First-Order Predicate Logic (exercises)

Throughout this exercise sheet, variables (x, y, z, ...) range over elements of a non-empty *domain* M. Predicate symbols' (P, Q, R, ...) interpretations range over *predicates* of M, i.e., functions of the form $\bar{P}\colon \underbrace{M \times \ldots \times M}_{k} \to \{0,1\}$, where k is the number of arguments of P.

1. Which of the following formulae are generally true (i.e., true on any M and under any interpretations of predicate symbols)?
   (a) $\forall x\,(P(x) \vee Q(x)) \to (\forall x\, P(x)) \vee (\forall x\, Q(x))$
   (b) $\forall x\,(P(x) \vee Q(x)) \to (\forall x\, P(x)) \vee (\exists x\, Q(x))$
   (c) $(\forall x\,(P(x) \to Q(x)) \wedge \neg\exists x\, Q(x)) \to \forall y\, \neg P(y)$
   (d) $\forall x\, \exists y\, R(x,y) \to \exists y\, \forall x\, R(x,y)$
   (e) $\exists x(D(x) \to \forall y\, D(y))$
2. Which of the following formulae are satisfiable (i.e., true on some M for some interpretation of predicate symbols)?
   (a) $\exists x\, \forall y\,(Q(x,x) \wedge \neg Q(x,y))$
   (b) $\exists x\, \exists y\,(P(x) \wedge \neg P(y))$
   (c) $\exists x\, \forall y\,(Q(x,y) \to \forall z\, R(x,y,z))$.
3. Show that the following formula could be true only on an infinite M:
$$(\forall x\, \exists y\, Q(x,y)) \wedge \forall x\, \forall y\, \forall z\,(\neg Q(x,x) \wedge (Q(x,y) \to (Q(y,z) \to Q(x,z)))).$$
4. Let $M = \mathbb{N}$ be the set of natural numbers, and let $R(a,b)$ be true if and only if $a < b$. Write a formula $\varphi(u,v)$ with two parameters, u and v, which is true if and only if $v = u + 1$.
5. Write a formula using a binary predicate symbol R which expresses the fact that the R relation is:
   (a) reflexive
   (b) transitive
   (c) symmetric
   (d) antisymmetric
6. Show that the following formula is true on $M = \{a,b,c\}$ for any interpretation of R:
$$(\forall x\, R(x,x)) \wedge \forall x\, \forall y\, \forall z\,(R(x,z) \to (R(x,y) \vee R(y,z))) \to \exists u\, \forall v\, R(u,v).$$