P & NP

Stepan Kuznetsov

Discrete Math Bridging Course, HSE University

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- The size of input, $|x|$ is the length of x in symbols.
- A decision problem is in the P class, if there exists an algorithm for solving it, whose **worst case** running time is bounded by $p(|x|)$.

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  - **Angelic choice:** if at least one execution trajectory yields “yes,” then the answer is “yes.”
  - One can implement **non-deterministic guess** (say, guess the satisfying assignment for a 3-CNF or guess a Hamiltonian cycle in a graph).

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  - Examples of y: the satisfying assignment; the Hamiltonian cycle; ...

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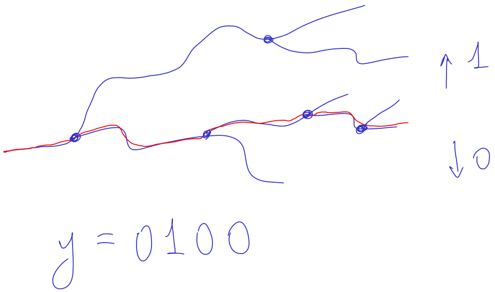

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- Informally, NP-complete problems are the **hardest possible** problems in NP.
  - In particular, if an NP-complete problem is solvable in poly time, then $P = NP$.
  - Contraposition: if $P \neq NP$ (which is highly likely), then any NP-complete problem is not in P.

NP-Completeness

- **m-reduction** (Carp reduction): A is reducible to B ($A \leq_m^P B$), if there exists a polytime computable function $f\colon \Sigma^* \to \Sigma^*$, such that $\boxed{A(x) = 1 \iff B(f(x)) = 1.}$

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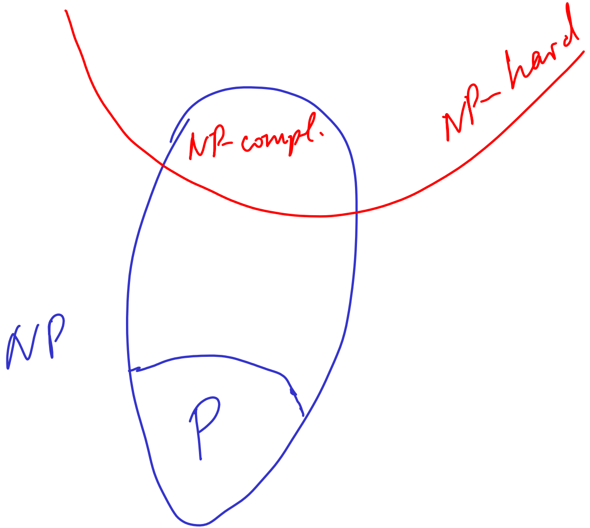
- A problem B is **NP-hard** if $A \leq_m^P B$ for any $A \in$ NP.
- B is **NP-complete** if $B \in$ NP and B is NP-hard.

Complexity Picture

(if $P \neq NP$)



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- The common method of proving NP-hardness is **backwards reduction.**
  - Suppose we know A to be already NP-hard.
  - In order to prove NP-hardness of a problem B, we reduce the **old** problem A to B.
- But how to bootstrap and obtain the first example of an NP-complete problem?

Cook – Levin Theorem

Theorem

SAT (satisfiability of arbitrary Boolean formulae) NP-complete.

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- The problem is **3-colorability of graphs.**
- Let us first recall what a graph is.

Graphs

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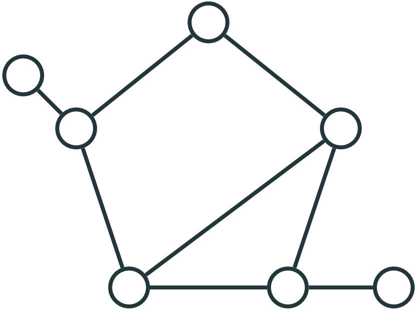
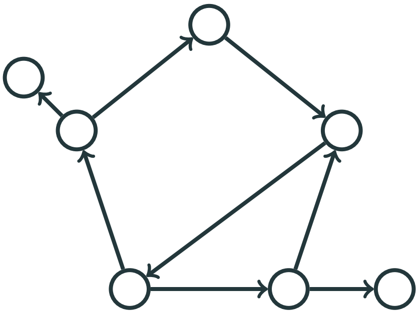


Graphs

In a **directed** graph, edges have arrows on them:



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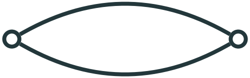
- *Parallel edges:* two vertices connected by more than one edge.



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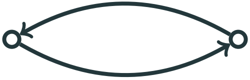
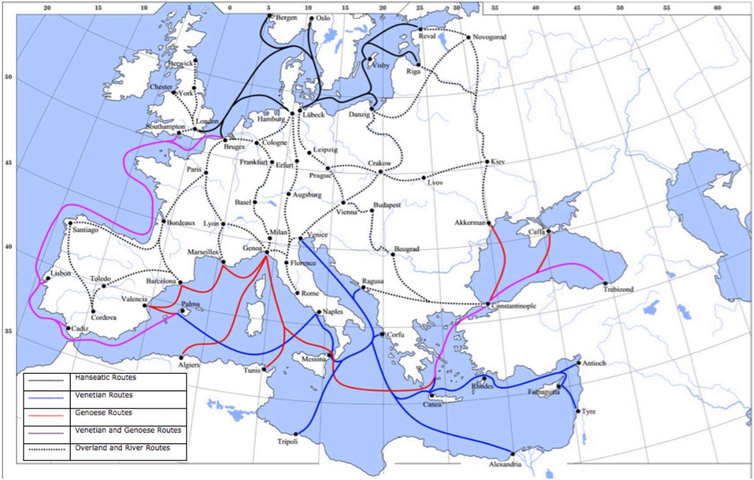
- Note that in a directed graph edges connecting two vertices in different directions are **not** considered parallel.



Applications of Graphs

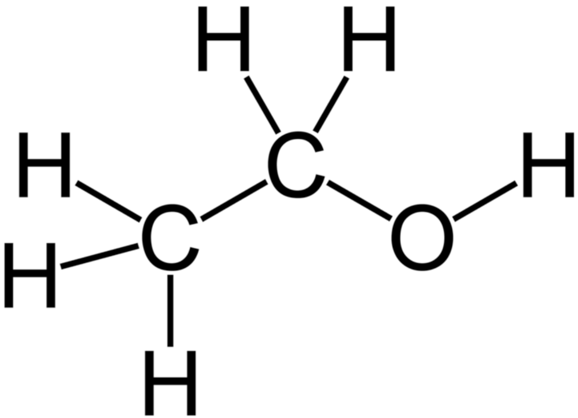
Maps (GIS): vertices = cities, edges = routes.



Lampman @ Wikipedia

Applications of Graphs

Chemistry: graphs of molecular structure.



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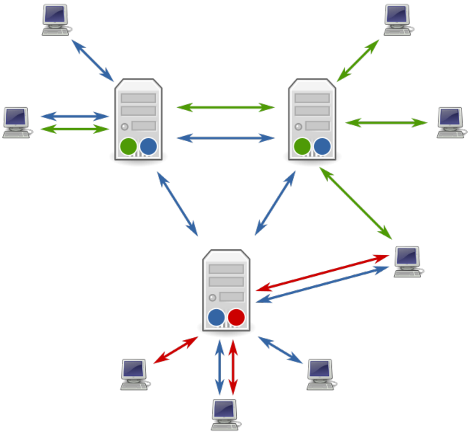
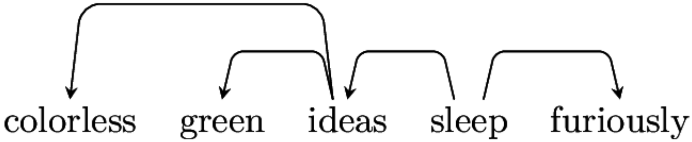
Internet: network topology.



Benjamin D. Esham @ Wikipedia

Applications of Graphs

Linguistics: syntactic dependencies.



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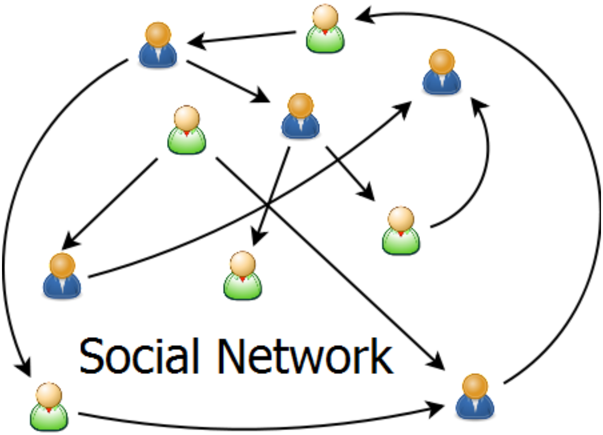

Zigomitros Athanasios – Thor4bp @ Wikipedia

Graph: Formal Definition

- A pseudograph can be formally defined as $G = (V, E)$, where V is the set of vertices (arbitrary finite set) and $E \subseteq V \times V = \{(u, v) \mid u, v \in V\}$ is the set of edges, such that $(u, v) \in E \iff (v, u) \in E$.

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    - The formal definition of multigraph is more involved.

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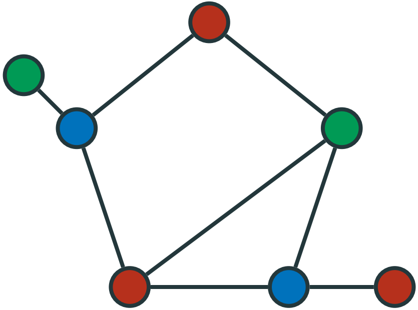
- Formally, $c\colon V \to C$, and if $(u, v) \in E$, then $c(u) \neq c(v)$.
- Example: 3-coloring $c\colon V \to \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$.

3-Coloring

For example, this graph is 3-colorable:



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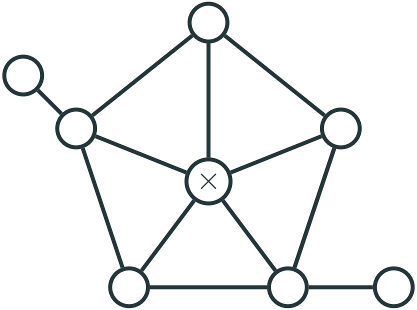
… and this one is not:



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- The 3-colorability problem (given a graph G, answer whether it is 3-colorable) clearly belongs to the NP class.
- Indeed, we can non-deterministically guess the coloring ("hint") and then check its correctness in poly time.
- Thus, a particular case of Cook – Levin theorem states that 3-COLOR $\leq_m^P$ SAT.

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- For any graph G we can construct a Boolean formula φ_G, which is satisfiable if and only if G is 3-colorable.

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- The reducing function $f\colon G \mapsto \varphi_G$ will be poly-time computable.
- Moreover, each correct 3-coloring of G will correspond to a satisfying assignment of φ_G, and vice versa.

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  r_i "v_i is colored red"
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- φ_G will represent natural conditions on these propositions.

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$$\varphi_G = \bigwedge_{v_i \in V} \Big((r_i \vee g_i \vee b_i) \wedge$$

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- By the way, φ_G is a 3-CNF.

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- Thus, if SAT were solvable in poly time, so would have been 3-COLOR.
- In reality, however, we **do not know** a polynomial algorithm for SAT, and such reductions give some evidence **against** its existence.
- The idea of Cook – Levin theorem is that **any** NP guessing can be represented as guessing a satisfying assignment for a Boolean formula.

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    - One of the reasons is that we measure **worst case** complexity, and instances which appear in practice could avoid such cases.
- Cook – Levin style reductions allow to use SAT solvers for other NP problems.

Cook – Levin Theorem

Theorem

SAT is NP-complete, that is, if $A \in$ NP, then A is m-reducible to SAT.

Turing Machines

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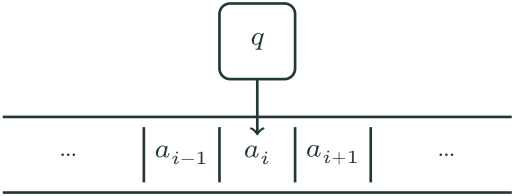
- This requires a formal notion of what an algorithm is, that is, a formal **model of computation.**
- Let us define one such model, namely, **Turing machines.**

Turing Machines

- A Turing machine is a tuple $\mathfrak{M} = \langle \Sigma, \Gamma, Q, q_0, q_F, \Delta \rangle$, where:
  - Σ is the *external alphabet* (in which input and output are formulated);
  - $\Gamma \supseteq \Sigma$ is the *internal alphabet* (used in the computation process);
  - Q is a finite set of *states*;
  - q_0 is the starting state and q_F is the final one;
  - Δ is the set of *rules* (also finite).

Turing Machines

- At each step of its run, the machine keeps one of the states (from Q) in its internal memory, and observes one of the cells of an infinite tape:



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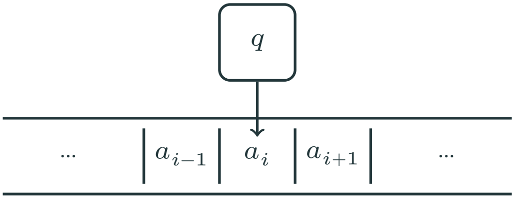

  - At each moment, only a finite part of the tape is populated by meaningful symbols; the rest is padded by "blank" symbols $B \in \Gamma - \Sigma$.

Turing Machines

- **Rules** of $\mathfrak{M}$ (elements of Δ) are of the form $\langle p, a\rangle \to \langle q, b, d\rangle$, where $p, q \in Q$, $a, b \in \Gamma$, and $d \in \{L, R, N\}$.

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- Such a rule is executed as follows. If $\mathfrak{M}$ keeps p in its internal memory and observes a on the tape, then the following move is performed:
    1. replace a with b in the cell;
    2. replace p with q in the internal memory;
    3. if $d = L$, move one cell left; if $d = R$, move one cell right; if $d = N$, stay on the same cell.

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Turing Machines

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- Once a machine runs into state q_F, it stops successfully, and the word on the tape is the output.
- It is also possible to stop unsuccessfully or to run infinitely long.

NP and co-NP

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- Example: SAT is in NP, while TAUT (checking whether a Boolean formula is a tautology) is in co-NP.

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- Moreover, if the computation is polynomial, it can be performed also polynomially on the Turing machine.
    - The degree of the polynomial could change.

Cook – Levin Theorem

Theorem

SAT is NP-complete, that is, if $A \in$ NP, then A is m-reducible to SAT.