First-Order Predicate Logic (exercises)

Throughout this exercise sheet, variables (x, y, z, ...) range over elements of a non-empty *domain* M. Predicate symbols' (P, Q, R, ...) interpretations range over *predicates* of M, i.e., functions of the form $\bar{P}: \underbrace{M \times \ldots \times M}_{k} \to \{0, 1\}$, where k is the number of arguments of P.

1. Which of the following formulae are generally true (i.e., true on any M and under any interpretations of predicate symbols)?

   (a) $\forall x\, (P(x) \lor Q(x)) \to (\forall x\, P(x)) \lor (\forall x\, Q(x))$

   (b) $\forall x\, (P(x) \lor Q(x)) \to (\forall x\, P(x)) \lor (\exists x\, Q(x))$

   (c) $(\forall x\, (P(x) \to Q(x)) \land \neg \exists x\, Q(x)) \to \forall y\, \neg P(y)$

   (d) $\forall x\, \exists y\, R(x, y) \to \exists y\, \forall x\, R(x, y)$

   (e) $\exists x (D(x) \to \forall y\, D(y))$

2. Which of the following formulae are satisfiable (i.e., true on some M for some interpretation of predicate symbols)?

   (a) $\exists x\, \forall y\, (Q(x, x) \land \neg Q(x, y))$

   (b) $\exists x\, \exists y\, (P(x) \land \neg P(y))$

   (c) $\exists x\, \forall y\, (Q(x, y) \to \forall z\, R(x, y, z))$.

3. Show that the following formula could be true only on an infinite M:

$$(\forall x\, \exists y\, Q(x, y)) \land \forall x\, \forall y\, \forall z\, (\neg Q(x, x) \land (Q(x, y) \to (Q(y, z) \to Q(x, z)))).$$

4. Let $M = \mathbb{N}$ be the set of natural numbers, and let $R(a, b)$ be true if and only if $a < b$. Write a formula $\varphi(u, v)$ with two parameters, u and v, which is true if and only if $v = u + 1$.

5. Write a formula using a binary predicate symbol R which expresses the fact that the R relation is:

   (a) reflexive

   (b) transitive

   (c) symmetric

   (d) antisymmetric

6. Show that the following formula is true on $M = \{a, b, c\}$ for any interpretation of R:

$$(\forall x\, R(x, x)) \land \forall x\, \forall y\, \forall z\, (R(x, z) \to (R(x, y) \lor R(y, z))) \to \exists u\, \forall v\, R(u, v).$$