# Infinitary Substructural Logics

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- Reading formulae as **resources** [Girard 1987], however, refutes contraction and weakening.
- Non-commutative (without exchange) resources are **actions.**

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- We shall use **sequent calculi.** A sequent is an object of the form $A_1, \ldots, A_n \vdash B$, meaning that the formula $B$ is entailed by $A_1, \ldots, A_n$ (in the given order and quantity).

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- In the sequent formalism, structural rules are represented as follows:

$$\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ \mathsf{C} \qquad \frac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ \mathsf{W} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathsf{E}$$

(Where $\Gamma$ and $\Delta$ are sequences of formulae.)

# Logical Rules

- The usual sequent calculus rules (in intuitionistic logic) for implication are as follows [Gentzen 1934]:

$$\frac{\Pi \vdash A \quad \Gamma, B \vdash C}{\Gamma, \Pi, A \to B \vdash C} \to\mathsf{L} \qquad \frac{A, \Pi \vdash B}{\Pi \vdash A \to B} \to\mathsf{R}$$

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- In the non-commutative setting (no exchange rule), implication is **directed,** left or right:

$$\frac{\Pi \vdash A \quad \Gamma, B, \Delta \vdash C}{\Gamma, \Pi, A \setminus B, \Delta \vdash C} \setminus\mathsf{L} \qquad \frac{A, \Pi \vdash B}{\Pi \vdash A \setminus B} \setminus\mathsf{R}$$

$$\frac{\Pi \vdash A \quad \Gamma, B, \Delta \vdash C}{\Gamma, B / A, \Pi, \Delta \vdash C} /\mathsf{L} \qquad \frac{\Pi, A \vdash B}{\Pi \vdash B / A} /\mathsf{R}$$

  - In Lambek's tradition, directed implications are denoted as **divisions:** $\setminus, /$.

# Logical Rules

- Similarly, in the absence of contraction we have two versions of conjunction: $\otimes$ (**multiplicative conjunction,** or **product**) and $\wedge$ (**additive conjunction,** or **meet**):

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C} \otimes\mathsf{L} \qquad \frac{\Pi \vdash A \quad \Delta \vdash B}{\Pi, \Delta \vdash A \otimes B} \otimes\mathsf{R}$$

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- In the resouce understanding, $A \otimes B$ means a bundle of two resources (in the given order), while $A \wedge B$ is one resource which can act both as $A$ and as $B$.
- Contraction makes $A \otimes B$ and $A \wedge B$ equivalent.

# The Lambek Calculus

- Our minimal associative substructural logic, in the multiplicative language $(\backslash, /, \otimes)$ is the **Lambek calculus** **L**:

$$\frac{}{A \vdash A}\ \mathsf{Id} \qquad \frac{\Pi \vdash A \quad \Gamma, A, \Delta \vdash C}{\Gamma, \Pi, \Delta \vdash C}\ \mathsf{Cut}$$

$$\frac{\Pi \vdash A \quad \Gamma, B, \Delta \vdash C}{\Gamma, \Pi, A \backslash B, \Delta \vdash C}\ \backslash\mathsf{L} \qquad \frac{A, \Pi \vdash B}{\Pi \vdash A \backslash B}\ \backslash\mathsf{R} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}\ \otimes\mathsf{L}$$

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- The Cut rule is eliminable.
- In the original formulation of **L**, left-hand sides were required to be non-empty (Lambek's non-emptiness restriction); we do not impose this restiction.

# The Full Lambek Calculus

- The **multiplicative-additive,** or **full** Lambek calculus **FL** is obtained from **L** by adding additive conjunction and disjunction and constants:

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \quad \frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \wedge\mathsf{L} \qquad \frac{\Pi \vdash A \quad \Pi \vdash B}{\Pi \vdash A \wedge B} \wedge\mathsf{L}$$

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$$\frac{}{\Gamma, 0, \Delta \vdash C}\ 0\mathsf{L} \qquad \frac{\Gamma, \Delta \vdash C}{\Gamma, 1, \Delta \vdash C}\ 1\mathsf{L} \qquad \frac{}{\vdash 1}\ 1\mathsf{R}$$

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- **FL** can be further extended by structural rules: e.g., $\mathbf{FL}_{\mathrm{ec}}$ is the non-distributive relevant logic.

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- Syntactic types are associated to words (lexemes) in the **lexicon** of a Lambek categorial grammar.
- Having $np$ (noun phrase), $n$ (common noun), and $s$ (sentence) as basic (primitive) types, we can start building our lexicon:

$$\begin{array}{rr} \textit{John}, \textit{Mary} \rhd np & \textit{loves} \rhd (np \setminus s) / np \\ \textit{girl} \rhd n & \textit{smiles} \rhd np \setminus s \\ \textit{the} \rhd np / n & \textit{charmingly} \rhd (np \setminus s) \setminus (np \setminus s) \end{array}$$

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- Grammatical correctness of sentences is validated by derivability of the corresponding sequents in the Lambek calculus:

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| *Mary smiles charmingly* | $np, np \setminus s, (np \setminus s) \setminus (np \setminus s) \vdash s$ |
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- The right rules ($\setminus$R, /R) are used for *hypothetical reasoning* which models dependent clauses:

  *the girl whom John loves*
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- Word order matters (at least in English), so we do not have exchange.

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- Example from Old Georgian [Michaelis, Kracht 1996]:

| *govel-i* | *igi* | *sisxl-i* | *saxl-isa-j* | *m-is* | *Saul-is-isa-j* |
|---|---|---|---|---|---|
| all-Nom | Art=Nom | blood-Nom | house-Gen-Nom | Art-Gen | Saul-Gen-Gen-Nom |

*'all the blood of the house of Saul'*

For $k$ stacked $np$'s, the number of genitive suffixes (*isa*, *is*) is bounded by $\frac{k(k-1)}{2}$, which is not semilinear.

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  - More traditionally: $\textit{ball} \rhd n$; $\textit{big}, \textit{red} \rhd n \,/\, n$.
  - The alternative way allows more fine-grained control over adjectives. For example, with

    $$\textit{tall} \rhd a_R; \quad \textit{American} \rhd a_A; \quad \textit{student} \rhd (a_R^* \otimes a_A^*) \setminus n,$$

    one validates *tall American student* and invalidates *American tall student*.

# Kleene Star

- In the aforementioned examples, the following right rule for Kleene star is used:

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  - In fact, it is an infinitary series of rules, for each $n \geq 0$.
- Example: derivation for *John, Bill, Mary, and Suzy:*

$$\dfrac{np \vdash np \qquad \dfrac{\dfrac{np \vdash np \quad np \vdash np}{np, np \vdash np^*}\, *\mathsf{R}_2 \qquad np \vdash np}{np, np, np \vdash np \otimes np^*}\, \otimes\mathsf{R} \qquad np \vdash np}{np, np, np, (np \otimes np^*) \setminus np \vdash np}\, \setminus\mathsf{L}}{np, np, np, ((np \otimes np^*) \setminus np) \,/\, np, np \vdash np}\, /\mathsf{L}$$

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- Infinite *upward paths*, however, are still disallowed (**well-foundedness**).

# Infinitary Action Logic

- **FL** extended with $*\mathsf{R}_n$ and $*\mathsf{L}_\omega$ is **infinitary action logic** $\mathbf{ACT}^\omega$ [Palka 2007].

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  - $\mathbf{ACT}^\omega$ is "**PDL** without propositions" (but with internalised implications).

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- Example: 'pure induction' [Pratt 1991]

$$(a \setminus a)^* \vdash a \setminus a$$

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- For the infinitary system $\mathbf{ACT}^\omega$, even enumerability is not guaranteed, since here proofs can be infinite objects.

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- In order to prove undecidability of $\mathbf{ACT}^{\omega}_{\mathrm{e}}$ [K. 2020], we reduce the **non-halting** (infinite run) problem.
- As our computation model, we shall use counter (Minsky) machines, following [Lincoln et al. 1992].

# Counter Machines

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- Commands (instructions):

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| $\textsc{inc}(p, \mathtt{a}_i, q)$ | $p \to q; \quad \mathtt{a}_i := \mathtt{a}_i + 1;$ |
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- $k = 3$ is sufficient for a Turing-complete computational model: three-counter machines compute all computable functions.
- **Non-halting:** given a three-counter machine, decide whether it runs forever (e.g., on zero input).

# Encoding Counter Machines in $\mathbf{ACT}_{\mathrm{e}}^{\omega}$

- Each **configuration** of a three-counter machine is encoded by the following sequence of variables:

$$q, \underbrace{\mathtt{a}_1, \ldots, \mathtt{a}_1}_{n_1}, \underbrace{\mathtt{a}_2, \ldots, \mathtt{a}_2}_{n_2}, \underbrace{\mathtt{a}_3, \ldots, \mathtt{a}_3}_{n_3}$$

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- We also introduce 'pseudo-states' $z_i$ for zero-checks on $\mathtt{a}_i$.
- The following formula is a regular expression for an arbitrary configuration (where in the pseudo-state $z_i$ the corresponding $\mathtt{a}_i$ should be zero):

$$D = \big((q_1 \vee \ldots \vee q_m) \otimes \mathtt{a}_1^* \otimes \mathtt{a}_2^* \otimes \mathtt{a}_3^*\big) \\ \vee (z_1 \otimes \mathtt{a}_2^* \otimes \mathtt{a}_3^*) \vee (z_2 \otimes \mathtt{a}_1^* \otimes \mathtt{a}_3^*) \vee (z_3 \otimes \mathtt{a}_1^* \otimes \mathtt{a}_2^*).$$

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- For each instruction $I$ we introduce the corresponding formula $A_I$:

$$A_{\mathrm{INC}(p,\mathtt{a}_i,q)} = p \backslash (q \otimes \mathtt{a}_i)$$

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- $E = \bigwedge_I A_I \wedge \bigwedge_{i=1}^{3} (z_i \setminus z_i)$ is the formula encoding the machine.

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- $E = \bigwedge_I A_I \wedge \bigwedge_{i=1}^{3} (z_i \setminus z_i)$ is the formula encoding the machine.

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*The machine runs infinitely if and only if the following sequent is provable in* $\mathbf{ACT}_{\mathrm{e}}^{\omega}$*:*

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- Here $E^*$, by the $*\mathsf{L}_\omega$ rule, instantiates $E^n$ for **any** $n$.
- The "if" direction (from proof to computation) is not so straightforward and requires some infinitary proof analysis.

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  - $\Sigma_1^0$ / $\Pi_1^0$ means one unrestricted $\exists$ / $\forall$ quantifier over natural numbers.
- $\mathbf{ACT}_{\mathrm{e}}^{\omega}$ is therefore **$\Pi_1^0$-hard,** but we still lack an upper bound (maybe it is even harder?).

# Non-Well-Founded Proofs

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- In this formulation, denoted by $\mathbf{ACT}^{\infty}_{(\mathrm{e})}$, the rules for Kleene star are replaced by the following ones:

$$\frac{\Gamma, \Delta \vdash C \quad \Gamma, A, A^*, \Delta \vdash C}{\Gamma, A^*, \Delta \vdash C}\ {*}\mathsf{L} \qquad \frac{}{\vdash A^*}\ {*}\mathsf{R}_0 \qquad \frac{\Pi \vdash A \quad \Delta \vdash A^*}{\Pi, \Delta \vdash A^*}\ {*}\mathsf{R},$$

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- Now all rules are finitary, but we allow **infinite paths** from the root upwards the tree.
- In the presence of Cut, unrestricted usage of such constructions immediately yields absurd:

$$\dfrac{\dfrac{p \vdash p \qquad \dfrac{\vdots}{p \vdash q}}{p \vdash q}\ \mathsf{Cut} \qquad p \vdash p}{p \vdash q}\ \mathsf{Cut}$$

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- This is the key lemma for the harder $\mathbf{ACT}^{\infty}_{(\mathrm{e})} \subseteq \mathbf{ACT}^{\omega}_{(\mathrm{e})}$ direction.
- The opposite one is basically modelling $*\mathsf{L}_\omega$ by an non-well-founded proof.

# Non-Well-Founded Proofs

$$\frac{\Gamma, \Delta \vdash C \quad \Gamma, A, \Delta \vdash C \quad \Gamma, A, A, \Delta \vdash C \quad \ldots}{\Gamma, A^*, \Delta \vdash C} *\mathsf{L}_\omega$$

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- Dually, a sequent is **not** provable if and only if one fails to find an $n$-proof for some $n$.
- For a fixed $n$, there is only a finite number of possible $n$-proofs, so the question whether such exists is algorithmically decidable.
- Thus, $n$ serves as a certificate that a sequent is not provable, and therefore $\mathbf{ACT}^{\omega}_{(\mathrm{e})}$ is in $\Pi^0_1$ (co-enumerable).

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- In fact, each computation step is modelled by a subderivation from one configuration to another.
- For example, $\textsc{inc}(p, \mathtt{a}_1, q)$ is modelled as follows:

$$\dfrac{p, \mathtt{a}_1^{n_1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D \qquad \dfrac{\dfrac{\dfrac{E^*, q, \mathtt{a}_1^{n_1+1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}{E^*, q \otimes \mathtt{a}_1, \mathtt{a}_1^{n_1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\otimes\mathsf{L} \quad p \vdash p}{E^*, p \setminus (q \otimes \mathtt{a}_1), p, \mathtt{a}_1^{n_1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\setminus\mathsf{L}}{E, E^*, p, \mathtt{a}_1^{n_1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\wedge\mathsf{L}}{E^*, p, \mathtt{a}_1^{n_1}, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}*\mathsf{L}$$

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- For zero-check, $\textsc{jzdec}(p, \mathtt{a}_1, q_0, q_1)$, where the value of $\mathtt{a}_1$ is zero, the derivation is a bit more involved:

$$\dfrac{\dfrac{p, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D \qquad \dfrac{\dfrac{\dfrac{E^*, q_0, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D \quad E^*, z_1, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}{E^*, q_0 \vee z_1, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\vee\mathsf{L} \qquad p \vdash p}{E^*, p \setminus (q_0 \vee z_1), p, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\setminus\mathsf{L}}{E, E^*, p, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}\wedge\mathsf{L}}{E^*, p, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D}*\mathsf{L}$$

# Circular Proofs

- For $E^*, z_1, \mathtt{a}_2^{n_2}, \mathtt{a}_3^{n_3} \vdash D$, we use the following non-well-founded proof, which is in fact a **circular** one:

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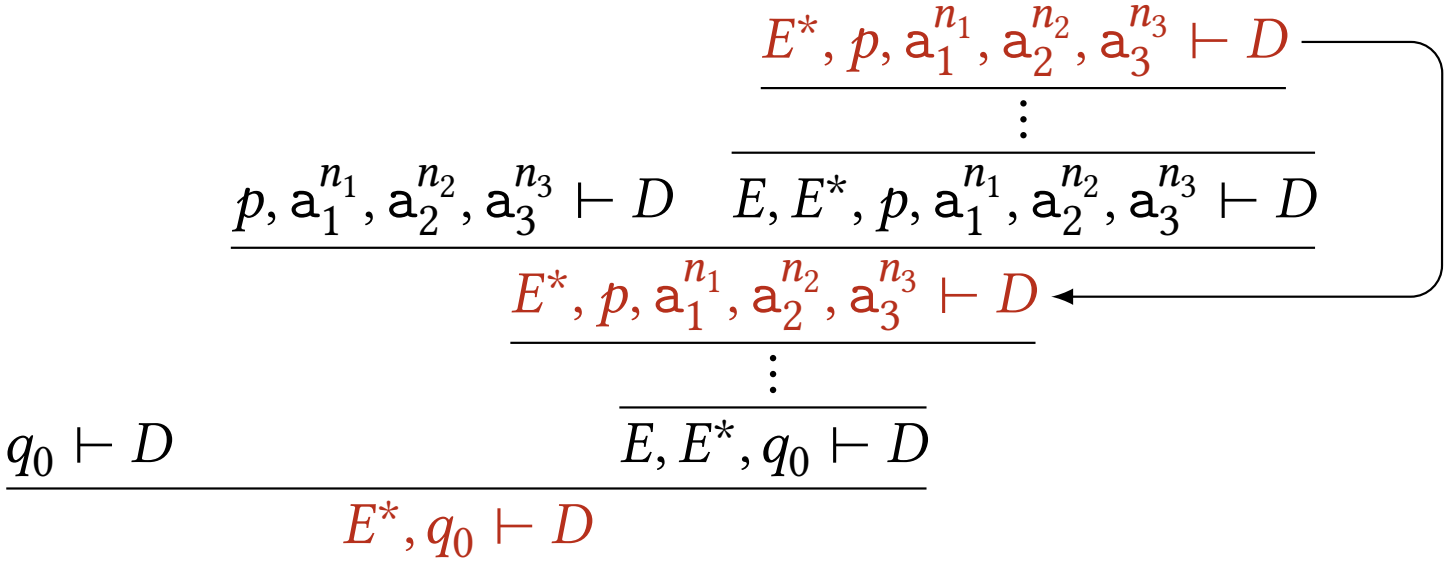
- Circular proofs are correct (surprise!), as a special case of non-well-founded proofs.
- In fact, at each iteration of the cycle, the proof makes *progress* by passing through $*\mathsf{L}$.

# Circular Proofs

- Circular proofs are capable of modelling circular (periodic) computations on counter machines:



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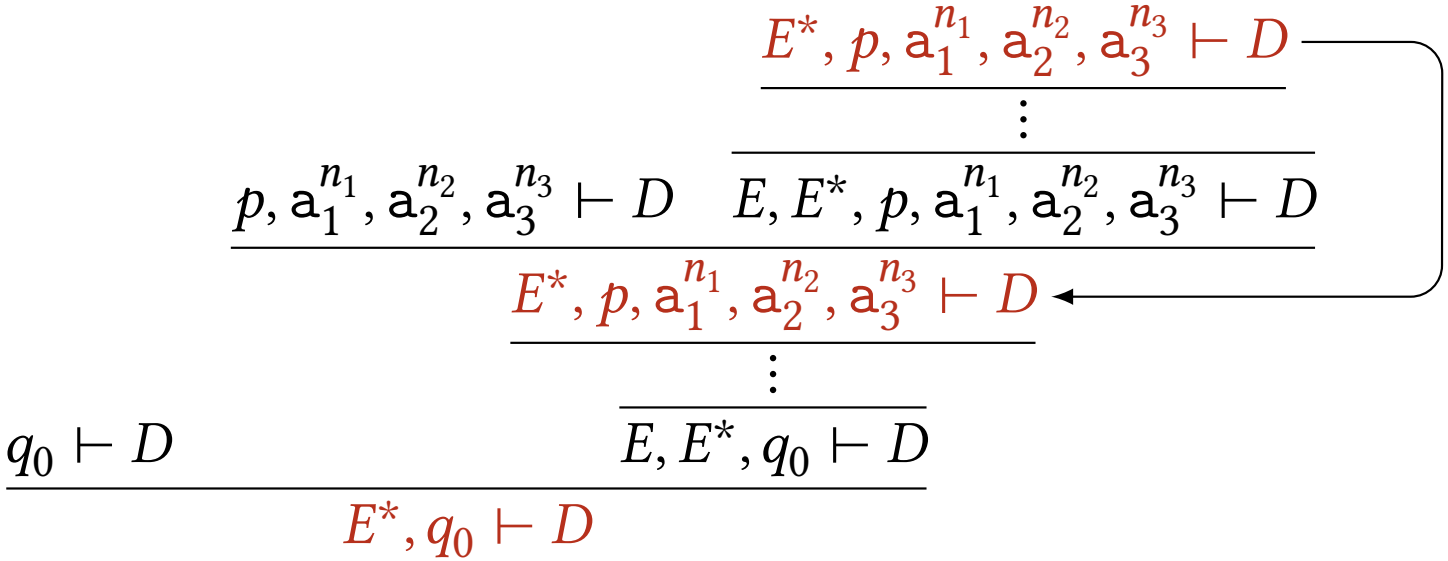


- On the other hand, circular proofs cannot model arbitrary infinitary computations, as they form an enumerable logic.

# Circular Proofs

- In fact, the circular fragment of $\mathbf{ACT}^{\infty}_{(c)}$ is equivalent to (commutative) **action logic** $\mathbf{ACT}_{(c)}$, where $A^*$ has a fixed point (inductive) axiomatisation [Pratt 1991; Kozen 1994]:

$$\frac{\vdash B \quad A, B \vdash B}{A^* \vdash B}\ {*\mathsf{L}_{\mathsf{fix}}} \qquad \frac{}{\vdash A^*}\ {*\mathsf{R}_0} \qquad \frac{\Pi \vdash A \quad \Delta \vdash A^*}{\Pi, \Delta \vdash A^*}\ {*\mathsf{R}}$$

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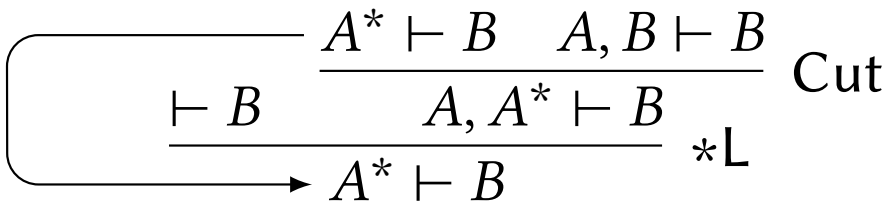
- If it is enumerable, then it is $\Sigma^0_1$-complete (effective inseparability).

# Circular Proofs and Induction

- Modelling $*\mathsf{L}_{\text{fix}}$ via circular proof:



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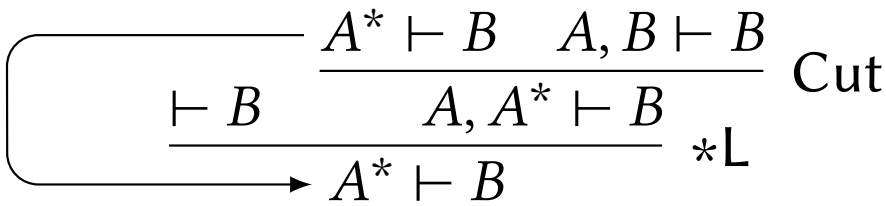


- In circular proofs, in general, we allow cuts. Correctness condition: each cyclic backlink points to the conclusion of a 'validating' $*\mathsf{L}$, and the cycle goes through its $2^{\text{nd}}$ premise.

# Circular Proofs and Induction

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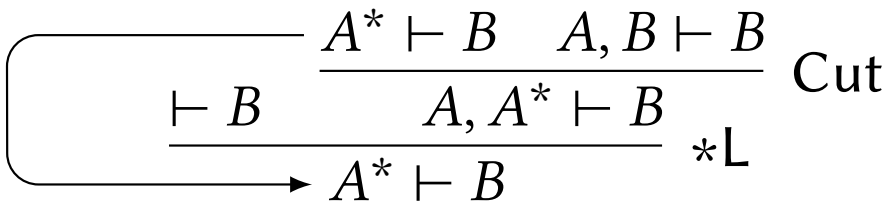


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- Let us track the 'active' $A^*$ and denote the sequents where it occurs by $\Gamma_i, A^*, \Delta_i \vdash C_i$, where $i = 1, \ldots, n$.

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- Thus, $A^* \vdash E$, hence $\Gamma, A^*, \Delta \vdash C$.

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- We shall need the easier part: from cf-grammars to Lambek grammars.

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- This gives undecidability of $\mathbf{ACT}^\omega$ [Buszkowski 2007]; for **ACT**, we again use inseparability [K. 2019].

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- This is also valid in the commutative case, but one should take extra care of the zero-checks.

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- This is proved by encoding well-foundedness of computable relations on $\omega$.

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- Now totality is encoded as $E, \mathtt{a}^* \vdash F$ (derivable from $\mathscr{H}$).

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- The 'key-and-lock' technique allows encoding the whole **arithmetical hierarchy,** i.e., algorithmic properties of the form:

  $P(x) = \forall y_1\, \exists y_2\, \ldots\, \forall y_{2k+1}\, \big(\text{the machine halts on } \mathtt{a}_1^{y_1}\mathtt{a}_2^{y_2}\ldots\mathtt{a}_{2k+1}^{y_{2k+1}}\mathtt{b}^x\big).$

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- In fact, even this is not all! Reasoning from $*$-free hypothesis in $\mathbf{ACT}^\omega$ is $\Sigma^0_{\omega^\omega}$-complete [K., Pshenitsyn, Speranski 2025]. ($\omega^\omega$ iteration of Turing jump, or infinitary quantifier alternation.)

# Systems with Contraction

- As noticed yesterday, adding the **contraction** rule raises complexity: $\mathbf{FL}_{\mathrm{c}}$ is undecidable [Chvalovský, Horčík 2016] and $\mathbf{FL}_{\mathrm{ec}}$ is decidable [Kripke 1959], but not primitive recursive.

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- The argument by Chvalovský and Horčík follows the general idea of encoding word rewriting systems, but faces two challenges.

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- The argument by Chvalovský and Horčík follows the general idea of encoding word rewriting systems, but faces two challenges.
  - Contraction is local, so there is no way of moving the code of a production rules. Solution: conditional rewriting systems, i.e., context-free grammars with regular constraints on contexts.

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- In contrast, $\mathbf{ACT}_{\mathrm{ec}}^{\omega}$ is $\Pi_1^0$-complete [Greati, Ramanayake 2026].

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