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ELLIPTIC AFFINE LIE ALGEBRAS

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In [1] I. M. Krichever and S. P. Novikov introduced a natural generalization of the Kac–Moody loop (current) algebras and considered their central extensions. Specifically, let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, Γ be a compact algebraic curve over $\mathbf{C}$ with two distinguished points $P_{\pm}$, and let $\mathcal{A}^{\Gamma}$ denote the algebra of the meromorphic functions on Γ that are holomorphic off $P_{\pm}$. The algebra of meromorphic loops (currents) on the curve Γ is defined to be the following Lie algebra G:

$$G = \mathfrak{g} \otimes_{\mathbf{C}} \mathcal{A}^{\Gamma}. \tag{0.1}$$

If Γ is a curve of genus 0, then $\mathcal{A}^{\Gamma}$ is the algebra of Laurent polynomials in one variable, and G is isomorphic to one of the Kac–Moody loop algebras. In this paper we consider the case of curves of genus 1. Following the traditional pattern of construction of affine Lie algebras [3, 4], in Sec. 1 we study 2-dimensional extensions of the algebras (0.1), in which one of the dimensions corresponds to the center, while the second corresponds to some vector field e on Γ. We consider some distinguished extensions, denoted below by $\tilde{G} = \tilde{G}(e)$.

In Sec. 2 we consider invariant symmetric bilinear forms on G. On an affine Lie algebra there is a canonical invariant form specified by the condition of orthogonality of the Laurent monomials with the sum of degrees different from zero. For algebras of the type (0.1) that condition does not admit a straightforward generalization in view of the absence of a grading (one has only a structure of quasi-graded Lie algebra [1]). We show in Sec. 2 that one can replace it by the condition of extendability of an invariant form to the 2-dimensional extension $\tilde{G}(e)$. If e has m zeros in the domain $\Gamma\backslash\{P_{\pm}\}$, then on $\tilde{G}(e)$ there exist m + 1 independent invariant symmetric bilinear forms.

In Sec. 3 we establish a correspondence between loop algebras of the type (0.1) and complex Coxeter crystallographic groups (CCC-groups for short), introduced and classified in [5, 6]. Specification of a CCC-group and of $P_{\pm}$ determines the algebra (0.1) uniquely up to an isomorphism of quasi-graded algebras (Theorem 3.1). One can conjecture that the CCC-group is connected with the Weyl group of the algebra $\tilde{G}(e)$.

In Sec. 4 we consider the orbits of the adjoint action of a loop group. There we develop the ideas of the papers [4, 7] and we exhibit a connection between the orbits of the adjoint action and the monodromy equation on the elliptic curve Γ. We obtain a sufficient condition for membership of two elements in the same orbit in terms of the monodromy group of that equation (Theorem 4.1). We also consider the connection between orbits and CCC-groups.

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1. Algebras of Meromorphic Loops and Their Extensions

Let Γ be an analytic curve with periods 2ω and $2\omega'$. Then $\mathcal{A}^\Gamma$ can be represented as a space of elliptic functions holomorphic off the points $z_\pm = \pm z_0$, and one can introduce in it a basis $\{A_i\}$, where i runs through the half-integers [1]:

$$A_i(z) = \frac{\sigma^{i-1/2}(z - z_0)\,\sigma(z + 2iz_0)}{\sigma^{i+1/2}(z + z_0)\,\sigma((2i+1)z_0)}\,\sigma^{i+1/2}(2z_0), \quad i \neq -\frac{1}{2}. \tag{1.1}$$

Here $\sigma(z)$ is the Weierstrass σ-function. The function $A_{-1/2}$ can be chosen to be of the form

$$A_{-1/2}(z) = \frac{\sigma^2(z)\,\sigma^2(2z_0)}{\sigma(z + z_0)\,\sigma(z - z_0)\,\sigma^2(z_0)}. \tag{1.2}$$

The central extensions of the algebra $\mathcal{A}^\Gamma$ are described by means of cocycles of the form [1]

$$\gamma(A_i, A_j) = \frac{1}{2\pi i}\oint_C A_i\,dA_j. \tag{1.3}$$

Consider the class of contours that are homologous to a small contour which surrounds one (any) of the points $z_\pm$. We denote this class by C_0 and call it the class of separating contours. The separating contours, and only them, enjoy the property that the corresponding cocycle γ is local in the sense that $\gamma(A_i, A_j) = 0$ for $|i + j| > 1$ [1]. In what follows we shall consider only this cocycle, i.e., we put $C = C_0$.

Let $(\cdot, \cdot)$ denote the Killing–Cartan form of the finite-dimensional Lie algebra $\mathfrak{g}$. Then on the Lie algebra G one can define a cocycle $\hat\gamma$ by the rule

$$\hat\gamma(xA_i, yA_j) = (x, y)\,\gamma(A_i, A_j) \tag{1.4}$$

(where $x, y \in \mathfrak{g}$), and then use it to define the central extension

$$\hat G = G \oplus \mathbf{C}c \tag{1.5}$$

of G, in which the commutator is specified by the relations

$$[xA_i, yA_j] = [x, y]\,A_iA_j + \hat\gamma(xA_i, yA_j)\,c, \quad [xA_i, c] = 0 \text{ for all i.} \tag{1.6}$$

In the Kac–Moody theory one considers extensions of the algebra $\hat G$ by means of the operator $z\partial/\partial z$. An analogue for the present situation is the following assertion.

Proposition 1.1. Let e be a meromorphic vector field on Γ that is holomorphic off $z_\pm$. Then the space

$$\tilde G = G \oplus \mathbf{C}c \oplus \mathbf{C}e \tag{1.7}$$

with the operation $[\cdot, \cdot]$ specified by relations (1.6) and the relations

$$[e, xA_i] = -[xA_i, e] = x\,(eA_i),\ [e, c] = 0 \tag{1.8}$$

is an $(m + 4)$-graded Lie algebra, where m is the number of zeros of e in the domain $\Gamma\backslash\{z\pm\}$ (here eA_i is meant as the standard action of a vector field on a function).

Proof. Let us verify the Jacobi identity. A straightforward computation shows that

$$[e, [xA_i, yA_j]] = [x, y]\,(e\,(A_iA_j)),$$
$$[[e, xA_i], yA_j] + [xA_i, [e, yA_j]] = [x, y]((eA_i)\,A_j + A_i\,(eA_j)) + (x, y)(\gamma(eA_i, A_j) + \gamma(A_i, eA_j))$$

for all i and j.

Therefore, a necessary and sufficient condition for the fulfillment of the Jacobi identity is

$$\gamma(eA_i, A_j) + \gamma(A_i, eA_j) = 0, \quad i, j = -\infty, \infty. \tag{1.9}$$

Let us show that this condition is indeed satisfied. The general form of a meromorphic vector field e on Γ that is holomorphic off $z_\pm$ is [1]

$$e(z) = E(z)\frac{\partial}{\partial z}, \tag{1.10}$$

where $E(z) \in \mathcal{A}^\Gamma$. Using (1.10) and the definition (1.3), we obtain relation (1.9) by a simple integration by parts on a closed contour.

We assign to the vector field e of the Lie algebra $\tilde{G}$ the grade 0. Suppose e has a zero of multiplicity p at z_+ and a pole of multiplicity q at z_-. A count of zeros and poles shows that eA_j can be written as a linear combination of the functions A_k with $k = j + p - n, \ldots, j + q + 1$, where $n = 0, 1$ (see [1, Sec. 3]). The number of indices in the indicated linear combination is $q - p + 3 + n$. But by the theorem on the number of zeros and poles of a meromorphic function, $q - p = m$. The proposition is proved.

Let us examine the structure of $\tilde{G}$ in more detail. Let $e_1, \ldots, e_{n-1}, h_1, \ldots, h_{n-1}, f_1, \ldots, f_{n-1}$ be canonical generators of the Lie algebra $\mathfrak{g}$, $A = (A_{ij})$ be the Cartan matrix of the affine Lie algebra corresponding to $\mathfrak{g}$ and the identity automorphism of its Dynkin scheme [3]. Set $e_n = f_\theta A_{3/2}$, $f_n = e_\theta A_{-3/2}$, where, as customary in the theory of affine Lie algebras, θ is the highest root of the algebra $\mathfrak{g}$, e_θ is the corresponding root vector, $f_\theta = e_{-\theta}$, $2h_\theta = [e_\theta, f_\theta]$, and $\gamma = \gamma(A_{3/2}, A_{-3/2})$ is the value of the cocycle (1.3). Also, set $h_{\pm 1/2} = h_\theta A_{\pm 1/2}(z)$.

Proposition 1.2. 1°. The elements e_i, f_i, h_i $(i = 1, \ldots, n)$, $h_{\pm 1/2}$, e, c generate the lie algebra $\tilde{G}$.

2°. The following relations hold:

$$[h_i, h_j] = [h_i, h_{\pm 1/2}] = [h_{1/2}, h_{-1/2}] = 0,$$
$$[h_i, e_j] = A_{ij}e_j,\ [h_i, f_j] = -A_{ij}f_j,$$
$$(\mathrm{ad}\, e_i)^{-A_{ij}+1}e_j = (\mathrm{ad}\, f_i)^{-A_{ij}+1}f_j = 0$$

for all $i, j = 1, \ldots, n$, where $h_n = h_{1/2}$.

$[e_i, f_j] = \delta_{ij}h_j$ except for the case $i = j = n$; $[e_n, f_n] = -\alpha h_{1/2} - \beta h_{-1/2} + \gamma c$, where α and β are determined from the relation $A_{3/2}A_{-3/2} = \alpha + \beta A_{-1/2}$, which holds in the algebra $\mathcal{A}^\Gamma$ [1].

Further,

$$[e, e_i] = [e, f_i] = [e, h_i] = [e, h_{1/2}] = 0 \quad (i = 1, \ldots, n-1).$$

And finally,

$$[h_{-1/2}[h_{-1/2}, e_j^{(\pm)}]] = \frac{1}{2}A_{nj}(\pm \varkappa [e_j^{(\pm)}f_1^{l_1}\ldots f_{n-1}^{l_{n-1}}f_n] \pm 2\lambda [h_{-1/2}, e_j^{(\pm)}] \mp$$
$$\mp \nu [e_j^{(\pm)}e_1^{l_1}\ldots e_{n-1}^{l_{n-1}}e_n]) + A_{nj}^2\mu e_j^{(\pm)},\ j = 1, \ldots, n-1,$$

where κ, λ, μ, ν are determined from the relations [1] $A_{-1/2}^2 = \kappa A_{-3/2} + \lambda A_{-1/2} + \mu + \nu A_{3/2}$; $e_j^{(+)} = e_j$, $e_j^{(-)} = f_j$, $\theta = \ell_1\alpha_1 + \ldots + \ell_{n-1}\alpha_{n-1}$ is the decomposition of the highest root θ into simple roots, and $[\ldots]$ denotes the chain of commutators of the form $[\cdot, [\cdot, [\cdot, [\cdot\cdot] \ldots\cdot]\ \cdot]\ \cdot]$.

In addition to the relations listed above there are two more, which define the action of e on e_n, f_n, and $h_{-1/2}$, whose explicit form we omit. They follow from structure decompositions of the form

$$eA_j = \sum_s r_j^s A_{j+s} + r_j, \tag{1.11}$$

the precise formulation of which can be found in [1, Sec. 3].

The problem of the completeness of the listed relations remains open for the moment.

The proof of 1.2.1° reduces to the observation that the functions $A_{\pm 1/2}$, $A_{\pm 3/2}$ generate the algebra $\mathcal{A}^\Gamma$, which in turn is obvious. Assertion 1.2.2° follows from the Cartan relations in the Lie algebra $\mathfrak{g}$ and the structure formulas for the algebra $\mathcal{A}^\Gamma$ [1].

A Cartan subalgebra of $\tilde{G}$ is defined to be a subalgebra of the form

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus Z \oplus \mathbf{C}e, \tag{1.12}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We remark that in $\tilde{G}$ there is another maximal commutative subalgebra, not conjugate to it, namely $\mathfrak{h}_{-1/2} = \mathfrak{h} \oplus A_{-1/2}\mathfrak{h} \oplus Z$.

For a special choice of the vector field e the algebra $\tilde{G}$ possesses an analogue of a Borel subalgebra. Let $e = A_{3/2}(z)\,\partial/\partial z$, where $A_{3/2}(z)$ is given by formula (1.1) (then $e(z) = z(1 + O(z))\,\partial/\partial z$ in the neighborhood of the point z_+). Let $\mathcal{A}_\pm^\Gamma$ denote the subrings of $\mathcal{A}^\Gamma$ generated by the functions A_j, $\pm j \geq 3/2$.

Proposition 1.3. 1°. In the case of the vector field e chosen as above the adjoint representation of the subalgebra $\tilde{\mathfrak{h}}$ on $\tilde{G}$ leaves invariant the subspace $\mathfrak{g} \otimes \mathcal{A}_+^\Gamma$, and its matrix in the basis $\{x_\alpha A_j \mid \alpha \in R(\mathfrak{g}),\ j \geq 3/2\}$ of this subspace is triangular (here $R(\mathfrak{g})$ is the root system of the Lie algebra $\mathfrak{g}$).

2°. The diagonal elements of the matrix of the adjoint representation of the subalgebra $\tilde{\mathfrak{h}}$ coincide with the affine roots $\alpha \in R(\mathfrak{g})$ (including $\alpha = 0$).

3°. The matrix of the adjoint action of the element $e = A_{3/2}\,\partial/\partial z$ has the minimal number of diagonals above the principal diagonal among the matrices of all fields that have the property 1°.

Proof. 1° follows from formulas (1.11) for the action of vector fields on functions [1], according to which

$$eA_i = (i - 1/2)\,A_i + \ldots,\ i \geqslant 3/2,$$

where the dots denote a sum of a finite number of terms with $j > i$.

2°. Let $x_\alpha \in \mathfrak{g}$ be a root vector belonging to the root $\alpha \in R(\mathfrak{g})$. Then for $i = n + 1/2$, $x_\alpha A_i$ is a weight vector of weight $\alpha + n$ of the subalgebra $\tilde{\mathfrak{h}}$ modulo a finite sum of loops involving A_j with $j > i$ ($x_\alpha \in \mathfrak{h}$ for $\alpha = 0$).

3°. In view of Proposition 1.1, this assertion reduces to the field e having one zero in the domain $\Gamma\backslash\{z_\pm\}$ (the only field with no zeros in that domain, $\partial/\partial z$, does not leave invariant the subspace $\mathfrak{g} \otimes \mathcal{A}_+^\Gamma$). The proposition is proved.

Remark 1. It is not difficult to show that $e = A_{3/2}(z)\,\partial/\partial z$ is the unique (up to proportionality) vector field on which the minimum of the number of diagonals of the matrix of the adjoint representation of the subalgebra $\tilde{\mathfrak{h}}$ is realized. In fact, let p [q] be the order of the field at the point z_+ [resp., z_-]. If to the field e corresponds the minimal number of diagonals, then, in view of the proof of assertion 3° of Proposition 1.3, e has one zero in the domain $\Gamma\backslash\{z_\pm\}$, and hence $p + q + 1 = 0$. Suppose $j \geq 3/2$. The orders of the function eA_j at the points z_+ and z_- equal $p_+ = j + p - 3/2$, and respectively $p_- = -j + q - 3/2 = -j - p - 5/2$. For sufficiently large j, $p_+ > 0 > p_-$ and $|p_-| > |p_+|$. Therefore, $|p_+|$ and $|p_-|$ are the orders at the point z_+ of the terms with the minimal and respectively the maximal indices in the decomposition $eA_j = \sum \lambda_i A_i$. The index k of the minimal term is found from the relation $p_+ = k - 1/2$, i.e., $k = j + p - 1$. From the condition that the action of the field e be triangular it follows that $k = j$, i.e., $p = 1$. Finally $q = -p - 1 = -2$. The vector field e with orders $p = 1$, $q = -2$ at the points z_+, z_- is uniquely determined and coincides with the field $A_{3/2}\,\partial/\partial z$ [1].

2. Invariant Symmetric Forms

On the Lie algebra G there are infinitely many linearly independent invariant symmetric bilinear forms. Indeed, to every meromorphic differential $d\omega$ on Γ that is holomorphic off the points $z_\pm$ there corresponds the symmetric bilinear form

$$B_\omega(xA_i, yA_j) = \frac{(x, y)}{2\pi i} \oint_{C_0} A_i A_j\, d\omega. \tag{2.1}$$

LEMMA 2.1. The symmetric bilinear form B_ω is G-invariant.

Proof. By definition,

$$B_\omega([xA_i, zA_k], yA_j) = B_\omega([x, z]A_iA_k, yA_j) = \frac{([x, z], y)}{2\pi i}\oint A_iA_jA_k d\omega,$$

$$B_\omega(xA_i, [zA_k, yA_j]) = B_\omega(xA_i, [z, y]A_kA_j) = \frac{(x, [z, y])}{2\pi i}\oint A_iA_jA_k d\omega.$$

Therefore, the G-invariance of the form B_ω follows from the $\mathfrak{g}$-invariance of the form $(\cdot, \cdot)$. The lemma is proved.

In this section we show that the condition of prolongation to the algebra $\tilde{G}$ allows one to single-out a unique invariant form or a finite number of such forms depending on the number of zeros of the vector field e in the domain $\Gamma\backslash\{z_\pm\}$.

Proposition 2.1. For any meromorphic vector field $e = e(z)$ on Γ that is holomorphic and has no zeros off the points $z_\pm$ there is exactly one invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\tilde{G}$ with the following properties:

1) $\langle xA, c\rangle = \langle xA, e\rangle = 0$ for all $x \in \mathfrak{g}$, $A \in \mathcal{A}^\Gamma$;

2) $\langle c, e\rangle = 1$;

3) the form $\langle\cdot, \cdot\rangle$ on the algebra G admits the representation $\langle xA, yB\rangle = (x, y)\langle A,B\rangle_\Gamma$, where $x, y \in \mathfrak{g}$, $A, B \in \mathcal{A}^\Gamma$, and $\langle\cdot, \cdot\rangle_\Gamma$ is a symmetric bilinear form on $\mathcal{A}^\Gamma$ with the following properties:

4) $\langle AB, C\rangle_\Gamma = \langle A, BC\rangle_\Gamma$ for all $A, B, C \in \mathcal{A}^\Gamma$;

5) $\langle eA, C\rangle = -\langle A, eC\rangle$ for all $A, C \in \mathcal{A}^\Gamma$.

Moreover, $\langle\cdot, \cdot\rangle_\Gamma$ is necessarily of the form

$$\langle A, B\rangle_\Gamma = \frac{1}{2\pi i}\oint_{C_0} A(z)B(z)\frac{dz}{E(z)}, \tag{2.2}$$

where E(z) is determined from the relation (1.10).

Let us prove the following lemma.

LEMMA 2.2 Under conditions 1)-5) of Proposition 2.1, the form $\langle\cdot, \cdot\rangle$ is $\tilde{G}$-invariant if and only if the relation

$$\langle eA, B\rangle_\Gamma = \gamma(A, B)$$

holds for all $A, B \in \mathcal{A}^\Gamma$, where γ is defined by formula (1.3).

Proof. Pick arbitrary $x, y, z \in \mathcal{A}^\Gamma$, $a_i, b_i \in \mathbf{C}$ ($i = 1, 2, 3$). Denote $X = xA + a_1c + b_1e$, $Y = yB + a_2c + b_2e$, $Z = zC + a_3c + b_3e$. A computation shows that

$$\langle [X, Y], Z\rangle = ([x, y], z)\langle AB, C\rangle_\Gamma + b_1(y, z)\langle eB, C\rangle_\Gamma - \\ - b_2(x, z)\langle eA, C\rangle_\Gamma + b_3(x, y)\gamma(A, B), \tag{2.3}$$

$$\langle X, [Y, Z]\rangle = (x, [y, z])\langle A, BC\rangle_\Gamma + b_2(x, z)\langle A, eC\rangle_\Gamma - \\ - b_3(x, y)\langle A, eB\rangle_\Gamma + b_1(y, z)\gamma(B, C). \tag{2.4}$$

The invariance of the form $\langle\cdot, \cdot\rangle$ means that the left-hand sides of the equalities (2.3) and (2.4) coincide. In view of the $\mathfrak{g}$-invariance of the form $(\cdot, \cdot)$ and properties 1)-5) the right-hand sides of equalities (2.3) and (2.4) coincide if and only if the relation of Lemma 2.2 holds for all $A, B \in \mathcal{A}^\Gamma$. The lemma is proved.

The proof of Proposition 2.1 follows readily from Lemma 2.2 upon observing that in view of the definition (1.3) of the cocycle γ there is a unique symmetric bilinear form on $\mathcal{A}^\Gamma$ satisfying the condition of Lemma 2.2, namely, the form (2.4).

The unique vector field with no zeros in the domain $\Gamma\backslash\{z_\pm\}$ is

$$e(z) = \frac{\partial}{\partial z}. \tag{2.5}$$

By Proposition 2.1, to e there corresponds the unique invariant symmetric form on $\tilde{G}$

$$\langle xA + a_1c + b_1e, yB + a_2c + b_2e \rangle = a_1b_2 + b_1a_2 + \frac{(x, y)}{2\pi i} \oint_{C_0} AB\, dz. \tag{2.6}$$

In the general case, when the field e is of the form $e = E(z)\, \partial/\partial z$ and the function $E(z)$ has m zeros in the domain $\Gamma \setminus \{z_\pm\}$ there are m + 1 invariant symmetric bilinear forms

$$\langle xA + a_1c + b_1e, yB + a_2c + b_2e \rangle = a_1b_2 + b_1a_2 + \frac{(x, y)}{2\pi i} \oint_{C} AB \frac{dz}{E(z)}, \tag{2.7}$$

where C runs through the homology classes of separating cycles on the curve Γ with the points $z_\pm$ and the zeros of the function E removed.

3. Elliptic Affine Lie Algebras and Crystallographic Groups

Turning now to the discussion of the connections between elliptic affine Lie algebras and crystallographic groups we introduce the notions of a Bernstein–Shvartsman system and of a Coxeter crystallographic group (CCC-group [5, 6]).

Let $R(\mathfrak{g})$ be the root system of the Lie algebra $\mathfrak{g}$, W be its Weyl group, and $l = \operatorname{rank} \mathfrak{g}$.

Definition 3.1. A Bernstein–Shvartsman system is a collection consisting of two l-dimensional W-modules M_1 and M_2, two full-rank lattices $T_1 \subset M_1$ and $T \subset M_2$, an operator $A: M_2 \to M_1$, and a complex number τ, $\operatorname{Im} \tau > 0$, such that

1°. the representation of W in the space M_i (i = 1, 2) is equivalent to the standard representation of W in $\mathbf{C}^l$;

2°. the semi-direct product $W_i = WT_i$ is an affine Weyl group in the space M_i (i = 1, 2);

3°. A is an isomorphism of W-modules and $AT_2 \subset T_1$;

4°. $\tau x = A^{-1}x$ for all $x \in M_1$.

The operator A is uniquely determined by condition 3° up to an integral constant factor. Let us make the convention that A is the "minimal" operator with these properties.

The Bernstein–Shvartsman systems were introduced in [5, 6] under the name of bases; see also [10]. According to the classification obtained in [5, 6], the pair of lattices T_1, T_2 can be of two types: $T_1 \simeq T_2 \simeq L(S)$ and $T_1 \simeq L(S)$, $T_2 \simeq L(S^\vee)$, where S is a finite root system, $S^\vee$ is the dual root system, and $L(S)$, $L(S^\vee)$ are the lattices generated by S and $S^\vee$, respectively. Here we shall consider only systems of the first type:

$$T_1 \cong T_2 \cong L(S). \tag{3.1}$$

For given A and τ, in the space $M_1 \oplus M_2$ there is a unique complex structure for which condition 4° of Definition 3.1 is satisfied [10]. We shall consider $M_1 \oplus M_2$ as being endowed with that complex structure.

Definition 3.2 [5]. The CCC-group corresponding to a Bernstein–Shvartsman system satisfying condition (3.1) is the group generated by the reflections in the hyperplanes $\tau\alpha(x) = m$, where $x \in M_1 \oplus M_2$ and α is an arbitrary root of the form $\alpha = \bar{\alpha} + n$ ($\bar{\alpha} \in S$), $m, n \in \mathbf{Z}$.

As an abstract group the CCC-group is equal to the semi-direct product of W and the lattice $T_1 \oplus T_2$ [6]. As a crystallographic group the CCC-group is determined by a class of Bernstein–Shvartsman systems with modularly-equivalent numbers τ (and all the other parameters identical) [5, 6].

Let us show how one can attach a Bernstein–Shvartsman system and its CCC-group to a loop algebra G.

As it follows from results of [2], the dual space G^* of the loop algebra G consists of the $\mathfrak{g}^*$-valued meromorphic differentials on Γ that are holomorphic off the points $z_\pm$. Let $H \subset G^*$ be the subspace of the $\mathfrak{h}^*$-valued differentials that are holomorphic everywhere on Γ, where $\mathfrak{h}^*$ is the dual space of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Q denote the lattice generated in $\mathfrak{h}^*$ by the roots of the Lie algebra $\mathfrak{g}$. Now consider the lattice generated by the periods of the differentials of the form $\lambda dz \in H$ with $\lambda \in Q$ (it is isomorphic to $Q \oplus L$, where L is the lattice of periods of the elliptic curve Γ), and the space generated by this lattice (it is isomorphic to $\mathfrak{h}^* \oplus_Z L$). Set $L = L_1 \oplus L_2$, where L_1 and L_2 are the lattices of the a-periods and respectively the b-periods of the elliptic curve Γ.

As the modules M_1 and M_2 we take $M_i = \mathfrak{h}^* \otimes_Z L_i$, i = 1, 2. The action of the Weyl group on $\mathfrak{h}^*$ carries over to M_1 and M_2. Set $\tau = \omega'/\omega$, where ω and ω' are the half-periods of the curve Γ. Define the operator A: $M_2 \to M_1$ by the condition

$$A^{-1}: \lambda \otimes l \to \lambda \otimes (\tau l) \qquad (\lambda \in \mathfrak{h}^*, \quad l \in L_1).$$

It is readily seen that A commutes with the action of W, i.e., we indeed produced a Bernstein–Shvartsman system. To a modular transformation of the number τ there corresponds a change of the canonical basis of cycles on the curve Γ, so that our constructioin uniquely associates a CCC-group to the Lie algebra G.

THEOREM 3.1. There exists a bijective correspondence between the loop algebras of the form (0.1) on elliptic curves, given up to an isomorphism of quasi-graded Lie algebras, and the collections consisting of a CCC-group that satisfies condition (3.1), and a complex number z_0, given modulo the numbers 1, τ (where τ corresponds to the CCC-group).

Proof. We showed above that to the loop algebra G there corresponds in unique manner a CCC-group satisfying condition (3.1).

Conversely, a class of equivalent CCC-groups that satisfy condition (3.1) is specified by a finite root system and a complex number τ, $\operatorname{Im}\tau > 0$, given up to a modular transformation. From these data one recovers in unique manner an elliptic curve Γ with periods 1, τ. Also, from the given z_0 one uniquely recovers the pair of distinguished points $z_\pm = \pm z_0$ on Γ. The elliptic curve together with the pair of distinguished points uniquely define a loop algebra of the form (0.1). The theorem is proved.

4. Orbits

The elements X = X(z) of the algebra G will be referred to as elliptic loops (currents) in the algebra $\mathfrak{g}$.

Definition 4.1. A group loop (loop in the group $\exp \mathfrak{g}$) is defined to be a map g: $\Gamma \to \exp \mathfrak{g}$ that is holomorphic everywhere, except possibly for the points $\pm z_0$ and the zeros of the vector field e.

The group loops form a group, denoted T_G. Clearly, T_G contains all functions of the form exp X, $X \in G$.

In this section we develop the ideas of the papers [4, 7], where a connection is exhibited between the adjoint action of a loop group of an affine Lie algebra and the gauge transformations of the monodromy equations corresponding to loops in the algebra. Therein the orbit of an element is given by the monodromy operator of the corresponding equation, regarded to within conjugation.

Let $g \in T_G$. Let us define the operator of the adjoint action, Ad g.

Definition 4.2. If the vector field e has no zeros off the points $\pm z_0$, we put

$$\begin{aligned}(Adg)(ac + be + X) = ac + be + gXg^{-1} - bEg'g^{-1} + \\ + (\langle Eg^{-1}g', X\rangle - \frac{b}{2}\langle Eg'g^{-1}, Eg'g^{-1}\rangle)\, c,\end{aligned} \tag{4.1}$$

where $X \in G$, a, $b \in \mathbf{C}$, $e = E(z)\,\partial/\partial z$. We wish to emphasize that in the case where the vector field e has no zeros in $\Gamma\backslash\{\pm z_0\}$ on the algebra $\tilde{G}$ there is only one invariant symmetric bilinear form $\langle\cdot, \cdot\rangle$. The motivation behind Definition 4.2 follows from Theorem 3.1.5 of [4], where formula (4.1) is a consequence of the definition $\operatorname{Ad}\exp X = \exp \operatorname{ad} X$.

Now let us consider the case where the field e has m zeros $z_1, \ldots, z_m$ (m > 0) on $\Gamma\backslash\{\pm z_0\}$. Then, as shown in Sec. 2, there exist m + 1 independent invariant symmetric bilinear forms $\langle\cdot, \cdot\rangle_l$ (l = 1, ..., m + 1) on $\tilde{G}$, which correspond to the homology classes of separating cycles on $\Gamma\backslash\{\pm z_0, z_1, \ldots, z_m\}$. The right-hand side of (4.1) depends in essential manner on the choice of one of the forms $\langle\cdot, \cdot\rangle_l$. In order to get a well-defined adjoint action, we introduce independent central elements $c_1, \ldots, c_{m+1}$, which correspond to one and the same cocycle $\hat{\gamma}$ (Sec. 1).

Definition 4.3. If the vector field e has m zeros off the points $\pm z_0$, we set

$$(\mathrm{Ad}\, g)\,(a^l c_l + be + X) = a^l c_l + be + gXg^{-1} - bEg'g^{-1} + \\ + (\langle Eg^{-1}g',\, X\rangle_l - \frac{b}{2}\langle Eg'g^{-1}, Eg'g^{-1}\rangle_l)c_l,$$

where X, a, b, e have the same meaning as in Definition 4.2, and summation is carried out over the repeated index l. All ensuing arguments will use Definition 4.2, keeping in mind that they all generalize in obvious manner to the mase m > 0. Here we should mention that the group loop g, constructed below in the proof of Theorem 4.1, is analytic off the points $\pm z_0$ for m = 0, and has singularities at the points $z_1, \ldots, z_m$ for m > 0, and therefore it is indeed necessary to consider such loops.

To each element X = X(z) + ac + be we associate the corresponding monodromy equation on the curve Γ:

$$bE(z)\, u'(z) = u(z)\, X(z). \tag{4.2}$$

The point z = 0 is a regular point of the loop X(z), and so we may consider the germ of the solution of Eq. (4.2) with initial condition u(0) = I (I is the identity matrix). We term this solution fundamental.

Pick an arbitrary contour $z = z(\tau)$ on the curve Γ, which starts and terminates at the point z = 0, and continue the fundamental solution analytically along this contour. This results in a solution of the same equation, but with a different initial condition

$$u_1(0) = g_{z(\tau)}, \text{where } g_{z(\tau)} \in \exp \mathfrak{g}.$$

Definition 4.4. The element $g_{z(\tau)} \in \exp \mathfrak{g}$ is called the monodromy operator of Eq. (4.2) along the contour $z = z(\tau)$.

Definition 4.5. The group generated by the monodromy operators along all closed contours with distinguished point z = 0 on the curve Γ is called the monodromy group of Eq. (4.1) on Γ.

From the general theory of monodromy equations [8, 9] it is known that the monodromy operator depends only on the homotopy class of the contour $z = z(\tau)$ on the elliptic curve Γ with the singularities of Eq. (4.2) removed; these singularities include the poles of the elliptic loop X(z), the zeros of the vector field $e = E(z)\,\partial/\partial z$ – among them, the points $\pm z_0$. As one can easily show, the monodromy group is finitely generated, with generators M_ω, $M_{\omega'}$, $M_\pm$, $M_1, \ldots, M_m$ and relations

$$M_\omega M_{\omega'} M_\omega^{-1} M_{\omega'}^{-1} = M_+ M_- M_1 \cdot \ldots \cdot M_m, \tag{4.3}$$

where M_ω, $M_{\omega'}$ are the monodromy operators along the basis cycles of the curve Γ, $M_\pm$ are the monodromy operators at the points $\pm z_0$, and $M_1, \ldots, M_m$ are the monodromy operators at the zeros of the field e. By obvious homotopy considerations, the order of the factors in the right-hand side of relation (4.3) is immaterial, and consequently the operators $M_\pm$, $M_1, \ldots,$ M_m pairwise commute. An effective construction of the monodromy at singular points can be found in [9].

The Theorem 4.1 given below allows one to specify an orbit of the adjoint action in the Lie algebra $\tilde{G}$ by means of a finite number of parameters and is a generalization of Theorem 3.2.10 (ii) of [4]. A preliminary observation is that for an element $X = X(z) + a^\ell c_\ell + be$ the expressions

$$\langle X, X\rangle_l = 2a^l b + \langle X(z), X(z)\rangle_l$$

are invariants of the adjoint action for any $l = 1, \ldots, m + 1$ and any value of b. This follows from the invariance of the forms $\langle\cdot, \cdot\rangle_\ell$ and Definition 4.3. Let $X = X(z) + a_1^l c_l + b_1 e$, $Y = Y(z) + a_2^\ell c_\ell + b_2 e$, where X(z) and Y(z) are elliptic loops in the Lie algebra $\mathfrak{g}$.

Let G_X and G_Y be the monodromy groups of Eq. (4.2) for the loops X(z) and Y(z), respectively.

THEOREM 4.1. If the groups G_X and G_Y are conjugate with respect to the group $\exp \mathfrak{g}$ and the following relations are satisfied:

$$2a_1^l b_1 + \langle X(z), X(z)\rangle_l = 2a_2^l b_2 + \langle Y(z), Y(z)\rangle_l \tag{4.4}$$

for all $\ell = 1, \ldots, m+1$ and

$$b_1 = b_2 \neq 0, \tag{4.5}$$

then the elements X and Y belong to the same orbit of the adjoint representation in the Lie algebra G.

Proof. In the proof of Theorem 3.1.5 of [4] it is shown that when the relations (4.4), (4.5) are satisfied, the elements X and Y are conjugate if and only if the loops X(z) and Y(z) are connected by a gauge transformation

$$Y(z) = g(z)\, X(z)\, g(x)^{-1} - bE(z)\, g'(z)\, g(z)^{-1}. \tag{4.6}$$

In fact, suppose X(z) and Y(z) are related as in (4.6). Then from (4.4) and (4.5) it follows that

$$a_2^l = a_1^l + \langle X, g^{-1}Eg'\rangle_l - \frac{b_1}{2}\langle Eg'g^{-1}, Eg'g^{-1}\rangle_l. \tag{4.7}$$

By Definition 4.3, Y = (Ad g)X. Conversely, Definition 4.3 and relations (4.4), (4.5) imply (4.7).

Thus, we need to show that the conjugacy of G_X and G_Y implies (4.6).

Suppose that there is a $g_0 \in \exp \mathfrak{g}$ such that $g_0 G_X g_0^{-1} = G_Y$. Let u_X and u_Y be analytic continuations of fundamental solutions of the monodromy equations for the loops X(z) and Y(z), respectively. By analogy with the proof of Proposition 3.2.5 of [4], we put $g(z) = u_Y^{-1}(z)\, g_0 u_X(z)$. Whereas $u_X(z)$ and $u_Y(z)$ are not uniquely defined, g(z) is a single-valued function. Indeed, let us check this last assertion, for example, for a circuit along a small contour γ surrounding the point z_0. Let $M_+(X)$ and $M_-(Y)$ denote the monodromy operators corresponding to the loops X(z) and Y(z). Parametrize the contour γ by a segment [0, T], setting $g(\tau) = g(\gamma(\tau))$ $(0 \le \tau \le T)$. Then

$$g(\tau + T) = u_Y^{-1}(\tau + T)\, g_0 u_X(\tau + T) = u_Y^{-1}(\tau)\, M_+(Y)^{-1} g_0 M_+(X)\, u_X(\tau).$$

Since $M_+(Y) = g_0 M_+(X)\, g_0^{-1}$, we have

$$g(\tau + T) = u_Y^{-1}(\tau)\,(g_0 M_+(X)^{-1} g_0^{-1})\, g_0 M_+(X)\, u_X(\tau) = u_Y^{-1}(\tau)\, g_0 u_X(\tau) = g(\tau),$$

i.e., $g(\tau)$ is periodic on the contour γ, and similarly on any other closed contour on the curve Γ. From here it follows, in an almost standard manner, that g(z) is path-independent. In fact, let γ_1 and γ_2 be two paths from the point 0 to the point z. For fixed z let $g(\gamma)$ and $u(\gamma)$ denote the values of g(z) and u(z) corresponding to a path γ from 0 to z. By what we proved above,

$$g(\gamma_1\gamma_2^{-1}) = g(0). \tag{4.8}$$

Since $u_X(0) = u_Y(0) = I$, one has $g(0) = g_0$. On the other hand, $g(\gamma_1\gamma_2^{-1}) = u_Y(\gamma_1\gamma_2^{-1})^{-1} \times g_0 u_X(\gamma_1\gamma_2^{-1})$. Using the known multiplicative properties of the solutions of the monodromy equation with respect to multiplication of paths, we have $g(\gamma_1\gamma_2^{-1}) = u_Y(\gamma_1\gamma_2^{-1}) g_0\, u_X(\gamma_1\gamma_2^{-1}) = u_Y(\gamma_2) u_Y(\gamma_1^{-1})\, g_0 u_X(\gamma_1)\, u_X(\gamma_2)^{-1}$. In view of (4.8), we get $u_Y(\gamma_1)^{-1}\, g_0 u_X(\gamma_1) = u_Y(\gamma_2)^{-1} \times g_0 u_Y(\gamma_2)$, i.e., $g(\gamma_1) = g(\gamma_2)$. The single-valuedness of the function g(z) is thus established.

Next, from the conjugacy of the monodromy operators it follows that g(z) is double periodic.

Finally, by analogy with Theorem 3.2.5 (iii) of [4], we obtain

$$gX(z)\, g^{-1} - bEg'g^{-1} = u_Y^{-1} g_0\,(u_X X(z))\, u_X^{-1} g_0^{-1} u_Y + \\ + bEu_Y^{-1} u_Y' u_Y^{-1} g_0 u_X u_X^{-1} g_0^{-1} u_Y - bEu_Y^{-1} g_0 u_X' u_X^{-1} g_0^{-1} u_Y.$$

By Eq. (4.2), we can replace $u_X X(z)$ in first term by bEu'_X, and replace $bEu_Y^{-1}u_Y'$ in the second term by Y(z), and thus get $gX(z)g^{-1} - bEg'g^{-1} = Y(z)$, as needed. The theorem is proved.

Let us examine the connection between the CCC-group constructed in Sec. 3 and orbits. Denote the CCC-group by W_C. Let us define an action of W_C on the space of pairs $\{\ln M_\omega, \ln M_{\omega'}\}$, where M_ω and $M_{\omega'}$ are semisimple elements. To this end we represent M_ω and $M_{\omega'}$ in the form $M_\omega = g_\omega h_\omega g_\omega^{-1}$, $M_{\omega'} = g_{\omega'} h_{\omega'} g_{\omega'}^{-1}$, where $h_\omega, h_{\omega'} \in \mathfrak{h}$. By definition, we have $\ln M_\omega = g_\omega \ln h_\omega g_\omega^{-1}$, $\ln M_{\omega'} = g_{\omega'} \ln h_{\omega'} g_{\omega'}^{-1}$, where the elements $\ln h_\omega, \ln h_{\omega'} \in \mathfrak{h}$ are defined up to translations by elements of the lattice Q generated by a root system in the Cartan subalgebra $\mathfrak{h}$. On the space of pairs $\{\ln h_\omega, \ln h_{\omega'}\}$ there is the standard action of the group W_C, defined in Sec. 3: if $w_C \in W_C, q_1 \oplus q_2$, where $w \in W$, $q_1 \oplus q_2 \in Q \oplus Q$, and $T_{q_1 \oplus q_2}$ is the translation operator by the element $q_1 \oplus q_2$, and the action is given by the rule

$$w_C (\ln h_\omega \oplus \ln h_{\omega'}) = w ((\ln h_\omega + q_1) \oplus (\ln h_{\omega'} + q_2)). \tag{4.9}$$

Proposition 4.1. 1°. Two pairs of semisimple elements of the form $\{M_\omega, M_{\omega'}\}$ are conjugate with respect to the group $\exp \mathfrak{g}$ if and only if the corresponding pairs $\{\ln h_\omega, \ln h_{\omega'}\}$ belong to the same W_C-orbit in the space $\mathfrak{h} \oplus \mathfrak{h}$.

2°. The set of all pairs $\{\ln h_\omega, \ln h_{\omega'}\}$ corresponding to a given orbit of the adjoint action of the loop group is a W_C-orbit in the space $\mathfrak{h} \oplus \mathfrak{h}$.

Proof. By Theorem 4.1, assertions 1 and 2 are equivalent. Two pairs of semisimple elements $\{\ln h_\omega^{(i)}, \ln h_{\omega'}^{(i)}\}$ (i = 1, 2) are W_C-conjugate if and only if there exists a $w \in \exp \mathfrak{g}$ such that $(\mathrm{Ad}\, w)\, \mathfrak{h} = \mathfrak{h}$, and for some choice of the value of the logarithm, $(\mathrm{Ad}\, w) \times \ln h_\omega^{(1)} = \ln_\omega^{(2)}$, $(\mathrm{Ad}\, w) \ln h_{\omega'}^{(1)} = \ln h_{\omega'}^{(2)}$. Then $w h_\omega^{(1)} w^{-1} = h_\omega^{(2)}$ and $w h_{\omega'}^{(1)} w^{-1} = h_{\omega'}^{(2)}$, which obviously implies the conjugacy of the pairs $\{M_\omega^{(i)}, M_{\omega'}^{(i)}\}$ with respect to $\exp \mathfrak{g}$.

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