Graph Colorings

By k-COLOR, for a fixed k, we denote the following algorithmic problem: given a graph G, determine whether vertices of G can be colored in k colors such that adjacent vertices are colored differently.

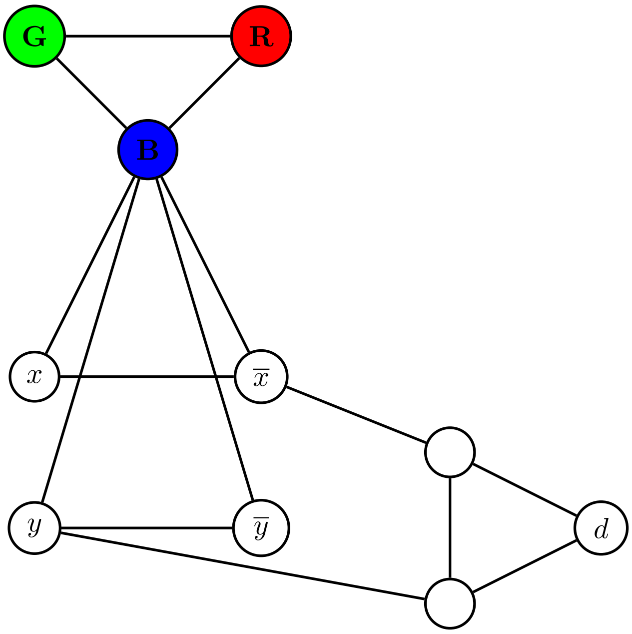
1. Show that 2-COLOR is polynomially decidable.

2. (a) Show that for any k there exists a graph which is k-colorable, but not $(k-1)$-colorable.

   (b) Show that if a graph is k-colorable, but not $(k-1)$-colorable, then its number of edges is at least $k(k-1)/2$.

3. Show that k-COLOR belongs to NP for any k.

4. (a) Consider a graph fragment of the following form, which is partially colored in red, green, blue.



   Now let us also color the vertices x, $\overline{x}$, y, and $\overline{y}$, so that no adjacent vertices have the same color. Let Boolean variable x be true if x is green and false if it is red (then $\overline{x}$ is green); the same for y. Write down a Boolean formula with variables x and y which is true if and only if the remaining three white vertices can be correctly colored so that d becomes green.

   (b) Extend the construction of Task 4(a) to three variables x, y, z (and their negations $\overline{x}$, $\overline{y}$, $\overline{z}$).

   (c) For a given Boolean formula φ in 3-CNF, construct a graph G_φ which is 3-colorable if and only if φ is satisfiable.

   (d) Show that 3-COLOR is NP-complete.