P & NP

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- For convenience, let the input data be a word over an alphabet: $x \in \Sigma^*$.
- The size of input, $|x|$ is the length of x in symbols.
- A decision problem is in the P class, if there exists an algorithm for solving it, whose **worst case** running time is bounded by $p(|x|)$.

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- Def. 1: non-deterministic computations.
    - The computation process may **branch:** at some point of execution, there could be more than one (but a finite number of) possibilities to perform the next step.
    - **Angelic choice:** if at least one execution trajectory yields "yes," then the answer is "yes."
    - One can implement **non-deterministic guess** (say, guess the satisfying assignment for a 3-CNF or guess a Hamiltonian cycle in a graph).

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  - Examples of y: the satisfying assignment; the Hamiltonian cycle; …

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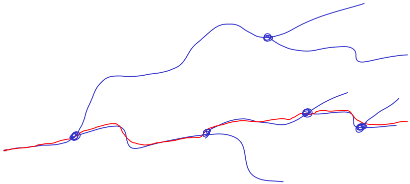

$\uparrow 1$

$\downarrow 0$

$y = 0100$

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- Informally, NP-complete problems are the **hardest possible** problems in NP.
    - In particular, if an NP-complete problem is solvable in poly time, then $P = NP$.
    - Contraposition: if $P \neq NP$ (which is highly likely), then any NP-complete problem is not in P.

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- **m-reduction** (Carp reduction): A is reducible to B ($A \leq_m^P B$), if there exists a polytime computable function $f\colon \Sigma^* \to \Sigma^*$, such that $\boxed{A(x) = 1 \iff B(f(x)) = 1.}$

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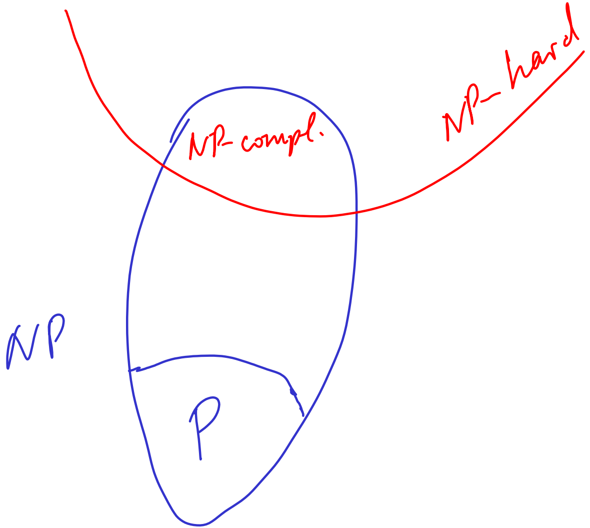
- A problem B is **NP-hard** if $A \leq_m^P B$ for any $A \in$ NP.
- B is **NP-complete** if $B \in$ NP and B is NP-hard.

Complexity Picture

(if $P \neq NP$)



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  - In order to prove NP-hardness of a problem B, we reduce the **old** problem A to B.
- But how to bootstrap and obtain the first example of an NP-complete problem?

Cook – Levin Theorem

Theorem

SAT (satisfiability of arbitrary Boolean formulae) NP-complete.