Beyond NP-Completeness

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Euler and Hamiltonian Paths

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- An **Euler path** in a (multi)graph is a path which traverses each **edge** exactly once.
- A **Hamiltonian path** should traverse each **vertex** exactly once.
- The two notions look similar, but there is a complexity gap: finding an Euler path is polynomial, while existence of a Hamiltonian one is NP-complete.

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- Finding a Euler cycle is easy (can be done in polynomial time).
- What if, for a specific class of graphs, the problem of finding a Hamiltonian cycle could be *reduced* to the problem of finding a Euler cycle?

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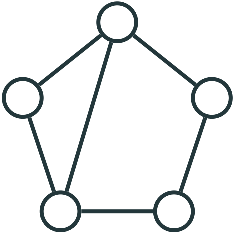

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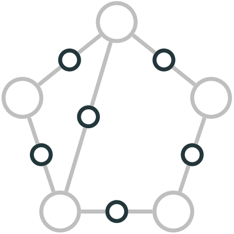

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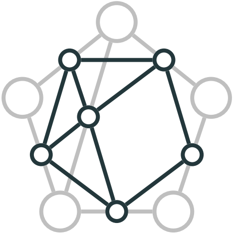

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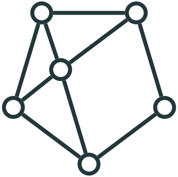

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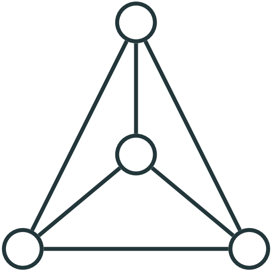

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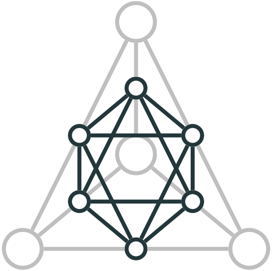

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- And, of course, not every Hamiltonian graph is a line graph of some other graph G.
- Nevertheless, in some practically important cases representation of a given graph as $L(G)$ allows efficient construction of Hamiltonian cycles.

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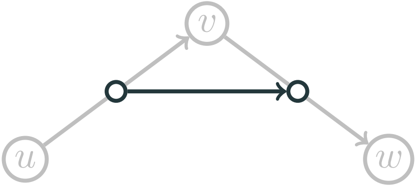
- Vertices of $L(G)$ are directed edges of G, and we connect $\langle u, v\rangle$ with $\langle v, w\rangle$, in the given direction:



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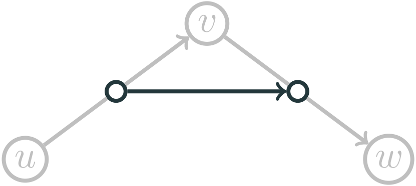

- Again, a directed Euler path in G induces a directed Hamiltonian path in $L(G)$.

Application: Genome and Its Fragments

- The *genome* is, roughly a string of letters A, C, G, T (they encode *nucleotides:* adenine, cytosine, guanine, thymine).

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- Consider the following model situation. Experiment does not give us the complete genome, but rather all its fragments of length 3, in a random order:

  TGC, CTA, GCT, AGC, ACT, GCC, TAG, CTG

- Our goal is to reassemble the genome.

Reassembly as Hamiltonian Path

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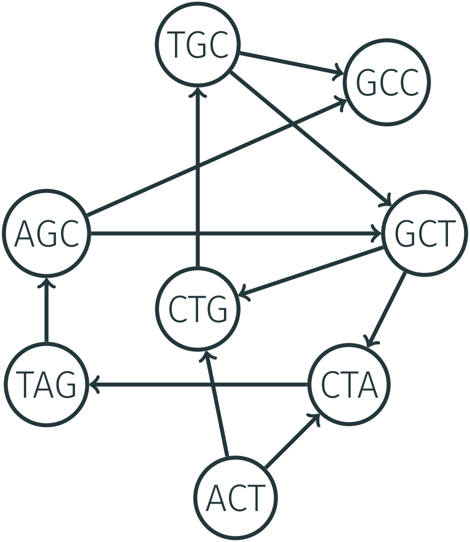
- In this graph, triplet u is connected to triplet v, if the last two letters of u are the first two letters of v:

$$\text{CTA} \rightarrow \text{TAG}.$$

Reassembly as Hamiltonian Path



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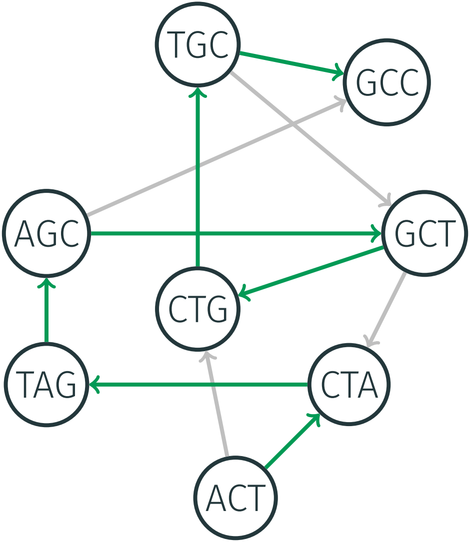

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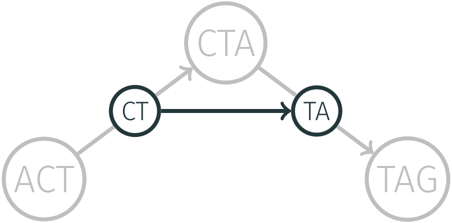
- It would be much better if we could use Euler path instead.

De Bruijn Graph

- Fortunately, the overlap graph is $L(G)$ for some other graph G:



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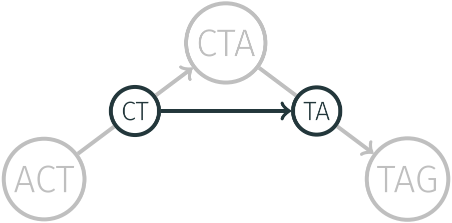
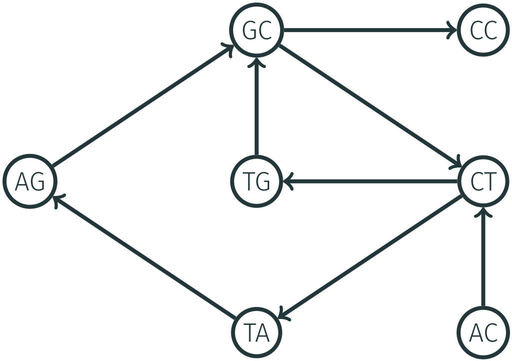


- After identifying vertices with the same annotation in G and adding AC and CC (start and end), we get **de Bruijn graph.**

De Bruijn Graph

A Euler path in de Bruijn graph induces a Hamiltonian path in overlap graph.

Reassembly as Euler Path



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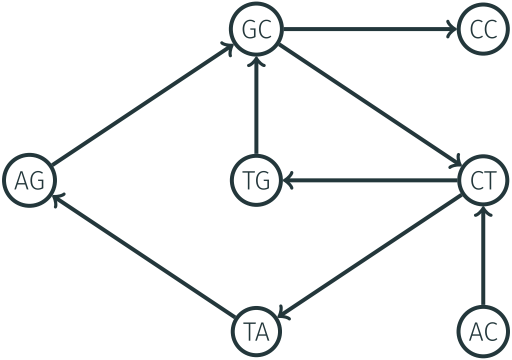

Two possible ways to reassemble: ACTAGCTGCC and ACTGCTAGCC.

Overlap vs. de Bruijn

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- De Bruijn graph is used in real-world genome assemblers.

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- Examples of y: the satisfying assignment; the Hamiltonian cycle; …

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- **Search problem:** yield a witness or say "no."
- **Counting problem** (the #P class): yield the **number** of witnesses.

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- Namely, if $\mathrm{P} = \mathrm{NP}$, then any search problem is also solvable in polynomial time.
  - E.g., searching for SAT can be done via dichotomy using decision for SAT.

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Search Problems

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- For example, let $R(\varphi, y)$ mean "$y = (a, b)$, where a is a satisfying assignment for φ or b is a satisfying assignment for $\neg\varphi$."
- Here the decision problem is trivial (always "yes"), but the search problem is equivalent to the one for SAT.

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Theorem

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- In order to prove theorems like this one, one has to develop the theory of #P-completeness.

Counting Reductions

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- A counting reduction consists of **two** functions, $f\colon \Sigma^* \to \Sigma^*$ on input data and $g\colon \mathbb{N} \to \mathbb{N}$ on counts (results).
- Recall that $\#A$ and $\#B$ are counting problems, that is,

$$\#A(x) = |\{y \mid R(x,y) = 1\}| \in \mathbb{N},$$

and the same for $\#B$.

Counting Reductions

- We say that $\#A \leq_c^P \#B$, if there exists a pair of polynomially computable reducing functions f and g such that for any input x we have

$$\#A(x) = g(\#B(f(x))).$$

- This indeed allows to **reduce** $\#A$ to $\#B$. Suppose we know how to solve $\#B$. Then, in order to solve $\#A$, we take x, apply f, then solve $\#B$ (yielding a natural number) and apply g.

#P-Completeness

- A counting problem $\#B$ is #P-complete, if for any other $\#A \in \#$P we have $\#A \leq_c^P \#B$...
- ... just as for NP-completeness.
- Now we can develop a theory of #P-complete problems, which is parallel to the theory of NP-completeness.

Parsimonious Reductions

- A counting reduction (f, g), where g is identity, $g(n) = n$, is called a **parsimonious** reduction.
- A parsimonious reduction is also a specific kind of m-reduction, since, in particular, $g(0) = 0$, thus, it conveys the answer to the decision problem.

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#SAT is #P-complete.

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- The sequence of configurations (protocol) of A on input x is encoded by a binary matrix (b_{ij}) of size $(m \cdot p(|x|)) \times p(|x|)$.

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- Next, we construct a formula φ_x with variables $b_{00}, b_{01}, \ldots$ which expresses the fact that this matrix represents a correct protocol of a successful execution.

Cook – Levin Theorem

φ_x is a conjunction of the following claims:

1. the first row represents the configuration with x on the tape, the machine observing its first letter;
2. each next row is obtained from the previous one by one of the rules of the machine;
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This is all expressible as Boolean formulae.

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- That is, any Boolean formula φ can be translated into a 3-CNF ψ, such satisfying assignments of ψ are in one-to-one correspondence with those for φ.
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- Thus, #3-SAT is also #P-complete.

Tseitin’s Transformations

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For any Boolean formula φ, there exists an equisatisfiable 3-CNF ψ of polynomial size.

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For any Boolean formula φ, there exists an equisatisfiable 3-CNF ψ of polynomial size.

- Equisatisfiability means that ψ is satisfiable iff so is φ.
- Constructing an *equivalent* 3-CNF of polynomial size is not always possible: even translation to CNF can lead to exponential blowup.

Tseitin's Transformations

- Tseitin's transformations look like translation into 3-address (Assembler-like) code:
  $(a+b)*(c+d)$ is translated to
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  $(a + b) * (c + d)$ is translated to
  "add $a\ b\ t_1$; add $c\ d\ t_2$; mul $t_1\ t_2\ r$"
- For each subformula we introduce a new variable and write the corresponding equivalences.

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Example: $(p \to q) \vee (q \to (p \to r))$

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Example: $(p \rightarrow q) \vee (q \rightarrow (p \rightarrow r))$

$$\begin{aligned}
&(t_1 \leftrightarrow (p \rightarrow q)) \wedge \\
&(t_2 \leftrightarrow (p \rightarrow r)) \wedge \\
&(t_3 \leftrightarrow (q \rightarrow t_2)) \wedge \\
&(t_4 \leftrightarrow (t_1 \vee t_3)) \wedge \\
&t_4
\end{aligned}$$

Tseitin's Transformations

Transform into 3-CNF by the following table:

$$
\begin{array}{l|l}
t_k \leftrightarrow (t_i \wedge t_j) & (\neg t_i \vee \neg t_j \vee t_k) \wedge (t_i \vee \neg t_k) \wedge (t_j \vee \neg t_k) \\
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For our example, we get:

$$(\neg p \vee q \vee \neg t_1) \wedge (p \vee t_1) \wedge (\neg q \vee t_1) \wedge$$
$$(\neg p \vee r \vee \neg t_2) \wedge (p \vee t_2) \wedge (\neg r \vee t_2) \wedge$$
$$(\neg q \vee t_2 \vee \neg t_3) \wedge (q \vee t_3) \wedge (\neg t_2 \vee t_3) \wedge$$
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- In this case, if P $\neq$ NP, we know that even A is not polynomially solvable, nothing to say about $\#A$.
- Using more general counting reductions, however, could give interesting results.

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- Interesting cases include situations when the decision problem is polynomially decidable, while the counting problem is hard.

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    - See A. Ben-Dor, S. Halevi (1993), “Zero-one permanent is #P-complete, a simple proof” and L.G. Valiant (1979), “The complexity of enumeration and reliability problems”.

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- Easily, DNF-SAT $\in$ P (as a decision problem).
- However, in the counting case we can reduce from CNF-SAT by duality:

$$f(\varphi) = \mathsf{DNF}(\neg\varphi)$$
$$g(n) = 2^k - n$$

(where k is the number of variables).

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- Thus, #CNF-SAT $\leq_c^P$ #DNF-SAT, and therefore #DNF-SAT is #P-complete.
- **Corollary:** if P $\neq$ NP, then #DNF-SAT is not polynomially solvable.
- Otherwise so would be #CNF-SAT, and therefore CNF-SAT, which implies P $=$ NP.

Permanent

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- There exist fast algorithms for computing the determinant, not by its definition (e.g., Gauss' diagonalization).

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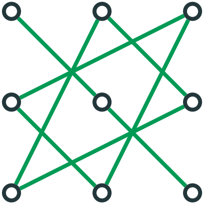

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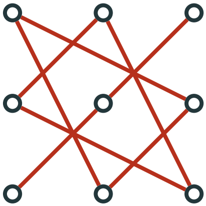

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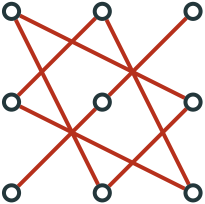

- The **permanent** is like the determinant, but without signs:

$$\operatorname{perm}(a_{i,j}) = \sum_{\sigma \in \mathbf{S}_n} a_{1,\sigma(1)} \cdot \ldots \cdot a_{n,\sigma(n)}$$

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- One example: computing the normalization constant for Markov random fields is equivalent to computing the permanent.
- However, computing the permanent by definition requires more than exponential time: namely, $n \cdot n!$.

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- This follows from the theory of #P-hardness.
- Let $a_{i,j}$ be zeroes and ones. Then $\mathrm{perm}(a_{i,j})$ can be seen as a counting problem: how many permutations from $\mathbf{S}_n$ give all ones?

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- The decision problem here ($\mathrm{perm} > 0$) is easy (polynomial), since it reduces to finding a perfect matching in a bipartite graph.
- The counting problem (computing perm) is #P-hard (see Valiant 1979).
- This problem is parsimoniously reducible to #2-SAT, so the latter is also #P-hard.