NP-Completeness
Cook – Levin Theorem

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The P Class

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- For convenience, let the input data be a word over an alphabet: $x \in \Sigma^*$.
- The size of input, $|x|$ is the length of x in symbols.
- A decision problem is in the P class, if there exists an algorithm for solving it, whose **worst case** running time is bounded by $p(|x|)$.

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- Def. 1: non-deterministic computations.
    - The computation process may **branch:** at some point of execution, there could be more than one (but a finite number of) possibilities to perform the next step.
    - **Angelic choice:** if at least one execution trajectory yields “yes,” then the answer is “yes.”
    - One can implement **non-deterministic guess** (say, guess the satisfying assignment for a 3-CNF or guess a Hamiltonian cycle in a graph).

NP-Completeness

- **m-reduction** (Carp reduction): A is reducible to B ($A \leq_m^P B$), if there exists a polytime computable function $f\colon \Sigma^* \to \Sigma^*$, such that $\boxed{A(x) = 1 \iff B(f(x)) = 1.}$

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- A problem B is **NP-hard** if $A \leq_m^P B$ for any $A \in$ NP.
- B is **NP-complete** if $B \in$ NP and B is NP-hard.

Cook – Levin Theorem

Theorem

SAT (satisfiability of arbitrary Boolean formulae) NP-complete, that is, if $A \in NP$, then A is m-reducible to SAT.

Example: Graph Coloring

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- The problem is **3-colorability of graphs.**

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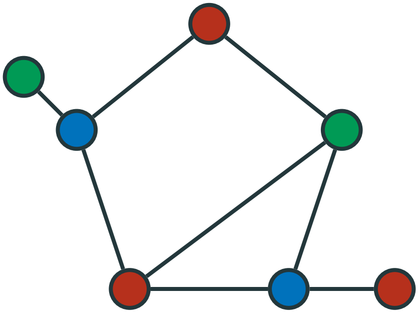
- Formally, $c\colon V \to C$, and if $(u, v) \in E$, then $c(u) \neq c(v)$.
- Example: 3-coloring $c\colon V \to \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$.

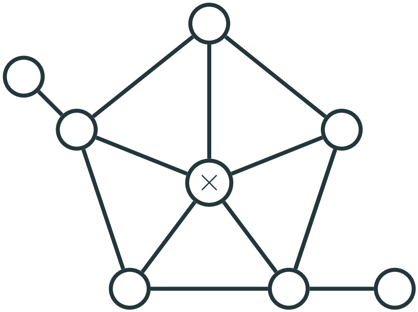
3-Coloring

For example, this graph is 3-colorable:



3-Coloring

… and this one is not:



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- The 3-colorability problem (given a graph G, answer whether it is 3-colorable) clearly belongs to the NP class.
- Indeed, we can non-deterministically guess the coloring (“hint”) and then check its correctness in poly time.
- Thus, a particular case of Cook – Levin theorem states that 3-COLOR $\leq_m^P$ SAT.

Reducing 3-COLOR to SAT

- For any graph G we can construct a Boolean formula φ_G, which is satisfiable if and only if G is 3-colorable.

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- The reducing function $f\colon G \mapsto \varphi_G$ will be poly-time computable.
- Moreover, each correct 3-coloring of G will correspond to a satisfying assignment of φ_G, and vice versa.

Reducing 3-COLOR to SAT

- For each vertex $v_i \in V$, introduce the following Boolean variables:
  r_i "v_i is colored red"
  g_i "v_i is colored green"
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  - r_i "v_i is colored red"
  - g_i "v_i is colored green"
  - b_i "v_i is colored blue"
- φ_G will represent natural conditions on these propositions.

Reducing 3-COLOR to SAT

$$\varphi_G = \bigwedge_{v_i \in V} \Big((r_i \vee g_i \vee b_i) \wedge$$

$$(\neg r_i \vee \neg g_i) \wedge (\neg r_i \vee \neg b_i) \wedge (\neg b_i \vee \neg g_i) \Big) \wedge$$

$$\bigwedge_{(v_i, v_k) \in E} \Big((\neg r_i \vee \neg r_k) \wedge (\neg g_i \vee \neg g_k) \wedge (\neg b_i \vee \neg b_k) \Big)$$

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- 3-colorings of G and satisfying assignments of φ_G are in one-to-one correspondence.
- By the way, φ_G is a 3-CNF.

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- Thus, if SAT were solvable in poly time, so would have been 3-COLOR.
- In reality, however, we **do not know** a polynomial algorithm for SAT, and such reductions give some evidence **against** its existence.
- The idea of Cook – Levin theorem is that **any** NP guessing can be represented as guessing a satisfying assignment for a Boolean formula.

Cook – Levin Theorem

Theorem

SAT is NP-complete, that is, if $A \in$ NP, then A is m-reducible to SAT.

Turing Machines

- In order to prove Cook – Levin theorem, we need to show that $A \leq_m^P$ SAT for **any** $A \in$ NP.

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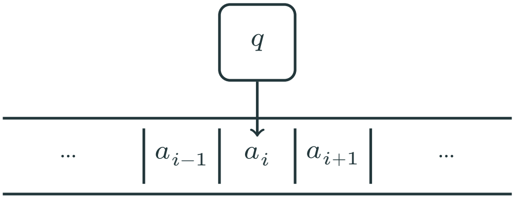
- In order to prove Cook – Levin theorem, we need to show that $A \leq_m^P$ SAT for **any** $A \in$ NP.
- This requires a formal notion of what an algorithm is, that is, a formal **model of computation.**
- Let us define one such model, namely, **Turing machines.**

Turing Machines

- A Turing machine is a tuple $\mathfrak{M} = \langle \Sigma, \Gamma, Q, q_0, q_F, \Delta \rangle$, where:
    - Σ is the *external alphabet* (in which input and output are formulated);
    - $\Gamma \supseteq \Sigma$ is the *internal alphabet* (used in the computation process);
    - Q is a finite set of *states*;
    - q_0 is the starting state and q_F is the final one;
    - Δ is the set of *rules* (also finite).

Turing Machines

- At each step of its run, the machine keeps one of the states (from Q) in its internal memory, and observes one of the cells of an infinite tape:



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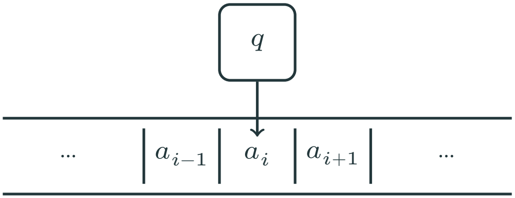

  - At each moment, only a finite part of the tape is populated by meaningful symbols; the rest is padded by “blank” symbols $B \in \Gamma - \Sigma$.

Turing Machines

- **Rules** of $\mathfrak{M}$ (elements of Δ) are of the form $\langle p, a \rangle \to \langle q, b, d \rangle$, where $p, q \in Q$, $a, b \in \Gamma$, and $d \in \{L, R, N\}$.

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- **Rules** of $\mathfrak{M}$ (elements of Δ) are of the form $\langle p, a \rangle \to \langle q, b, d \rangle$, where $p, q \in Q$, $a, b \in \Gamma$, and $d \in \{L, R, N\}$.
- Such a rule is executed as follows. If $\mathfrak{M}$ keeps p in its internal memory and observes a on the tape, then the following move is performed:
    1. replace a with b in the cell;
    2. replace p with q in the internal memory;
    3. if $d = L$, move one cell left; if $d = R$, move one cell right; if $d = N$, stay on the same cell.

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- A deterministic machine, on a given input, has a unique execution trajectory; in general, the trajectory may branch.
- Once a machine runs into state q_F, it stops successfully, and the word on the tape is the output.
- It is also possible to stop unsuccessfully or to run infinitely long.

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- This class is called co-NP.
- Example: SAT is in NP, while TAUT (checking whether a Boolean formula is a tautology) is in co-NP.

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- Moreover, if the computation is polynomial, it can be performed also polynomially on the Turing machine.
    - The degree of the polynomial could change.

Cook – Levin Theorem

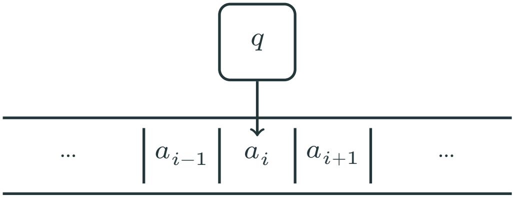
Theorem

SAT is NP-complete, that is, if $A \in$ NP, then A is m-reducible to SAT.

Cook – Levin Theorem

Proof sketch.

- Suppose $A \in$ NP, let us show $A \leq_m^P$ SAT.
- We encode each configuration of the non-deterministic Turing machine for A as a binary word:



$$0^m \quad a_1 \quad \dots \quad 0^m \quad a_{i-1} \quad q \quad a_i \quad 0^m \quad a_{i+1} \quad \dots$$

Cook – Levin Theorem

- The sequence of configurations (protocol) of A on input x is encoded by a binary matrix (b_{ij}) of size $(m \cdot p(|x|)) \times p(|x|)$.

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- Next, we construct a formula φ_x with variables $b_{00}, b_{01}, \ldots$ which expresses the fact that this matrix represents a correct protocol of a successful execution.

Cook – Levin Theorem

φ_x is a conjunction of the following claims:

1. the first row represents the configuration with x on the tape, the machine observing its first letter;
2. each next row is obtained from the previous one by one of the rules of the machine;
3. the last row includes state q_F and the answer "yes" (1).

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This is all expressible as Boolean formulae.

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- Since A was taken arbitrarily, we get NP-hardness of SAT.
- On the other hand, SAT is in NP, so it is NP-complete.

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- 3-SAT is a special version of SAT, where only 3-CNFs are allowed.
- Trivially, 3-SAT $\leq_m^P$ SAT… but we need the opposite reduction!
- Let us show that SAT $\leq_m^P$ 3-SAT.

Tseitin's Transformations

Theorem

For any Boolean formula A, there exists an equisatisfiable 3-CNF B of polynomial size.

- Equisatisfiability means that B is satisfiable iff so is A.

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For any Boolean formula A, there exists an equisatisfiable 3-CNF B of polynomial size.

- Equisatisfiability means that B is satisfiable iff so is A.
- Constructing an *equivalent* 3-CNF of polynomial size is not always possible: even translation to CNF can lead to exponential blowup.

Tseitin’s Transformations

- Tseitin’s transformations look like translation into 3-address (Assembler-like) code:
  $(a + b) * (c + d)$ is translated to
  “add a b t_1; add c d t_2; mul t_1 t_2 r”

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  $(a + b) * (c + d)$ is translated to
  "add a b t_1; add c d t_2; mul t_1 t_2 r"
- For each subformula we introduce a new variable and write the corresponding equivalences.

Tseitin's Transformations

Example: $(p \rightarrow q) \vee (q \rightarrow (p \rightarrow r))$

Tseitin's Transformations

Example: $(p \to q) \vee (q \to (p \to r))$

$$
\begin{aligned}
&(t_1 \leftrightarrow (p \to q)) \wedge \\
&(t_2 \leftrightarrow (p \to r)) \wedge \\
&(t_3 \leftrightarrow (q \to t_2)) \wedge \\
&(t_4 \leftrightarrow (t_1 \vee t_3)) \wedge \\
&t_4
\end{aligned}
$$

Tseitin's Transformations

Transform into 3-CNF by the following table:

$t_k \leftrightarrow (t_i \land t_j)$	$(\neg t_i \lor \neg t_j \lor t_k) \land (t_i \lor \neg t_k) \land (t_j \lor \neg t_k)$
$t_k \leftrightarrow (t_i \lor t_j)$	$(t_i \lor t_j \lor \neg t_k) \land (\neg t_i \lor t_k) \land (\neg t_j \lor t_k)$
$t_k \leftrightarrow (t_i \rightarrow t_j)$	$(\neg t_i \lor t_j \lor \neg t_k) \land (t_i \lor t_k) \land (\neg t_j \lor t_k)$
$t_k \leftrightarrow \neg t_i$	$(t_i \lor t_k) \land (\neg t_i \lor \neg t_k)$

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Transform into 3-CNF by the following table:

$t_k \leftrightarrow (t_i \wedge t_j)$	$(\neg t_i \vee \neg t_j \vee t_k) \wedge (t_i \vee \neg t_k) \wedge (t_j \vee \neg t_k)$
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$t_k \leftrightarrow \neg t_i$	$(t_i \vee t_k) \wedge (\neg t_i \vee \neg t_k)$

For our example, we get:

$(\neg p \vee q \vee \neg t_1) \wedge (p \vee t_1) \wedge (\neg q \vee t_1) \wedge$
$(\neg p \vee r \vee \neg t_2) \wedge (p \vee t_2) \wedge (\neg r \vee t_2) \wedge$
$(\neg q \vee t_2 \vee \neg t_3) \wedge (q \vee t_3) \wedge (\neg t_2 \vee t_3) \wedge$
$(t_1 \vee t_3 \vee \neg t_4) \wedge (\neg t_1 \vee t_4) \wedge (\neg t_3 \vee t_4) \wedge t_4$

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- The problem is as follows: given a graph G and a natural number k, determine whether there exists a set $S \subset V$ of k vertices, no two of which are connected.

Theorem

3-SAT $\leq_m^P$ INDSET, therefore, INDSET is NP-complete.

Independent Sets

- Reduction: given a 3-CNF φ, construct a graph G and give a number k (i.e., $(G, k) = f(\varphi)$), such that G has an independent set of k vertices iff φ is satisfiable.

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- Satisfiability is equivalent to taking, from each clause, **one literal,** so that the resulting set of literals is non-contradictory.

Independent Sets

- Example:

$$(z \vee \bar{w} \vee \bar{y}) \wedge (y \vee \bar{x} \vee \bar{w}) \wedge (x \vee w) \wedge (\bar{y} \vee \bar{x} \vee \bar{z})$$

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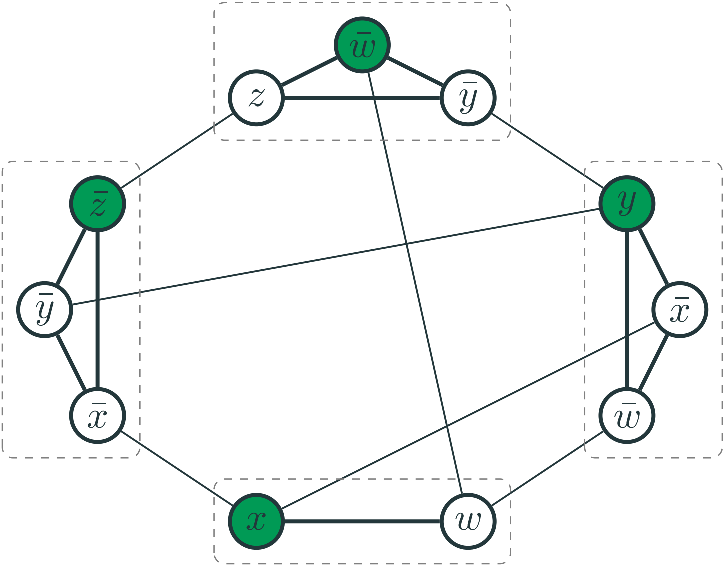
- We shall construct a graph in which each 3-clause will be modelled by a triangle (and each 2-clause by an edge, and each 1-clause by a vertex), and connect contradicting literals.
- k will be the number of clauses. Thus, we are forced to take exactly one vertex from each clause.

Independent Sets



Beyond Decision Problems

- An NP decision problem is the question **whether there exists** a *witness* y such that $R(x, y) = 1$.

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- Finally, we could ask for **all** witnesses.

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- The counting problem could be harder than the decision one (example: DNF-SAT).