Miscellanea

1. We know that 2-COLOR $\in$ P and 3-COLOR is NP-complete. What about 4-COLOR?

2. Is #2-COLOR a #P-complete problem?

3. By $T_{\mathfrak{M}}(n)$ let us denote $\max_{|x|=n} t_{\mathfrak{M}}(x)$, where $t_{\mathfrak{M}}(x)$ is the the number of steps $\mathfrak{M}$ performs when running on x (if it never stops, $t_{\mathfrak{M}}(x) = \infty$). Consider a Turing machine $\mathfrak{M}_2$ with two tapes. At each step, it operates on each tape. Show that there exists a one-tape Turing machine $\mathfrak{M}$ that computes the same function as $\mathfrak{M}_2$. Give an upper bound for $T_{\mathfrak{M}}(n)$ in terms of $T_{\mathfrak{M}_2}(n)$. Do the same for the more general case of a k-tape machine $\mathfrak{M}_k$.

4. A graph has 17 vertices, and the degree of each vertex is greater or equal than 8. Prove that such a graph is always connected.

5. A graph has 20 vertices, and the degree of each vertex is greater or equal than 10. Prove that such a graph always has a Hamiltonian path.

6. Vertices of graph G are colored in two colors (black and white). Each black vertex is connected to 3 white ones and each white vertex is connected to 4 black ones. Prove that the total number of vertices of G divides by 7.

7. Express the following property of a graph (a) by a first-order formula; (b) by a Boolean formula: graph G has no triangles, i.e., cliques of 3 vertices. (c) Does checking this property belong to P?

8. Is the following first-order formula generally true?

$$\big((\forall x\, R(x,x)) \wedge (\forall x \forall y\, (R(x,y) \to R(y,x)))\big) \to (\forall x \forall y \forall z\, (R(x,y) \wedge R(y,z)) \to R(x,z))$$

9. Let us denote by EXP (resp., NEXP) the class of decision problems that can be solved on a deterministic (resp., non-deterministic) Turing machine with running time bounded by $2^{\mathrm{poly}(n)}$, where $n = |x|$ is the input size. Prove that if EXP $\neq$ NEXP then P $\neq$ NP.