to as a collection of equations on α's, β's, γ's, $\varkappa$'s and the main parts of the expansions of L at P_i's.

M is defined by the same constrains as L, excluding $\beta^t \sigma \alpha = 0$ and $L_0 \alpha = k\alpha$, namely

$$M = \frac{M_{-2}}{(z - z_\gamma)^2} + \frac{M_{-1}}{z - z_\gamma} + M_0 + M_1(z - z_\gamma) + O((z - z_\gamma)^2)$$

where

$$\boxed{M_{-2} = \lambda \alpha \alpha^t \sigma} \quad \boxed{M_{-1} = (\alpha \mu^t + \varepsilon \mu \alpha^t)\sigma}$$

Hierarchy of commuting flows

$D := \sum m_i P_i$ ($i = 1, \ldots, N$), s.t. supp $D \cap \{\gamma\} = \varnothing$.

$\mathcal{L}^D := \{ L \in \mathcal{L} \mid (L) + D \geq 0 \text{ outside } \gamma\text{'s}\}$.

Upon a certain (effective) condition a Lax equation defines a flow on $\mathcal{L}^D$.

THEOREM: Given a generic L, there is a family of M-operators $M_a = M_a(L)$ ($a = (P_i, n, m), n > 0, \ m > -m_i$) uniquely defined up to normalization, such that outside the γ-points M_a has pole at the point P_i only, and in the neighborhood of P_i

$$M_a(w_i) = w_i^{-m} L^n(w_i) + O(1),$$

The equations

$$\partial_a L = [L, M_a], \ \partial_a = \partial/\partial t_a \tag{4}$$

define a family of commuting flows on $\mathcal{L}^D$.

$\circlearrowright \, \Omega \, \Omega \, \bigcirc$

Krichever–Phong symplectic structure

We define an external 2-form on $\mathcal{L}^D$. For $L \in \mathcal{L}^D$ let Ψ be a matrix-valued function formed by the eigenvectors of L: $\Psi L = K \Psi$ (K — diagonal).

$$\Omega := \mathrm{tr}(\Psi^{-1}\delta L \wedge \delta \Psi - \Psi^{-1}\delta\Psi \wedge \delta K) = \delta\mathrm{tr}(\Psi^{-1}L\delta\Psi)$$

where $\delta\Psi$ is the differential of Ψ in $\alpha, \beta, \ldots$.

Let ω_0 be a holomorphic 1-form on Σ and

$$\omega := \sum \mathrm{res}_{\gamma_s} \Omega\omega_0 + \sum \mathrm{res}_{P_i} \Omega\omega_0$$

THEOREM: ω is a symplectic form on a certain invariant manyfold $\mathcal{P}^D \subset \mathcal{L}^D$.

Hamiltonians

THEOREM: The equations of the above commutative family are Hamiltonian with respect to the *Krichever-Phong symplectic structure* on $\mathcal{L}^D$, with the Hamiltonians given by

$$H_a = -\frac{1}{n+1}\operatorname{res}_{P_i} tr(w_i^{-m}L^{n+1})dw_i$$

Example. Let $D = (\omega_0)$. Then H_a are Hitchin Hamiltonians.

Calogero-Moser systems, $\mathfrak{g} = \mathfrak{gl}(n)$:

Lax operator:

$$L_{i,j} := \frac{\sigma(z + q_i - q_j)\sigma(z - q_i)\sigma(q_j)}{\sigma(z)\sigma(z - q_j)\sigma(q_i - q_j)\sigma(q_i)} \ (i \neq j), \ \ L_{ii} = p_i$$

2d order Hamiltonian:

$$H = -\frac{1}{2}\operatorname{res}_{z=0}\operatorname{tr}(z^{-1}L^2) = -\frac{1}{2}\sum_{j=1}^{n}p_j^2 + \sum_{i<j}\wp(q_i - q_j).$$

Tyurin parameters: (q_i, e_i), $e_i = (\ldots, \delta_{ij}, \ldots)^t$, $\omega = \sum dp_i \wedge dq_i$.

Calogero-Moser systems, $\mathfrak{g} = \mathfrak{so}(2n)$:

Lax operator: $L = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{so}(2n)$, $B^t = -B$, $C^t = -C$.

$A_{i,j}$ is the same as $L_{i,j}$ above.

$$B_{ij} = \frac{\sigma(z + q_j + q_i)\sigma(z - q_j)}{\sigma(z)\sigma(z + q_i)\sigma(q_i + q_j)}, \ \ C_{j\,i} = -\frac{\sigma(z - q_j - q_i)\sigma(z + q_i)}{\sigma(z)\sigma(z - q_i)\sigma(q_i + q_j)}, \ i < j.$$

2d order Hamiltonian:

$$H = -\frac{1}{2} \operatorname{res} \operatorname{tr}(z^{-1}L^2) = -\sum_{i=1}^{n} p_i^2 + 2\sum_{i<j} \wp(q_i - q_j) + 2\sum_{i<j} \wp(q_i + q_j)$$

.

Other examples

1) $g = 0$, $\alpha = 0$ (i.e. $\Sigma = \mathbb{C}P^1$, the bundle is trivial), $P_1 = 0$, $P_2 = \infty$. Then $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ — <u>loop algebra</u>.

It yields a conventional Lax equation with a <u>rational</u> spectral parameter:

$$L_t = [L, M], \quad L, M \in \mathfrak{g} \otimes \mathbb{C}[\lambda^{-1}, \lambda), \quad \lambda \in \mathcal{D}^1$$

(I.Gelfand, L.Dikii, I.Dorfman, A.Reyman, M.Semenov-Tian - Shanskii, V.Drinfeld, V.Sokolov, V.Kac, P. van Moerbeke). Majority of known integrable cases of motion and hydrodynamics of a solid body.

2) Arbitrary genus, $D = (\omega_0)$ where ω_0 is a holomorphic 1-form. Then we obtain Hitchin systems.

SECTION 3. QUANTIZATION OF LAX INTEGRABLE SYSTEMS AND 2D CFT

Sheaf of conformal blocks

$\dot{L} = [L, M]$

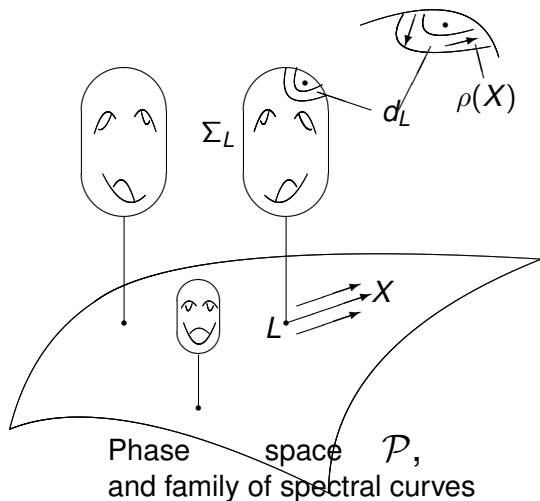$\mathcal{P} = \{L\}$ — phase space.

$\{\Sigma_L\}$ — family of spectral curves over $\mathcal{P}$

Over every $\{\Sigma_L\}$ take the canonical vacuum $\mathcal{L}$-module $\mathcal{F}_L^{\infty/2}$

$\mathcal{A}_L$ acts in $\mathcal{F}_L^{\infty/2}$ by means $\varkappa \to L(\varkappa) = \Psi^{-1} K(\varkappa) \Psi$

Conformal blocks:

$C = \mathcal{F}_L^{\infty/2} / \mathcal{A}_L^{reg} \mathcal{F}_L^{\infty/2}$.

Phase space $\mathcal{P}$, and family of spectral curves

Kodaira–Spencer cocycle

$X \in Vect(\mathcal{P})$, $d_L \in Diff_{loc}$ — transition function on Σ_L.
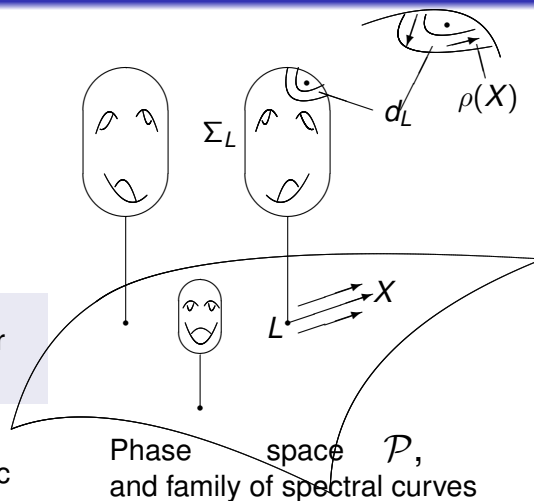
Kodaira–Spencer cocycle:

$$\rho(X) = d_L^{-1} \cdot \partial_X d_L$$

THEOREM (SCH-SH, '05):
$\rho(X)$ continues to a KN vector field on Σ_L

(i.e. a global meromorphic vector field on Σ_L holomorphic outside P-Q-points).

Notation: $\mathcal{V}_L$ is the Lie algebra of KN vector fields over L.

Phase space $\mathcal{P}$, and family of spectral curves

Sugawara construction

V – vacuum $\mathcal{L}$-module.   Example: $V = \mathcal{F}_L^{\infty/2}$

Let $\{A_j\}$, $\{\omega^j\}$ be the Krichever–Novikov bases of functions and 1-forms, resp., on Σ, $\{u_i\}$.

Notation: $a(m)$ is the representation operator of A_m in V.

Energy–momentum tensor: $T = \sum_{m,n} :a(m)a(n): \omega^m \omega^n$

For $e \in \mathcal{V}$ let $T(e) = \sum_{i \in I} \mathrm{res}_{P_i}(Te)$.

THEOREM:
1) $e \to T(e)$ defines a projective representation of $\mathcal{V}$:

$$T([e_1, e_2]) = [T(e_1), T(e_2)] + \lambda(e_1, e_2) id$$

2) $[T(e), a(m)] = const \cdot a(eA_m)$

Representation of the algebra $C^\infty(\mathcal{P})$ of observables

$f \in C^\infty(\mathcal{P})$, X_f — the corresponding Hamiltonian vector field,
$\nabla_{X_f} = \partial_{X_f} + T(\rho(X_f))$ — the corresponding KZ-type operator.

THEOREM: 1) $f \to \nabla_{X_f}$ is a projective representation of the Poisson algebra of observables;
2) if f, g depend only on the action variables then

$$[\nabla_{X_f}, \nabla_{X_g}] = 0.$$

3) $f \to \nabla_{X_f}$ is a unitary representation.

<u>Proof.</u> 1) follows from $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + \lambda(X, Y) \cdot id, \forall X, Y$.
2) by invariance of the complex structure on Σ_L along trajectories (the spectral curve is an integral of motion).
3) by invariance of the volume form $\omega^p/p!$, $p = \dim \mathcal{P}/2$ resp. Hamiltonian flows, and by unitarity of the Sugawara rep.

Seiberg–Witten prepotential

Partition function of a gauge theory: $\widetilde{\mathcal{F}} = \int e^{i\hbar S(\psi)} D\psi$ where

S is the Yang–Mills action, ψ runs over gauge fields.

Seiberg–Witten prepotential $\mathcal{F}$ is a low energy limit of $\widetilde{\mathcal{F}}$.

There is a family of pairs (Σ, λ) where Σ is a Riemann surface, λ is a 1-form on it. Let a_i, b_i be the periods of λ. Then $\mathcal{F}$ is a function of $\{a_i, b_i\}$ satisfying the following Seiberg–Witten equations:

$$b_i = \frac{\partial \mathcal{F}}{\partial a_i}.$$

Prepotential and integrable systems

Observation (Gorski–Krichever–Marshakov–Mironov –Morozov):
in all known cases the family of Riemann surfaces is just the
family of the spectral curves of an integrable system fibered
over the phase space of the latter. As for the 1-forms,

$$\lambda = \varkappa \cdot \omega_0$$

$\varkappa$ is the spectrum of $\mathcal{L}$ (function on Σ_L), ω_0 is fixed in Lecture 2.
Then

$$a_i = \oint_{A_i} \varkappa \cdot \omega_0, \quad b_i = \oint_{B_i} \varkappa \cdot \omega_0$$

are integrals of motion (depend only on the spectrum of Lax
operator). They are dependent, so that

$$\sum_i da_i \wedge db_i = 0.$$

Hence the 1-form $\sum_i b_i da_i$ has potential $\mathcal{F}$ obviously satisfying
the Seiberg–Witten equations.

O. Sheinman
Current algebras on Riemann surfaces
DeGruyter, 2012, 150 p.

M. Schlichenmaier
Higher genus conformal algebras
DeGruyter, in process.