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Lax Operator Algebras and Integrable Hierarchies

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Dedicated to S.P. Novikov
on the occasion of his 70th birthday

Abstract—We study applications of a new class of infinite-dimensional Lie algebras, called *Lax operator algebras*, which goes back to the works by I. Krichever and S. Novikov on finite-zone integration related to holomorphic vector bundles and on Lie algebras on Riemann surfaces. Lax operator algebras are almost graded Lie algebras of current type. They were introduced by I. Krichever and the author as a development of the theory of Lax operators on Riemann surfaces due to I. Krichever, and further investigated in a joint paper by M. Schlichenmaier and the author. In this article we construct integrable hierarchies of Lax equations of that type.

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1. INTRODUCTION

In [6, 7] I.M. Krichever and S.P. Novikov proposed a technique for finding high-rank finite-zone solutions to Kadomtsev–Petviashvili and Shrödinger equations. Based on the ideas of these works and on his own results on effective classification of high-rank pairs of commuting differential operators [8], Krichever proposed a theory of Lax operators with spectral parameter on a Riemann surface [4]. In [10] Krichever and the author found that these Lax operators form an associative algebra and constructed their orthogonal and symplectic analogs, which form Lie algebras. All of them were called *Lax operator algebras*. Lax operator algebras form a new class of one-dimensional current algebras.

In this article we give some applications of Lax operator algebras to integrable hierarchies of Lax equations.

The applications of current algebras to the theory of Lax equations have a long history. They were initiated in the works of I. Gelfand, L. Dikii, I. Dorfman, A. Reyman, M. Semenov-Tian-Shansky, V. Drinfeld, V. Sokolov, V. Kac, and P. van Moerbeke. Basically these applications are related to Kac–Moody algebras, which appear quite naturally in the context of Lax equations with *rational* spectral parameter. In [4] the theory of conventional Lax and zero-curvature representations with rational spectral parameter was generalized to the case of algebraic curves Σ of arbitrary genus g. Such representations arise in several ways in the theory of integrable systems (cf. [6], where a zero-curvature representation of the Krichever–Novikov equation is introduced, or [4], where a field analog of the Calogero–Moser system on an elliptic curve is presented). Lax operator algebras appear as an appropriate generalization of Kac–Moody algebras.

The concept of Lax operators on algebraic curves is closely related to A. Tyurin's results on the classification of holomorphic vector bundles on algebraic curves [17]. It uses *Tyurin data* modeled on *Tyurin parameters* of such bundles. The Turin data consist of points γ_s, $s = 1, \dots, ng$, and associated elements $\alpha_s \in \mathbb{C}P^n$, where g denotes the genus of the Riemann surface Σ and n corresponds to

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the rank of the bundle. In [4] the linear space of Lax operators is associated with a positive divisor $D = \sum_k m_k P_k$, $P_k \in \Sigma$. Lax operators are defined as meromorphic $n \times n$ matrix-valued functions on Σ that have poles of multiplicity at most m_k at the points P_k and at most simple poles at γ_s's. The coefficients of the Laurent expansions of these matrix-valued functions in the neighborhood of a point γ_s must obey certain constraints parameterized by α_s (relations (3.4) below). In [10] it was found that the Lax operators having poles of arbitrary orders at the points P_k form an algebra with respect to the usual pointwise multiplication, $\mathfrak{g}$-valued Lax operators were introduced, and it was shown that for $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$ over $\mathbb{C}$ the space of such operators forms a Lie algebra with respect to the pointwise bracket (the setup of [4] corresponds to the case $\mathfrak{g} = \mathfrak{gl}(n)$). We denote this algebra by $\overline{\mathfrak{g}}$. Considering $\mathfrak{g}$-valued Lax operators requires certain modifications of the above-mentioned constraints. It turns out that the orders of poles at γ_s's must be set equal to 2 for $\mathfrak{g} = \mathfrak{sp}(2n)$. There is no doubt that by means of appropriate modifications it is possible to construct Lax operator algebras for other classical Lie algebras.

In the absence of points γ_s (which corresponds to trivial vector bundles) we return to the known class of Krichever–Novikov algebras (see [15] for a review). If, in addition, the genus of Σ is equal to 0 and D is supported at two points, we obtain (up to isomorphism) loop algebras.

Like the Krichever–Novikov algebras, the Lax operator algebras possess an almost graded structure (Theorem 3.2 below) that generalizes the graded structure of affine algebras. A Lie algebra $\mathcal{V}$ is called *almost graded* if $\mathcal{V} = \bigoplus_i \mathcal{V}_i$, where $\dim \mathcal{V}_i < \infty$, and $[\mathcal{V}_i, \mathcal{V}_j] \subseteq \bigoplus_{k=i+j-k_0}^{k=i+j+k_1} \mathcal{V}_k$, where k_0 and k_1 do not depend on i and j.

The known results on central extensions of loop algebras remain valid for Lax operator algebras, while the technique of proof has undergone significant modifications. For graded current algebras the problem of classifying their central extensions is considered in a series of articles initiated by V. Kac [2], R. Moody [11], and H. Garland [1] (see [3, Comments to Ch. 7] for further references). For Krichever–Novikov algebras the problem was set up in [9] as a problem of classification of almost graded central extensions, and an outline of the proof was given. A complete classification of almost graded central extensions is given in [12, 13] for Krichever–Novikov algebras and in [10, 14] for Lax operator algebras (see [16] for a review).

The theory of Lax operators on Riemann surfaces proposed in [4] includes the construction of commuting hierarchies and the Hamiltonian theory of Lax and zero-curvature equations, the theory of Baker–Akhiezer functions, and an approach to the corresponding algebraic–geometric solutions. In [16] we addressed the problem of generalizing this theory to all Lax operator algebras and made the first step in this direction (Lemmas 4.1 and 4.2 below). In the present article we construct integrable hierarchies for such equations. Unfortunately, the result remains a conjecture for the symplectic algebra. We present all results in a new uniform way instead of treating every type of classical Lie algebras separately as it was earlier.

2. M-OPERATORS AND TIMES

Let Σ be a compact Riemann surface of genus g with two marked points P_+ and P_-. For $n \in \mathbb{N} \cup \{0\}$ we fix K additional points

$$W := \big\{\gamma_s \in \Sigma \setminus \{P_+, P_-\} \mid s = 1, \dots, K\big\} \tag{2.1}$$

(K will be specified in Section 5). To every point γ_s we assign a vector $\alpha_s \in \mathbb{C}^n$ given up to a scalar factor. The system

$$T := \big\{(\gamma_s, \alpha_s) \mid s = 1, \dots, K\big\} \tag{2.2}$$

is called *Tyurin data* below. These data are related to the moduli of holomorphic vector bundles over Σ. In particular, for generic values of (γ_s, α_s) with $\alpha_s \neq 0$ and $K = ng$ the Tyurin data parameterize semistable rank n and degree ng framed holomorphic vector bundles over Σ (see [17]).

In the following, let $\mathfrak{g}$ be one of the matrix algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$, or $\mathfrak{s}(n)$, where the last is the algebra of scalar matrices.

Let $M\colon \Sigma \to \mathfrak{g}$ be a meromorphic function. We require that M have the following expansion at a point $\gamma = \gamma_s$:

$$M = \frac{M_{-2}}{(z-z_\gamma)^2} + \frac{M_{-1}}{z-z_\gamma} + M_0 + \dots, \tag{2.3}$$

where z is a fixed local coordinate in the neighborhood of γ, z_γ is the coordinate of γ itself, $M_{-2}, M_{-1}, M_0, M_1, \ldots \in \mathfrak{g}$, and

$$M_{-2} = \lambda\alpha\alpha^{\mathrm{t}}\sigma, \qquad M_{-1} = (\alpha\mu^{\mathrm{t}} + \varepsilon\mu\alpha^{\mathrm{t}})\sigma, \tag{2.4}$$

where $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}^n$, σ is an $n\times n$ matrix, the upper t denotes the matrix transposition,

$$\begin{aligned} &\lambda \equiv 0, &&\varepsilon = 0, &&\sigma = \mathrm{id} &&\text{for } \ \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n),\\ &\lambda \equiv 0, &&\varepsilon = -1, &&\sigma = \mathrm{id} &&\text{for } \ \mathfrak{g} = \mathfrak{so}(n),\\ & &&\varepsilon = 1 && &&\text{for } \ \mathfrak{g} = \mathfrak{sp}(2n), \end{aligned} \tag{2.5}$$

and σ is a matrix of the symplectic form for $\mathfrak{g} = \mathfrak{sp}(2n)$. Here and below we omit the subscripts s and γ indicating the point γ, except for z_γ.

Every M-operator defines a dynamical system on the space of Tyurin data:

$$\dot z_\gamma = -\mu^{\mathrm{t}}\sigma\alpha, \qquad \dot\alpha = -M_0\alpha + \kappa\alpha, \tag{2.6}$$

where the dot denotes the time derivative. We comment on these equations in Lemma 4.1 and subsequent remarks (see below).

Lemma 2.1. *For any two M-operators M_a and M_b and the corresponding times the expression*

$$M_{ab} = \partial_a M_b - \partial_b M_a + [M_a, M_b]$$

is an M-operator too.

Proof. Let us verify that M_{ab} satisfies (2.4).

For an arbitrary $\mathfrak{g}$ from our list we have

$$M_a = \frac{\lambda_a\alpha\alpha^{\mathrm{t}}\sigma}{(z-z_\gamma)^2} + \frac{(\alpha\mu_a^{\mathrm{t}} + \varepsilon\mu_a\alpha^{\mathrm{t}})\sigma}{z-z_\gamma} + M_{0a} + \dots$$

and a similar expression for M_b, where λ_a, λ_b, ε, and $\sigma = \mathrm{id}$ are subject to (2.5). Next we have

$$\begin{aligned}\partial_a M_b = 2(\partial_a z_\gamma)\frac{\lambda_b\alpha\alpha^{\mathrm{t}}\sigma}{(z-z_\gamma)^3} &+ \frac{((\partial_a\lambda_b)\alpha\alpha^{\mathrm{t}} + \lambda_b\partial_a(\alpha\alpha^{\mathrm{t}}))\sigma + (\partial_a z_\gamma)M_{-1,b}}{(z-z_\gamma)^2}\\ &+ \frac{((\partial_a\alpha)\mu_b^{\mathrm{t}} + \varepsilon\mu_b(\partial_a\alpha^{\mathrm{t}}) + \alpha(\partial_a\mu_b^{\mathrm{t}}) + \varepsilon(\partial_a\mu_b)\alpha^{\mathrm{t}})\sigma}{z-z_\gamma} + \dots\end{aligned} \tag{2.7}$$

and a similar expression for $\partial_b M_a$.

For the commutator we have

$$\begin{aligned}[M_a, M_b] = {}&\frac{(1+\varepsilon^2)(\lambda_b\cdot\mu_a^{\mathrm{t}}\sigma\alpha - \lambda_a\cdot\mu_b^{\mathrm{t}}\sigma\alpha)\alpha\alpha^{\mathrm{t}}\sigma}{(z-z_\gamma)^3}\\ &+ \frac{(\lambda_a\partial_b - \lambda_b\partial_a)\alpha\alpha^{\mathrm{t}}\sigma + \lambda_{ab}\alpha\alpha^{\mathrm{t}}\sigma + (\mu_a^{\mathrm{t}}\sigma\alpha)M_{-1,b} - (\mu_b^{\mathrm{t}}\sigma\alpha)M_{-1,a}}{(z-z_\gamma)^2}\\ &+ \frac{((\partial_b\alpha)\mu_a^{\mathrm{t}} + \varepsilon\mu_a(\partial_b\alpha^{\mathrm{t}}))\sigma - ((\partial_a\alpha)\mu_b^{\mathrm{t}} + \varepsilon\mu_b(\partial_a\alpha^{\mathrm{t}}))\sigma}{z-z_\gamma} + \frac{(\alpha\mu_{ab}^{\mathrm{t}} + \varepsilon\mu_{ab}\alpha^{\mathrm{t}})\sigma}{z-z_\gamma} + \dots,\end{aligned} \tag{2.8}$$

where $\lambda_{ab} = 2\lambda_b\kappa_a - 2\lambda_a\kappa_b + \varepsilon(\mu_a^{\mathrm{t}}\sigma\mu_b - \mu_b^{\mathrm{t}}\sigma\mu_a)$ and $\mu_{ab} = \kappa_a\mu_b - \kappa_b\mu_a - \lambda_a M_{1b}\alpha + \lambda_b M_{1a}\alpha - M_{0b}\mu_a + M_{0a}\mu_b$. To obtain these relations, we used equations (2.6) and some additional relations, in particular, $\varepsilon\alpha^{\mathrm{t}}\sigma\mu = -\varepsilon^2\mu^{\mathrm{t}}\sigma\alpha$ and $\lambda\alpha^{\mathrm{t}}\sigma\alpha = 0$, which are fulfilled in all cases. When computing $[M_a, M_b]_{-2}$, we also used the relation

$$[M_{-1,a}, M_{-1,b}] = (\mu_a^{\mathrm{t}}\sigma\alpha)M_{-1,b} - (\mu_b^{\mathrm{t}}\sigma\alpha)M_{-1,a} + \varepsilon(\mu_a^{\mathrm{t}}\sigma\mu_b - \mu_b^{\mathrm{t}}\sigma\mu_a)\alpha\alpha^{\mathrm{t}}\sigma,$$

which can be verified using (2.4). To obtain $[M_a, M_b]_{-1}$ in the form (2.8), we substantially used the relation $M_{i,a}^{\mathrm{t}} = -\sigma M_{i,a}\sigma^{-1}$ for $\varepsilon \neq 0$ (which follows from (2.5)) and a similar relation for $M_{i,b}$.

Comparing (2.8) and (2.7) (and the corresponding relation for $\partial_b M_a$) and using (2.6), we obtain

$$M_{ab} = \frac{\widetilde{\lambda}_{ab}\alpha\alpha^{\mathrm{t}}\sigma}{(z - z_\gamma)^2} + \frac{(\alpha\widetilde{\mu}_{ab}^{\mathrm{t}} + \varepsilon\widetilde{\mu}_{ab}\alpha^{\mathrm{t}})\sigma}{z - z_\gamma} + \dots,$$

where $\widetilde{\lambda}_{ab} = \partial_a\lambda_b - \partial_b\lambda_a + \lambda_{ab}$ and $\widetilde{\mu}_{ab} = \partial_a\mu_b - \partial_b\mu_a + \mu_{ab}$. We observe that M_{ab} has the form (2.3), (2.4). In particular, the order -3 term vanishes because either $\lambda_a = \lambda_b = 0$ or $\varepsilon^2 = 1$ (which follows from (2.5)). $\square$

Remark. For this reason the expansions of the form (2.3) with the second-order pole are prohibited in the case of $\mathfrak{g} = \mathfrak{sl}(n)$. Indeed, this would require that $\varepsilon \neq 0$; hence $M_i^{\mathrm{t}} = -\sigma M_i\sigma^{-1}$, which holds only in the case of $\mathfrak{so}(n)$ or $\mathfrak{sp}(2n)$.

3. L-OPERATORS AND LAX OPERATOR ALGEBRAS

We define Lax operators (L-operators) as M-operators yielding trivial dynamics by (2.6). Thus, by definition, every L-operator is a meromorphic $\mathfrak{g}$-valued function L on Σ that is holomorphic outside $W \cup \{P_+, P_-\}$ and is such that, at a point $\gamma = \gamma_s$,

$$L = \frac{L_{-2}}{(z - z_\gamma)^2} + \frac{L_{-1}}{z - z_\gamma} + L_0 + \dots, \tag{3.1}$$

where z is a fixed local coordinate in the neighborhood of γ, z_γ is the coordinate of γ itself, $L_{-2}, L_{-1}, L_0, L_1, \ldots \in \mathfrak{g}$, and

$$L_{-2} = \nu\alpha\alpha^{\mathrm{t}}\sigma, \qquad L_{-1} = (\alpha\beta^{\mathrm{t}} + \varepsilon\beta\alpha^{\mathrm{t}})\sigma, \tag{3.2}$$

where $\nu \in \mathbb{C}$, $\beta \in \mathbb{C}^n$, σ is an $n \times n$ matrix,

$$\begin{aligned} &\nu \equiv 0, && \varepsilon = 0, && \sigma = \mathrm{id} && \text{for } \ \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \\ &\nu \equiv 0, && \varepsilon = -1, && \sigma = \mathrm{id} && \text{for } \ \mathfrak{g} = \mathfrak{so}(n), \\ & && \varepsilon = 1 && && \text{for } \ \mathfrak{g} = \mathfrak{sp}(2n), \end{aligned} \tag{3.3}$$

and σ is a matrix of the symplectic form for $\mathfrak{g} = \mathfrak{sp}(2n)$.

Further on, the requirement of the triviality of the dynamics (2.6) is expressed as

$$\beta^{\mathrm{t}}\sigma\alpha = 0, \qquad L_0\alpha = \kappa\alpha. \tag{3.4}$$

In addition, we assume that

$$\alpha^{\mathrm{t}}\alpha = 0 \qquad \text{for} \quad \mathfrak{g} = \mathfrak{so}(n) \tag{3.5}$$

and

$$\alpha^{\mathrm{t}}\sigma L_1\alpha = 0 \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sp}(2n). \tag{3.6}$$

Theorem 3.1 [10]. *The space $\overline{\mathfrak{g}}$ of Lax operators is a Lie algebra under the pointwise matrix commutator. For $\mathfrak{g} = \mathfrak{gl}(n)$ it is also an associative algebra under the pointwise matrix multiplication.*

The Lie algebra $\overline{\mathfrak{g}}$ is called a *Lax operator algebra*.

In [10] Theorem 3.1 is proven for every complex classic Lie algebra case by case by straightforward computation. In the context of the present paper, a large part of this computation can be omitted due to Lemma 2.1. This lemma implies that for every pair L_a, L_b of L-operators their bracket $[L_a, L_b]$ is an M-operator and as such is expressed as (2.3) and satisfies (2.4) and (2.5), where $\lambda = \lambda_{ab}$ and $\mu = \mu_{ab}$ are defined by (2.8). Interpreting this in terms of L, we obtain relations (3.1)–(3.3) for $[L_a, L_b]$ and expressions for the parameters ν and β of this bracket. We use these expressions and relations (3.4)–(3.6) for L_a and L_b to prove (3.4)–(3.6) for the bracket of these operators and complete the proof.

The algebra $\overline{\mathfrak{g}}$ depends on the choice of both the Tyurin parameters and the points P_+ and P_-, but we omit any indication of this dependence in our notation.

Consider $\overline{\mathfrak{gl}}(n)$ in more detail. In this case $L_{-2} = 0$ and $L_{-1} = \alpha\beta^{\mathrm{t}}$, where $\beta^{\mathrm{t}}\alpha = 0$ and $L_0\alpha = \kappa\alpha$. These constraints imply that the elements of the Lax operator algebra $\overline{\mathfrak{gl}}(n)$ can be considered as sections of the endomorphism bundle $\mathrm{End}(B)$, where B is the holomorphic vector bundle corresponding to the Tyurin data T.

The splitting $\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n)$ given by

$$X \mapsto \left(\frac{\mathrm{tr}(X)}{n} I_n,\ X - \frac{\mathrm{tr}(X)}{n} I_n\right), \tag{3.7}$$

where I_n is the $n \times n$ unit matrix, induces a corresponding splitting for $\overline{\mathfrak{gl}}(n)$:

$$\overline{\mathfrak{gl}}(n) = \overline{\mathfrak{s}}(n) \oplus \overline{\mathfrak{sl}}(n). \tag{3.8}$$

For $\overline{\mathfrak{s}}(n)$ all coefficients in (3.1) are scalar matrices. For this reason, the coefficients L_{-1} vanish for all $\gamma \in W$; hence, the elements of $\overline{\mathfrak{s}}(n)$ are holomorphic at W. Moreover, $L_{s,0}$, as a scalar matrix, has any α_s as an eigenvector. This means that, by definition,

$$\overline{\mathfrak{s}}(n) \cong \mathfrak{s}(n) \otimes \mathcal{A} \cong \mathcal{A} \tag{3.9}$$

as associative algebras.

Any Lax operator algebra $\overline{\mathfrak{g}}$ possesses an *almost graded structure* (see the Introduction for the definition).

Assume that all our marked points (including the points in W) are in generic position and $W \neq \varnothing$. Let us choose local coordinates $z_\pm$ at $P_\pm$ and z_s at γ_s, $s = 1, \dots, K$. Assume $\mathfrak{g}$ to be a simple Lie algebra from our list. For an arbitrary $m \in \mathbb{Z}$ consider the subspace

$$\begin{aligned}\overline{\mathfrak{g}}_m := \bigl\{L \in \overline{\mathfrak{g}} \mid \exists X_+, X_- \in \mathfrak{g} \text{ such that}\\ L(z_+) = X_+ z_+^m + O(z_+^{m+1}),\ L(z_-) = X_- z_-^{-m-g} + O(z_-^{-m-g+1})\bigr\}.\end{aligned} \tag{3.10}$$

For $\mathfrak{g} = \mathfrak{gl}(n)$ it is proven above that $\overline{\mathfrak{gl}}(n) = \overline{\mathfrak{sl}}(n) \oplus \mathcal{A} \cdot \mathrm{id}$, where $\mathcal{A}$ is the Krichever–Novikov function algebra. In this case we set

$$\overline{\mathfrak{gl}}(n)_m = \overline{\mathfrak{sl}}(n)_m \oplus \mathcal{A}_m \cdot \mathrm{id}, \tag{3.11}$$

where $\mathcal{A}_m$ is the corresponding homogeneous subspace for $\mathcal{A}$ [9]. If $W = \varnothing$, we are in the setup of Krichever–Novikov algebras and use the corresponding prescriptions [9, 15].

We call $\overline{\mathfrak{g}}_m$ a (*homogeneous*) *subspace of degree* m in $\overline{\mathfrak{g}}$.

Theorem 3.2 [10]. *The subspaces $\overline{\mathfrak{g}}_m$ give the structure of an almost graded Lie algebra on $\overline{\mathfrak{g}}$. More precisely,*

(1) $\dim \overline{\mathfrak{g}}_m = \dim \mathfrak{g}$;

(2) $\overline{\mathfrak{g}} = \bigoplus_{m\in\mathbb{Z}} \overline{\mathfrak{g}}_m$;

(3) $[\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_k] \subseteq \bigoplus_{h=m+k}^{m+k+M} \overline{\mathfrak{g}}_h$, *where $M = g$ for $\overline{\mathfrak{sl}}(n)$, $\overline{\mathfrak{so}}(n)$, and $\overline{\mathfrak{sp}}(2n)$ and $M = g+1$ for $\overline{\mathfrak{gl}}(n)$.*

Corollary 3.3. *Let X be an element of $\mathfrak{g}$. For each m there is a unique element X_m in $\overline{\mathfrak{g}}_m$ such that*

$$X_m = X z_+^m + O(z_+^{m+1}). \tag{3.12}$$

Proof. From statement (1) of Theorem 3.2, i.e., from $\dim \overline{\mathfrak{g}}_m = \dim \mathfrak{g}$, it follows that there is a unique linear combination of the basis elements such that (3.12) is true. $\square$

4. $\mathfrak{g}$-VALUED LAX EQUATIONS

In this section, we consider the consistency of Lax equations of the form

$$L_t = [L, M], \qquad L \in \overline{\mathfrak{g}}, \tag{4.1}$$

where L and M are an L-operator and an M-operator, respectively.

Following [4], let $\mathcal{L}^D = \{L \in \overline{\mathfrak{g}} \mid (L) + D \geq 0\}$ be a phase space of the Lax system, where $D = \sum_i m_i P_i$ is an effective divisor on Σ. Let the upper dot mean the time derivative.

Lemma 4.1. *At the weak singularity points, the equations for the main parts of L and M, which follow from* (4.1), *are fulfilled under the following sufficient conditions*:

$$\dot z_\gamma = -\mu^{\mathrm t}\sigma\alpha, \qquad \dot\alpha = -M_0\alpha + \kappa\alpha, \tag{4.2}$$

$$\begin{aligned}
\dot\beta &= M_0^{\mathrm t}\beta - L_0^{\mathrm t}\mu + \kappa_L\mu - \kappa\beta && \textit{for}\ \ \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n),\\
\dot\beta &= -M_0\beta + L_0\mu + \kappa_L\mu - \kappa\beta && \textit{for}\ \ \mathfrak{g} = \mathfrak{so}(n),\\
\dot\beta &= -M_0\beta + L_0\mu + \kappa_L\mu - \kappa\beta - \nu M_1\alpha + \lambda L_1\alpha && \textit{for}\ \ \mathfrak{g} = \mathfrak{sp}(2n),
\end{aligned} \tag{4.3}$$

$$\dot\nu = 2(\beta^{\mathrm t}\sigma\mu + \lambda\kappa_L - \nu\kappa) \qquad \textit{for}\ \ \mathfrak{g} = \mathfrak{sp}(2n), \tag{4.4}$$

where κ_L is defined by $L_0\alpha = \kappa_L\alpha$. Moreover, the conditions $\dot z_\gamma = -\mu^{\mathrm t}\sigma\alpha$ and (4.3) *are necessary.*

Proof. By a straightforward computation we have

$$\begin{aligned}
\dot L = 2\dot z_\gamma \frac{\nu\alpha\alpha^{\mathrm t}\sigma}{(z-z_\gamma)^3} &+ \frac{\dot\nu\alpha\alpha^{\mathrm t}\sigma + \nu\dot\alpha\alpha^{\mathrm t}\sigma + \nu\alpha\dot\alpha^{\mathrm t}\sigma + \dot z_\gamma(\alpha\beta^{\mathrm t} + \varepsilon\beta\alpha^{\mathrm t})\sigma}{(z-z_\gamma)^2}\\
&+ \frac{\dot\alpha\beta^{\mathrm t}\sigma + \alpha\dot\beta^{\mathrm t}\sigma + \varepsilon\dot\beta\alpha^{\mathrm t}\sigma + \varepsilon\beta\dot\alpha^{\mathrm t}\sigma}{z-z_\gamma} + (\dot L_0 - \dot z_\gamma L_1) + \dots .
\end{aligned} \tag{4.5}$$

Using (2.8) for $M_a = L$ and $M_b = M$, we obtain

$$\begin{aligned}
[L, M] = \frac{(1+\varepsilon)^2(-\nu\cdot\mu^{\mathrm t}\sigma\alpha)\alpha\alpha^{\mathrm t}\sigma}{(z-z_\gamma)^3} &+ \frac{\nu(\dot\alpha\alpha^{\mathrm t} + \alpha\dot\alpha^{\mathrm t})\sigma + \lambda_{ab}\alpha\alpha^{\mathrm t}\sigma - (\mu^{\mathrm t}\sigma\alpha)L_{-1}}{(z-z_\gamma)^2}\\
&+ \frac{(\dot\alpha\beta^{\mathrm t} + \varepsilon\beta\dot\alpha^{\mathrm t})\sigma + (\alpha\mu_{ab}^{\mathrm t} + \varepsilon\mu_{ab}\alpha^{\mathrm t})\sigma}{z-z_\gamma} + \dots .
\end{aligned} \tag{4.6}$$

Note that the second relation in (4.2) is used in deriving the last relation.

If $\nu \neq 0$ (i.e., $\mathfrak{g} = \mathfrak{sp}(2n)$), then the order -3 terms are equal if and only if

$$\dot z_\gamma = -\mu^{\mathrm t}\sigma\alpha.$$

If $\nu \equiv 0$, then the order -3 terms in (4.5) and (4.6) are both equal to 0. The order -2 terms are equal if and only if

$$\dot\nu\alpha\alpha^{\mathrm t}\sigma + \dot z_\gamma(\alpha\beta^{\mathrm t} + \varepsilon\beta\alpha^{\mathrm t})\sigma = \lambda_{ab}\alpha\alpha^{\mathrm t}\sigma - (\mu^{\mathrm t}\sigma\alpha)L_{-1}, \tag{4.7}$$

where λ_{ab} is defined after (2.8). By the previous relation we have

$$\dot\nu\alpha\alpha^{\mathrm t}\sigma = \lambda_{ab}\alpha\alpha^{\mathrm t}\sigma,$$

which is fulfilled if $\dot\nu = \lambda_{ab}$. Note that this relation is always trivial except for $\mathfrak g = \mathfrak{sp}(2n)$, in which case $\lambda_{ab} = 2(\lambda\kappa_L - \nu\kappa + 2\beta^{\mathrm t}\sigma\mu)$ and our relation coincides with (4.4).

In a similar way, comparing the terms of order -1 in relations (4.5) and (4.6), we observe that they are equal if $\dot\beta = \mu_{ab}$, where μ_{ab} are defined after (2.8). This gives relations (4.3). Since relations (2.8) themselves are derived under assumptions (4.2), we obtain the lemma. $\square$

Remark. Equation (4.3) for $\mathfrak g = \mathfrak{so}(n)$ follows from the corresponding equation for $\mathfrak{gl}(n)$ by the relations $M_0^{\mathrm t} = -M_0$ and $L_0^{\mathrm t} = -L_0$. This equation also follows from the equation for $\mathfrak g = \mathfrak{sp}(2n)$, with the corresponding replacement of the matrix σ, provided that $\lambda = \nu = 0$, which is indeed true for $\mathfrak g = \mathfrak{so}(n)$.

Remark. The second condition in (4.2) and conditions (4.3) are not necessary. The statement remains true if we take $\dot\alpha = -M_0\alpha$ in (4.2) and exclude the term $\kappa\beta$ from (4.3).

The next lemma shows that equations (4.3) and (4.4) can be discarded. Equations (4.2) are the most important ones. These are the equations of motion for the Tyurin parameters. They were substantially employed in [4]. Originally, the concept of moving Tyurin parameters was introduced in [6], where it was used for an effective solution of Kadomtsev–Petviashvili equations in certain cases.

Let $T_L\mathcal L^D$ denote the tangent space to $\mathcal L^D$ at a point L.

Lemma 4.2. $[L,M] \in T_L\mathcal L^D \Leftrightarrow ([L,M]) + D \ge 0$ *outside γ's, and equations* (4.2) *are fulfilled at every γ. In the case $\mathfrak g = \mathfrak{sp}(2n)$ this is true if $\alpha^{\mathrm t}\sigma M_1\alpha = 0$.*

Proof. In our proof we follow the lines of [4], where the lemma was formulated and proved for $\mathfrak g = \mathfrak{gl}(n)$.

Let z be a local coordinate in a (fixed) open set containing a weak singularity γ, and z_γ be the corresponding coordinate of γ.

We identify $T\mathcal L^D$ with the space $\mathcal T^D$ of all meromorphic $\mathfrak g$-valued functions T such that, at every weak singularity γ,

$$T = 2\dot z_\gamma\frac{\nu\alpha\alpha^{\mathrm t}\sigma}{(z-z_\gamma)^3} + \frac{\dot\nu\alpha\alpha^{\mathrm t}\sigma + \nu(\dot\alpha\alpha^{\mathrm t} + \alpha\dot\alpha^{\mathrm t})\sigma + \dot z_\gamma(\alpha\beta^{\mathrm t} + \varepsilon\beta\alpha^{\mathrm t})\sigma}{(z-z_\gamma)^2}$$
$$+ \frac{(\dot\alpha\beta^{\mathrm t} + \varepsilon\beta\dot\alpha^{\mathrm t} + \alpha\dot\beta^{\mathrm t} + \varepsilon\dot\beta\alpha^{\mathrm t})\sigma}{z-z_\gamma} + T_0 + \dots, \tag{4.8}$$

$$\dot\alpha^{\mathrm t}\sigma\beta + \alpha^{\mathrm t}\sigma\dot\beta = 0, \tag{4.9}$$

$$T_0\alpha = \kappa\dot\alpha + \dot\kappa\alpha - L_0\dot\alpha - \dot z_\gamma L_1\alpha, \tag{4.10}$$

where $\dot z_\gamma$ and $\dot\kappa$ are constants, $\dot\alpha$ and $\dot\beta$ are constant vectors satisfying relations (4.2), and the divisor of T outside the points γ is greater than or equal to $-D$.

Relation (4.8) is modeled on (4.5), which is obtained by the time derivation of (3.1). In particular,

$$T_0 = \dot L_0 - \dot z_\gamma L_1.$$

Together with the time derivation of (3.4) this gives (4.10). Thus, $T\mathcal L^D$ is embedded in $\mathcal T^D$. Let us check the coincidence of dimensions of these spaces. This can be done in quite a uniform way, so we show it in the most difficult case of $\mathfrak{g} = \mathfrak{sp}(2n)$. We have $(T) + \widetilde{D} \geq 0$, where $\widetilde{D} = D + 3\sum\gamma$ and $\deg\widetilde{D} = \deg D + 3K$. By the Riemann–Roch theorem $\dim\{T \mid (T) + \widetilde{D} \geq 0\} = (\dim\mathfrak{g})(\deg D + 3K - g + 1)$. The elements of $\mathcal{T}^D$ are distinguished in the space $\{T \mid (T) + \widetilde{D} \geq 0\}$ by the following relations. First, at every point γ we have

$$\begin{aligned} T_{-3} &= 2\dot{z}_\gamma \nu\alpha\alpha^{\mathrm{t}}\sigma, \\ T_{-2} &= \dot{\nu}\alpha\alpha^{\mathrm{t}}\sigma + \nu(\dot{\alpha}\alpha^{\mathrm{t}} + \alpha\dot{\alpha}^{\mathrm{t}})\sigma + \dot{z}_\gamma(\alpha\beta^{\mathrm{t}} + \beta\alpha^{\mathrm{t}})\sigma, \\ T_{-1} &= (\dot{\alpha}\beta^{\mathrm{t}} + \alpha\dot{\beta}^{\mathrm{t}} + \dot{\beta}\alpha^{\mathrm{t}} + \beta\dot{\alpha}^{\mathrm{t}})\sigma. \end{aligned} \tag{4.11}$$

Since the elements on the left-hand side belong to $\mathfrak{g}$, (4.11) gives $3\dim\mathfrak{g}$ relations. Taking account of (4.9) and (4.10) gives $2n+1$ relations, and (4.2) gives another $2n+1$ relations. Thus, we have $3\dim\mathfrak{g} + 4n + 2$ relations. These relations contain $4n+2$ free parameters $\dot{z}_\gamma$, $\dot{\nu}$, $\dot{\alpha}$, and $\dot{\beta}$. Thus, we actually obtained $3\dim\mathfrak{g}$ relations at every point γ, and the number of these points is K; hence, we have $3(\dim\mathfrak{g})K$ relations. We see that $\dim\mathcal{T}^D = (\dim\mathfrak{g})(\deg D - g + 1)$.

But $\mathcal{L}^D$ has the same dimension. We can count this in quite a similar way or make use of Theorem 3.2. Assume that we are in the two-point situation, i.e., $D = -m_+P_+ + (m_- + g)P_-$, where $m_- > m_+$ for simplicity. Then $\mathcal{L}^D = \mathfrak{g}_{m_+} \oplus \ldots \oplus \mathfrak{g}_{m_-}$. By Theorem 3.2, $\dim\mathcal{L}^D = (\dim\mathfrak{g})(m_- - m_+ + 1)$, which is exactly equal to $(\dim\mathfrak{g})(\deg D - g + 1)$. We conclude that $\dim\mathcal{T}^D = \dim T_L\mathcal{L}^D$; hence, these linear spaces coincide.

Next we prove that if L and M are as above, then $[L, M]$ possesses properties (4.8)–(4.10), i.e., it belongs to $\mathcal{T}^D$. The proof is straightforward again. For example, let us show (4.10). Denote the degree zero term $[L, M]_0$ of the commutator by T_0. Then in the case of $\mathfrak{g} = \mathfrak{gl}(n)$ we find by computation

$$T_0\alpha = \alpha(\beta^{\mathrm{t}}M_1 - \mu^{\mathrm{t}}L_1\alpha) + (L_0 - \kappa)M_0\alpha + L_1(\mu^{\mathrm{t}}\alpha). \tag{4.12}$$

If we replace $\mu^{\mathrm{t}}\alpha$ with $-\dot{z}_\gamma$, $M_0\alpha$ with $-\dot{\alpha}$, and denote $\beta^{\mathrm{t}}M_1 - \mu^{\mathrm{t}}L_1\alpha$ by $\dot{\kappa}$, we obtain (4.10). For other types of $\mathfrak{g}$ the expression for $T_0\alpha$ is more complicated, and we use relations (3.4)–(3.6) to identify it with (4.10). In the case of $\mathfrak{g} = \mathfrak{sp}(2n)$ we also make use of the relation $\alpha^{\mathrm{t}}\sigma M_1\alpha = 0$. □

Lemma 4.2 directly implies that if $([L, M]) + D \geq 0$ outside γ's and the equations of moving poles are fulfilled, the Lax equation (4.1) is consistent.

5. COMMUTING HIERARCHIES

For a divisor $D = \sum m_iP_i$ define a divisor $\widetilde{D} = D + \delta\sum_{s=1}^K \gamma_s$, where

$$K = \begin{cases} ng, & \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1), \mathfrak{sp}(2n), \\ (n+1)g, & \mathfrak{g} = \mathfrak{sl}(n), \end{cases}$$

and

$$\delta = \begin{cases} 1, & \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1), \\ 2, & \mathfrak{g} = \mathfrak{sp}(2n). \end{cases}$$

Let us define N^D as a space of M-operators such that $(M) + \widetilde{D} \geq 0$ (and $\mu^{\mathrm{t}}\alpha = 0$ for $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{g} = \mathfrak{so}(2n)$).

Lemma 5.1. $\dim N^D = (\dim\mathfrak{g})(\deg D + 1)$.

Proof. We compute $\dim N^D$ by the Riemann–Roch theorem taking account of additional relations at the points γ. These are relations determining M_{-2} and M_{-1}. The number of these relations at every point γ is equal to $\delta\dim\mathfrak{g}$. We also have free parameters λ and μ. Let r be the number of these parameters for a fixed γ and r_μ be equal to 1 if the relations $\mu^{\mathrm{t}}\alpha = 0$ are included in the definition of N^D and to 0 if they are not (for all γ simultaneously). We can think that at every γ there are $\delta\dim\mathfrak{g}-r+r_\mu$ relations.

Let us write K in the form $K=lg$, where l is n or $n+1$, depending on the type of the classical Lie algebra. We have

$$\begin{aligned}\dim N^D &= (\dim\mathfrak{g})(\deg D+\delta lg-g+1)-(\delta\dim\mathfrak{g}-r+r_\mu)lg\\ &= (\dim\mathfrak{g})(\deg D+1)-(\dim\mathfrak{g}-(r-r_\mu)l)g. \end{aligned}\tag{5.1}$$

Next, we verify that

$$\dim\mathfrak{g}-(r-r_\mu)l=0. \tag{5.2}$$

Indeed, for $\mathfrak{g}=\mathfrak{gl}(n)$ we have $r=n$, $r_\mu=0$, and $l=n$; hence, $(r-r_\mu)l=n^2$. If $\mathfrak{g}=\mathfrak{sl}(n)$, then $r=n$, $r_\mu=1$, $l=n+1$, and $(r-r_\mu)l=n^2-1$. If $\mathfrak{g}=\mathfrak{so}(2n+1)$ or $\mathfrak{g}=\mathfrak{sp}(2n)$, then $r=2n+1$, $r_\mu=0$, $l=n$, and $(r-r_\mu)l=(2n+1)n$. Recall that in the case of $\mathfrak{g}=\mathfrak{sp}(2n)$ the value $r=2n+1$ is the number of parameters coming from λ and μ, while in all other cases, only from μ. Finally, if $\mathfrak{g}=\mathfrak{so}(2n)$, then $r=2n$, $r_\mu=1$, $l=n$, and $(r-r_\mu)l=(2n-1)n$. In all cases (5.2) is true. □

Following [4], let us fix a point $P_0\in\Sigma$ and local coordinates w_0 and w_i in the neighborhoods of the points P_0 and P_i. Our next goal is to define gauge invariant functions M_a that satisfy the assumptions of Lemma 4.2. Let us define a as a triple

$$a=(P_i,k,m),\qquad k>0,\quad m>-m_i, \tag{5.3}$$

where k and m are integers, with $k\equiv 1 \pmod 2$ for $\mathfrak{g}=\mathfrak{so}(n)$ and $\mathfrak{g}=\mathfrak{sp}(2n)$.

By Lemma 5.1, for generic L there is a unique $\mathfrak{g}$-valued function M_a such that

(i) M_a is an M-operator;

(ii) outside the points γ it has a pole at the point P_i only, and

$$M_a(q)=w_i^{-m}L^n(q)+O(1);$$

i.e., the singular parts of M_a and $w_i^{-m}L^n$ coincide;

(iii) M_a is normalized by the condition $M_a(P_0)=0$.

Theorem 5.2. *For $\mathfrak{g}=\mathfrak{gl}(n),\mathfrak{sl}(n),\mathfrak{so}(2n),\mathfrak{so}(2n+1)$ the equations*

$$\partial_aL=[L,M_a],\qquad \partial_a=\frac{\partial}{\partial t_a},$$

define a hierarchy of commuting flows on an open subset of $\mathcal{L}^D$.

For $\mathfrak{g}=\mathfrak{gl}(n)$ the theorem is formulated and proved in [4].

Proof. It follows from (ii) that $([L,M_a])+D\ge 0$; hence, by Lemma 4.2, $[L,M_a]\in T_L\mathcal{L}^D$ and the equation $\partial_aL=[L,M_a]$ defines a flow on $\mathcal{L}^D$.

To prove the commutativity of such flows, it is sufficient to verify that $M_{ab}=\partial_aM_b-\partial_bM_a+[M_a,M_b]=0$ identically. By Lemma 2.1, M_{ab} is an M-operator. We prove that this M-operator is regular at the points of the divisor D. By Lemma 5.1 the space of such operators has the same dimension as $\mathfrak{g}$. Due to (iii) we obtain $M_{ab}=0$.

Let us prove that M_{ab} is regular at the points of the divisor D. We repeat here the corresponding part of the proof of [4, Theorem 2.1]. First, assume that the indices a and b correspond to the same point P_i; i.e., $a=(P_i,n,m)$ and $b=(P_i,n',m')$. Denote $M_a-w^{-m}L^n$ by M_a^- and $M_b-w^{-m'}L^{n'}$ by M_b^-; then, by (ii), M_a^- and M_b^- are regular in the neighborhood of P_i. We have

$$\partial_aM_b=w^{-m'}\partial_aL^{n'}+\partial_aM_b^-=w^{-m'}[L^{n'},M_a]+\partial_aM_b^-=w^{-m'}[L^{n'},M_a^-]+\partial_aM_b^-$$

and

$$[M_a, M_b] = [M_a^- + w^{-m}L^n, M_b^- + w^{-m'}L^{n'}] = w^{-m}[L^n, M_b^-] - w^{-m'}[L^{n'}, M_a^-] + [M_a^-, M_b^-].$$

Hence $M_{ab} = \partial_a M_b^- - \partial_b M_a^- + [M_a^-, M_b^-]$ at the point P_i, which is a regular expression at this point. By definition M_{ab} is also regular at the other points of D.

The proof is similar in the case when a and b correspond to different points of D. $\square$

The proof of Theorem 5.2 is basically valid for $\mathfrak{g} = \mathfrak{sp}(2n)$ too, except for the reference to Lemma 4.2, which holds only if $\alpha^{\mathrm{t}}\sigma M_1\alpha = 0$ in this case. The problem that remains is to prove the last property for the operators M_a.

Conjecture. Under appropriate assumptions Theorem 5.2 is also true for $\mathfrak{g} = \mathfrak{sp}(2n)$.

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