HW #1: Resolution Method and Parsing

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Satisfiability

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- A satisfying assignment is an assignment of 0's and 1's to variables, which makes the formula true (value = 1).
- Satisfiability is a model example of a very general situation of **finding** (more precisely: checking for existence) an object with given properties.

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Resolution Method

- Recall that **resolution method** is a method of determining whether a Boolean formula given in CNF is satisfiable.
- A **CNF** is a conjunction of clauses, where each clause is a disjunction of literals (e.g., $\overline{x} \vee y \vee \overline{z}$).
- The algorithm **saturates** the CNF by adding all clauses which can be generated by the **resolution rule:**

$$\frac{A \vee p \quad B \vee \overline{p}}{A \vee B}$$

Resolution Method

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- If the empty clause ($\perp$) got obtained, the CNF is not satisfiable (because the resolution rule keeps validity).
- Moreover, by **completeness theorem** this is a criterion: if the empty clause is not obtained, the CNF **is** satisfiable.

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- In other words, the method solves the **decision problems** (“yes”/“no”), but not the **search problem.**
- If we are lucky enough, and the CNF has only one satisfying assignment, then after saturation we get **isolated** literals (like x or $\overline{y}$, for example), which dictate the desired satisfying assignment (e.g., $x = 1$ or $y = 0$).

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- In particular, if $\mathcal{S}$ is satisfiable and includes neither x nor $\overline{x}$, we can make an **arbitrary choice** for the value of x.

Resolution Method

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- For example, the CNF $(x \vee \overline{y}) \wedge (x \vee z)$ is saturated, but choosing $x = 0$ (adding $\overline{x}$) allows new resolutions giving $\overline{y}$ and z, and thus dictating values for all other variables.

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- The proof of the proposition is easy.
- Indeed, new resolutions applied when we saturate $\mathcal{S} \wedge x$, should involve x.
- Therefore, if such a resolution generates $\perp$, there should have been $\overline{x}$ in the original $\mathcal{S}$.

Example

$$(\overline{p} \vee r \vee s), (\overline{r} \vee q), (\overline{s} \vee \overline{p} \vee z), (\overline{z} \vee t), p$$

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$$\ldots$$

Resolution for 2-CNF

- If clauses include at least 3 literals, resolution can lead to growth:

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- This makes saturation a potentially exponential procedure.
- However, for 2-CNF (each clause includes no more than 2 literals) the clauses do not grow:

$$\frac{x \vee p \qquad \overline{z} \vee \overline{p}}{x \vee \overline{z}}$$

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- New clauses are added to the bottom of the list.

Home Assignment # 1

- Satisfiability for 2-CNF will be your task for HW # 1.
- The **easy** version is to check satisfiability (using resolution method).
- The **full** task is to check satisfiability **and,** if the answer is "yes," to return one of the satisfying assignments.

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- The program (in Python) should implement two functions:
  1. `is_satisfiable`, which takes a CNF and answers `True` or `False`, depending on whether it is satisfiable.
  2. `sat_assignment`, which takes a CNF and returns a satisfying assignment as an associative array:

     `{ 'x': True, 'y': False, 'z': True }`

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- The CNF is a conjunction (`/\`) of clauses.

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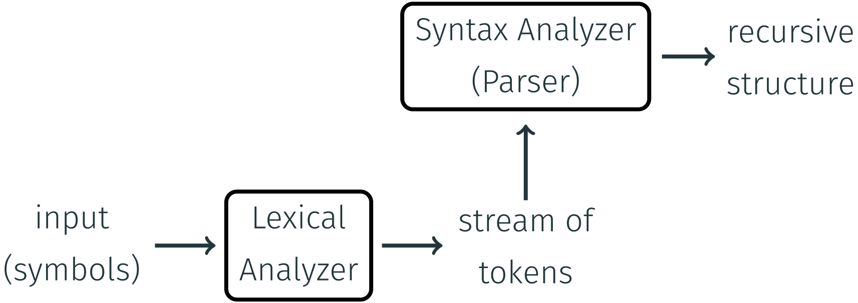
- We shall use specialized software, PLY (Python Lex & Yacc), in order to automatize the parsing process.

The Parsing Workflow



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    printf("Hello, World!\n");
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- Tokens are much more convenient to work with (in the grammar).

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- Input example:

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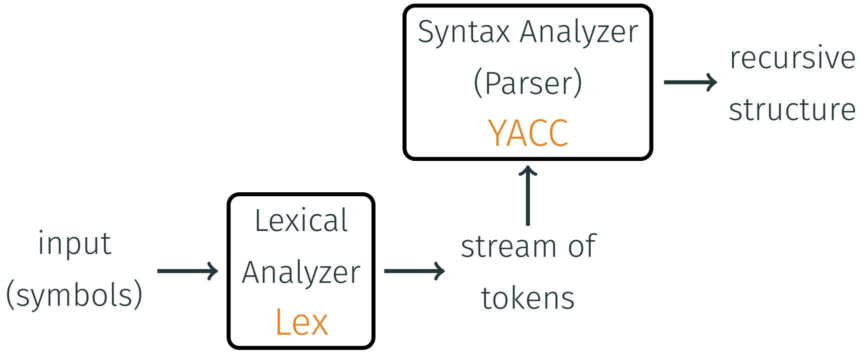
Implementation: Lex & Yacc



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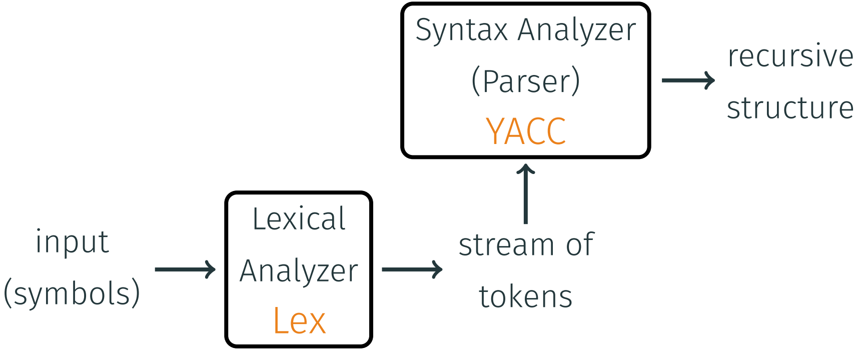

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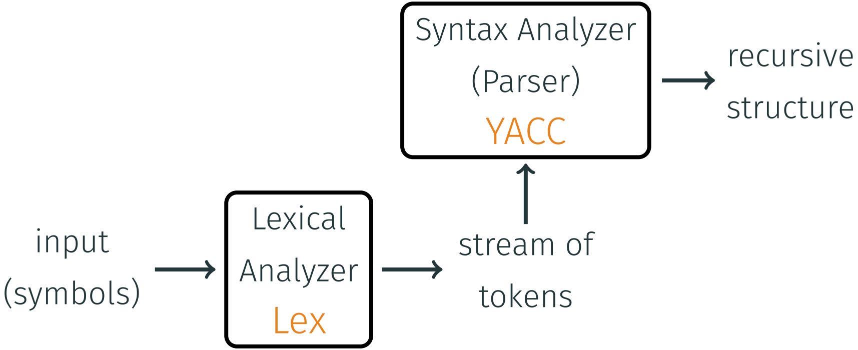

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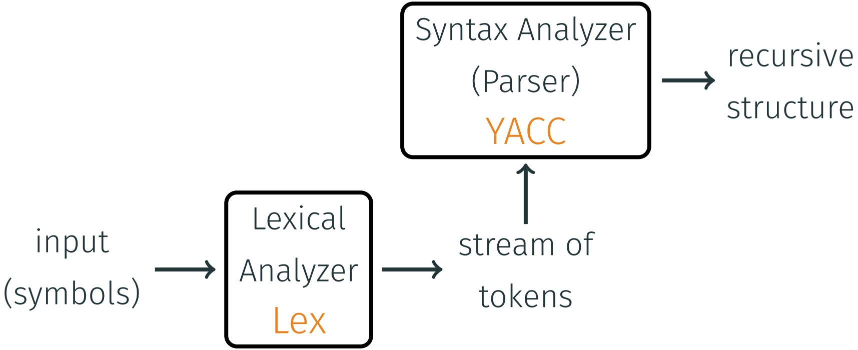

- YACC = Yet Another Compiler Compiler
- In Python, we use PLY (Python Lex & Yacc).

PLY Code for Lexical Analysis

- Declare tokens and literals (one-symbol tokens):

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literals = ['+','-','(',')','^','x']
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- For each token, declare a "`t_`"-function:

```
def t_INT(t):
    r'\d+'
    try:
        t.value = int(t.value)
    except ValueError:
        print "Too large!", t.value
        t.value = 0
    return t
```

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- Finally, build the lexer:

```
import ply.lex as lex
lex.lex()
```

PLY Code for Parsing

- Each rule of the grammar is implemented as a "`p_`"-function:

```
def polymult(p,q) :
    r = []
    for i in xrange(len(p)) :
        for j in xrange(len(q)) :
            safeadd(r,i+j,p[i]*q[j])
    return r
```

...

```
def p_tm_mult(p):
    "tm : tm '(' expr ')'"
    p[0] = polymult(p[1],p[3])
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- A “`p_`”-function generates an object `p[0]`, using `p[1]`, `p[2]`, ..., which are obtained from the lexer or recursively from parsing.

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- The code of PLY examples is available on the course's webpage:

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- For priorities of operations, see another example available on the webpage: calculator.

Good luck!

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- We prove this theorem using **induction** on the number of variables.
- That is, we establish it for zero variables (trivial) and then validate the **step** from n to $n+1$ variables.

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- Dually, take clauses without q and remove $\overline{q}$. This gives $\mathcal{S}^-$.

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  - Then $\mathcal{S}$ includes both q and $\overline{q}$, and therefore $\perp$. Contradiction.

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  - Since $\mathcal{S}^+$ and $\mathcal{S}^-$ use only $p_1, \dots, p_n$, we already know our theorem for them.
  - The one which does not include $\perp$ is satisfiable.

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- Clauses with $\overline{q}$ are satisfied by $\overline{q} = 1$.

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- If $\mathcal{S}^+$ is satisfiable, take the satisfying assignment and let $q = 0$.
- Clauses without $\overline{q}$ are already satisfied via $\mathcal{S}^+$.
- Clauses with $\overline{q}$ are satisfied by $\overline{q} = 1$.
- Dually, if $\mathcal{S}^-$ is satisfiable, take $q = 1$.

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- In order to allow richer expressive capabilities, more powerful logical languages were introduced.
- One of those is **first-order predicate logic,** which is usually used to formalize mathematics.

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- E.g., a two-argument P denotes a **binary relation** (say, $x < y$, written as $< (x, y)$).‘
- Besides propositional operations ($\rightarrow$, $\vee$, $\wedge$, $\neg$), there are **quantifiers** $\forall$ (forall) and $\exists$ (exists).

Predicate Logic: Example

$$\forall x \forall y (R(x, y) \to \exists z (R(x, z) \land R(z, y)))$$

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    - Again, universal truth and satisfiability are dual.

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  - This means that there is theoretically no algorithm for solving it, even without any time constraints.
- This motivates studying **decidable fragments** of predicate logic, where we restrict its expressivity in order to gain decidability.
  - Toy example: predicate logic with only unary predicates.

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- These systems are between propositional and predicate logics and are used in **knowledge representation.**
- Knowledge bases extend relational databases by a richer, logically enhanced language of queries. (This requires, obviously, fast algorithms.)