NP-Completeness of 3-COLOR

Stepan Kuznetsov

Discrete Math Bridging Course, HSE University

k-COLOR

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  $c \colon V \to C, \qquad$ where $|C| = k$
  (e.g., $C = \{1, \dots, k\}$)
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- The algorithmic problem k-COLOR: determine whether a given graph G is colorable in k colors.
- We have

$$k\text{-COLOR} \leq_m^P (k+1)\text{-COLOR},$$

  thus, complexity of k-COLOR grows with k.

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  but this is just a particular case of Cook–Levin.
- In order to prove NP-hardness of 3-COLOR, we need the **opposite** reduction.

3-COLOR

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- We need to construct a polynomially computable reducing function

$$f\colon \varphi \mapsto G_\varphi,$$

such that for any 3-CNF Boolean formula φ it is satisfiable if and only if the graph G_φ is colorable in 3 colors.

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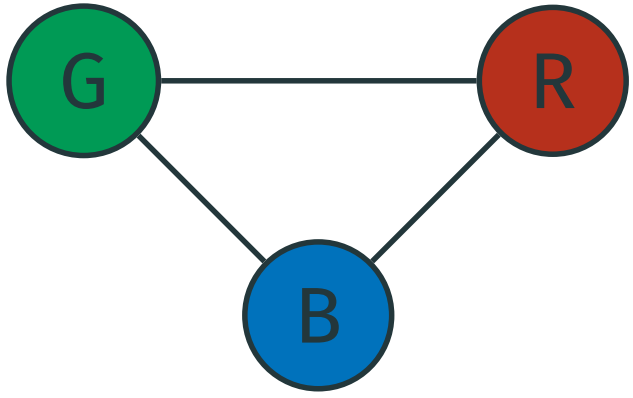

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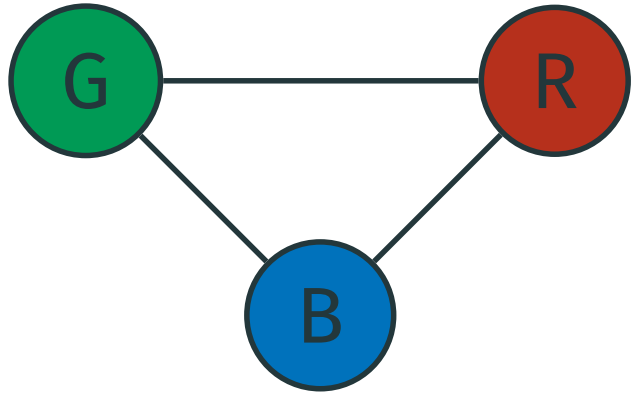
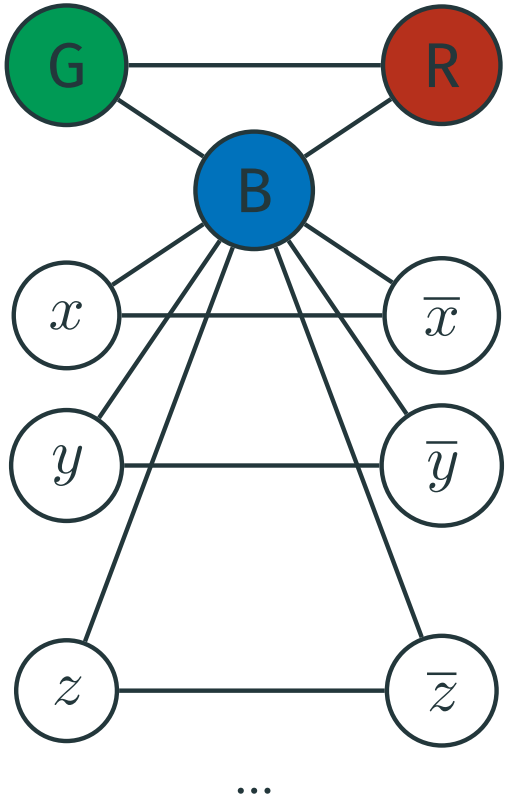


- We may suppose that the palette is colored as shown (otherwise rename the colors).

3-COLOR: Modelling Assignments

Next, we introduce a pair of vertices (x and $\bar{x}$) for each variable, and connect them as shown:



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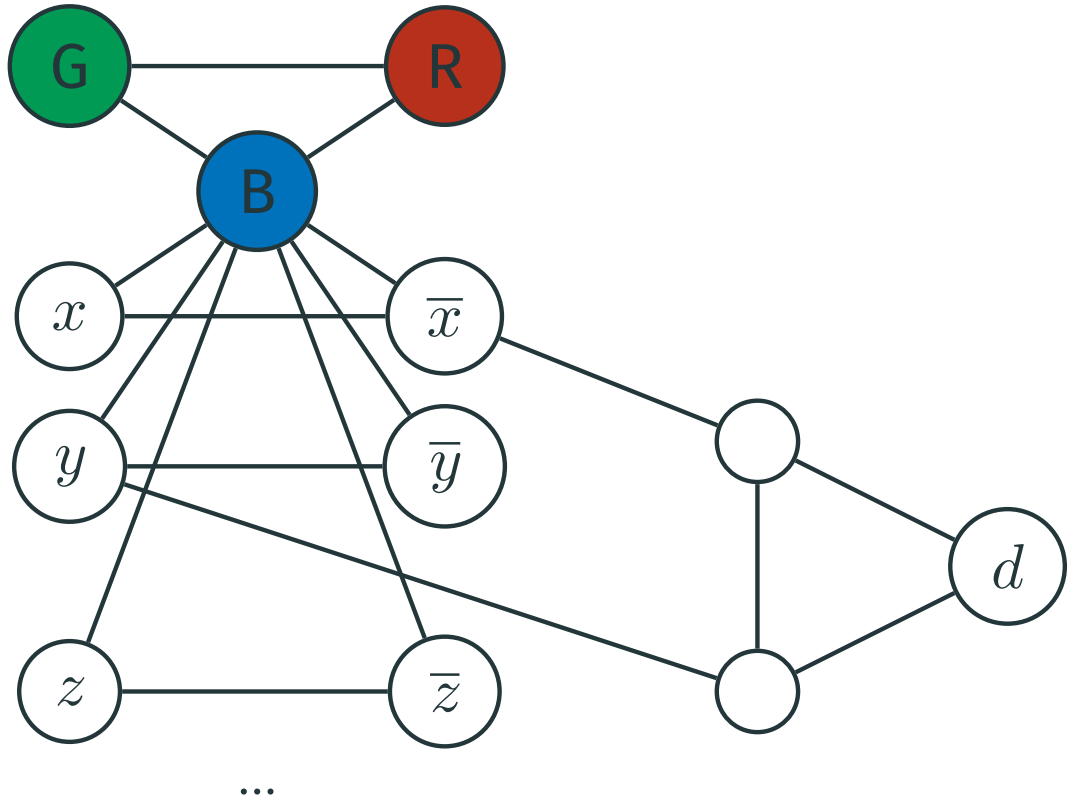
- Each correct 3-coloring of the graph we have so far corresponds to a truth assignment.
- Now we add **constraints** so that the assignment satisfies the 3-CNF φ.

3-COLOR: Satisfying a 2-Clause

We start with modelling a 2-clause, e.g., $\bar{x} \vee y$.



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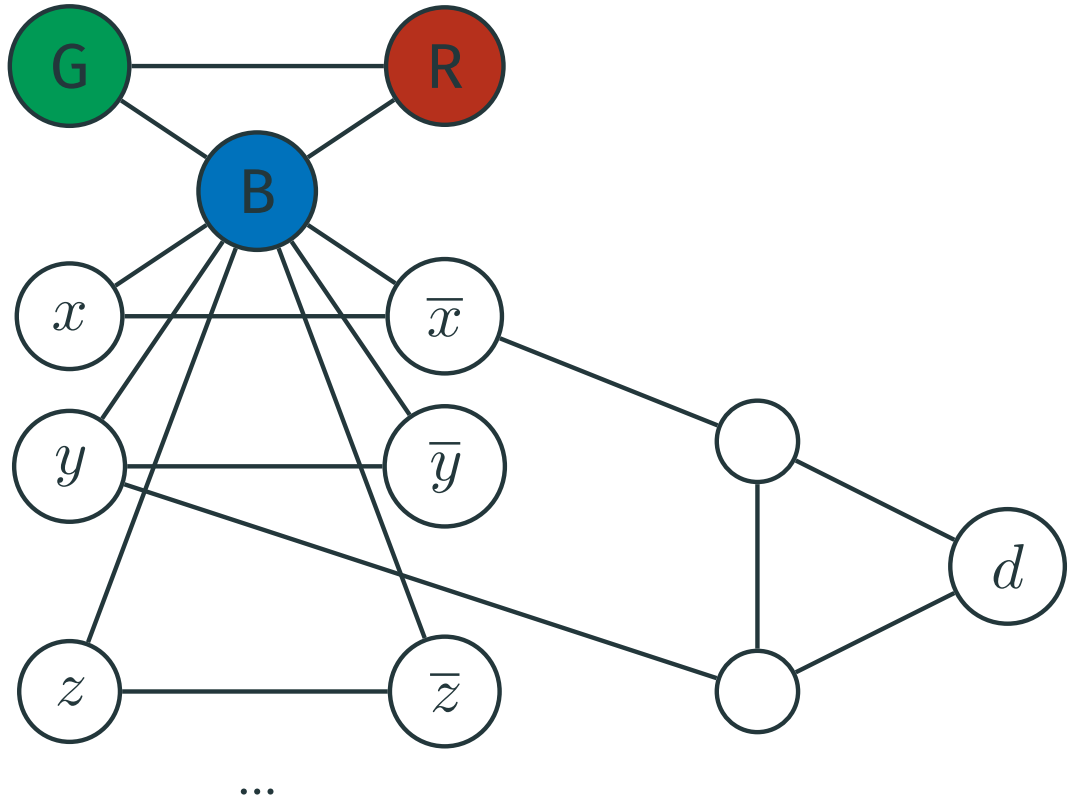
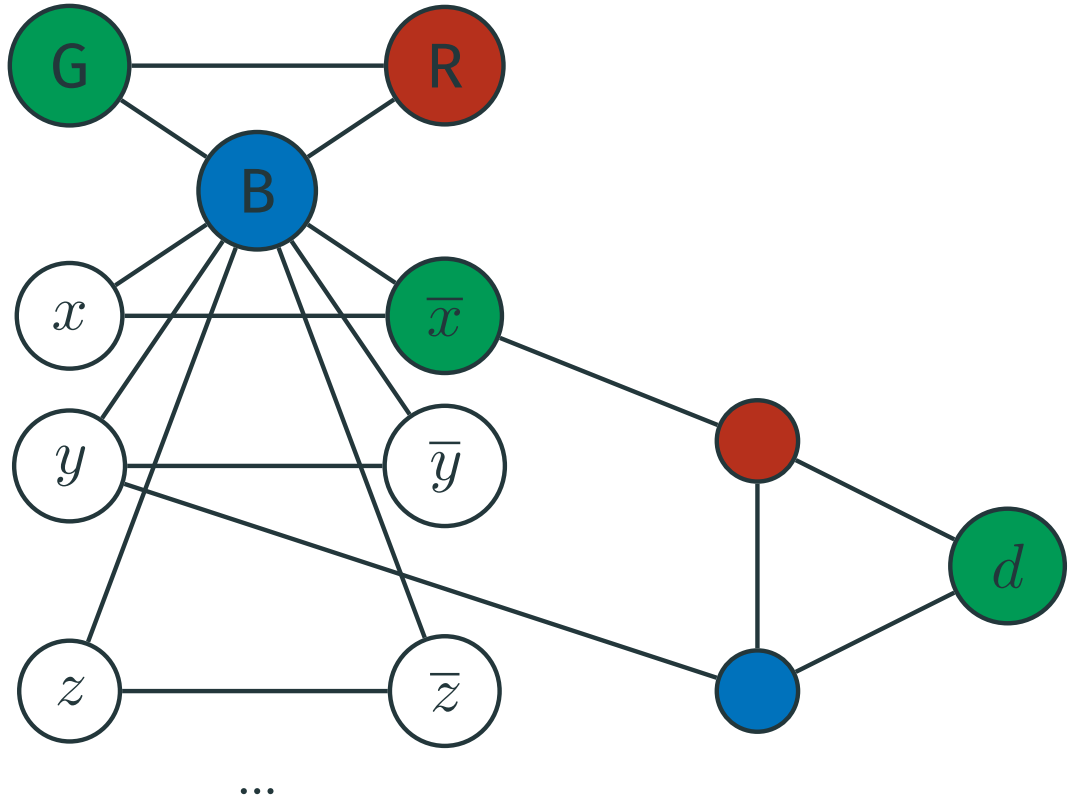


Claim: Vertex d **can be** colored in green if and only if $\bar{x}$ is green and y is green (maybe both).

3-COLOR: Satisfying a 2-Clause

If $\bar{x}$ or y is green, then we may color as follows:



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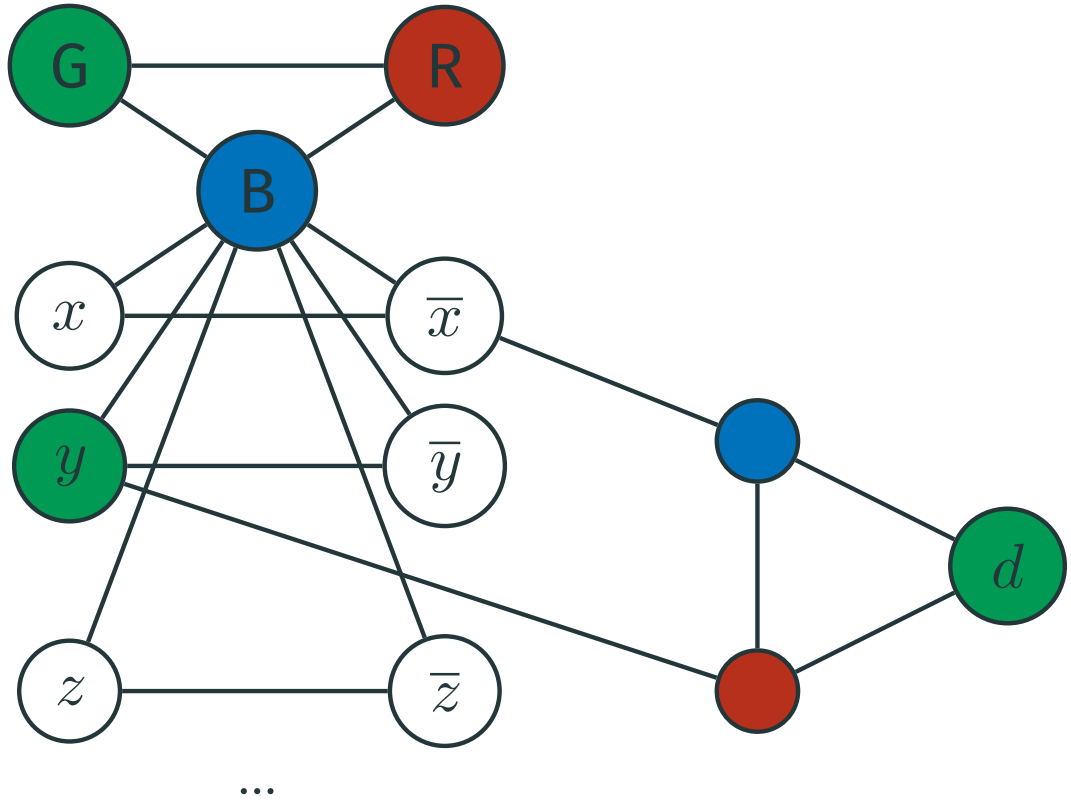

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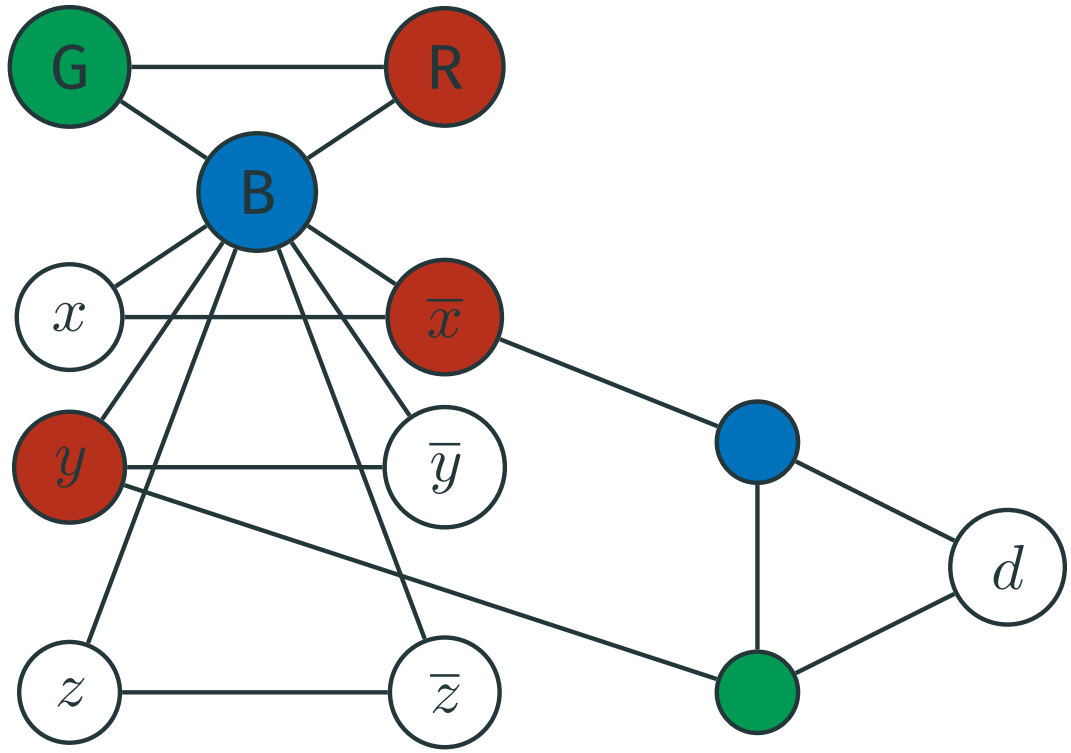
If both $\bar{x}$ and y are red, then one of the two unnamed vertices should be green, which prevents d from being green.



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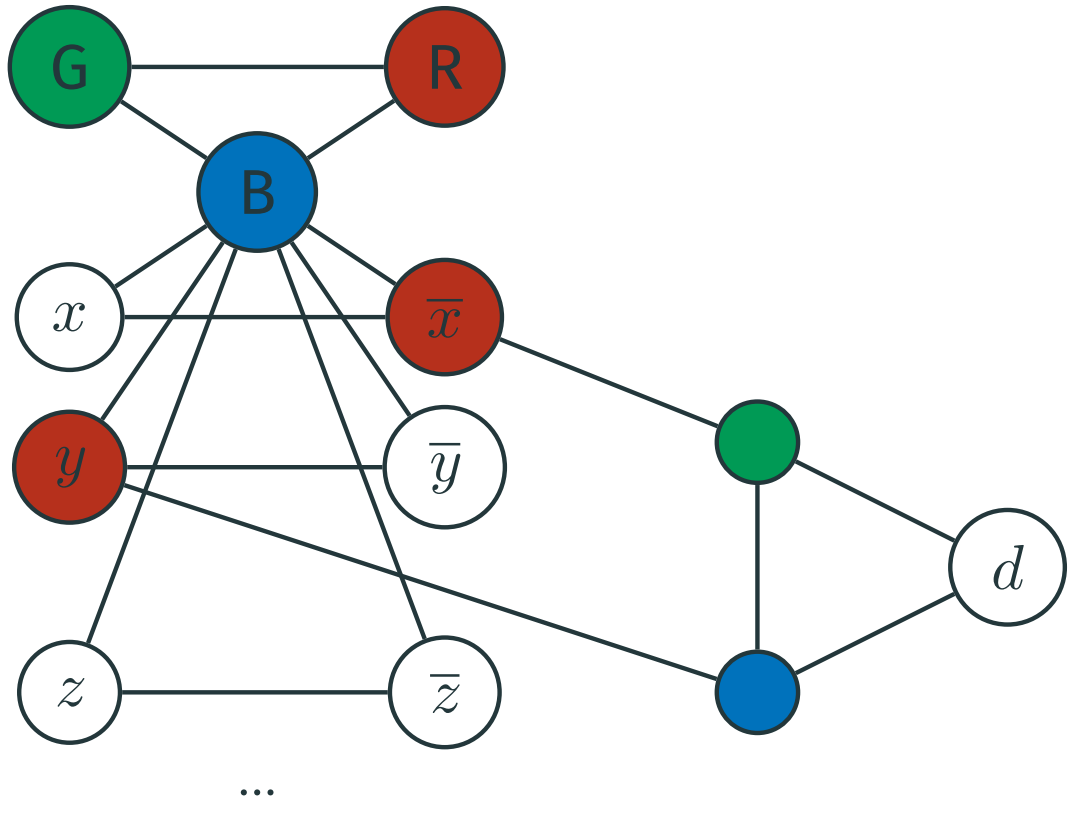
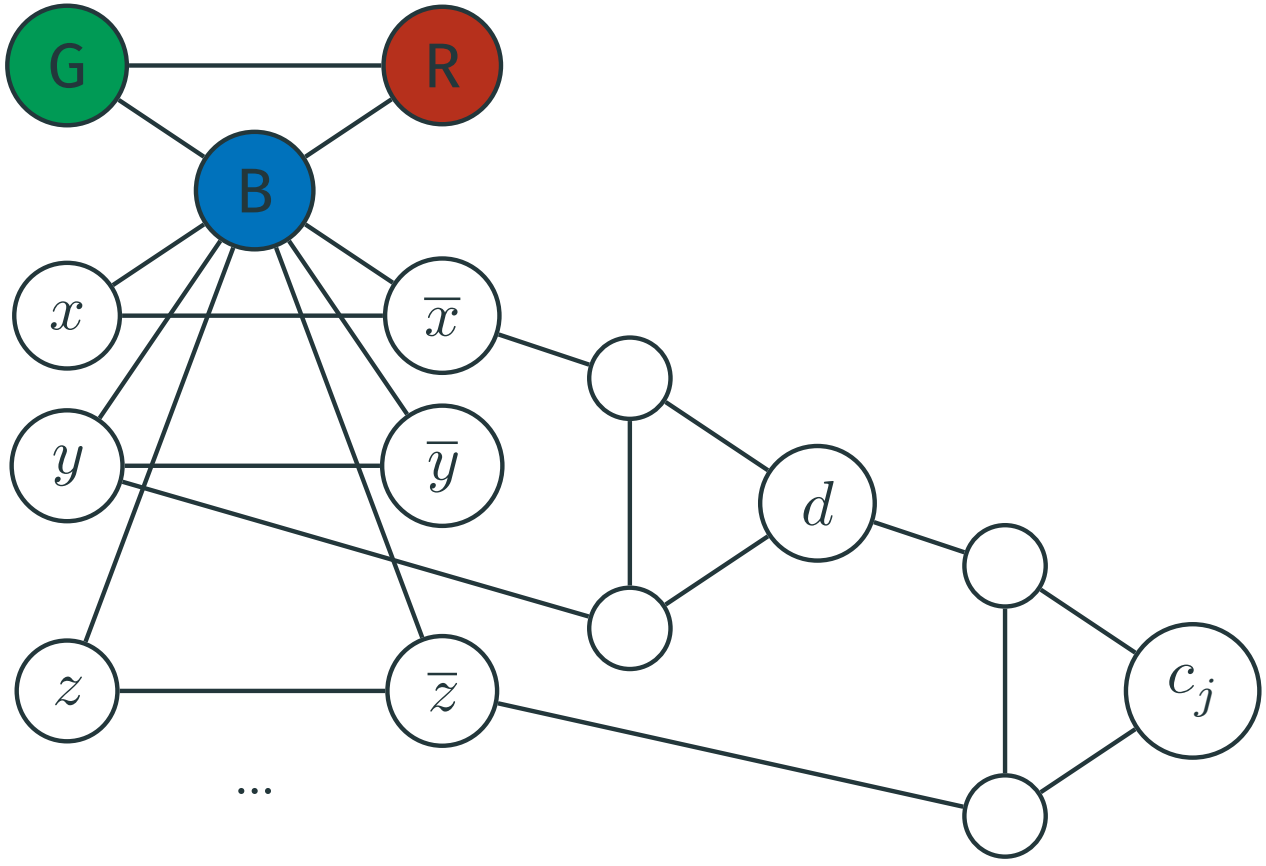


3-COLOR: Satisfying a 3-Clause

Now for a 3-clause, e.g., $C_j = \bar{x} \vee y \vee \bar{z}$, we copy the triangle once more:



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- Finally, we connect each c_j to both R and B, in order to enforce c_j being green.
- The resulting graph G_φ is 3-colorable if and only if φ is satisfiable.

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    - We have also considered more sophisticated behaviour, e.g., algorithms with delays.

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- We also take care of the **running time** of the algorithm.
- A reasonable notion of "effectivity" is **polynomial time computation**, where computing $f(x)$ takes $\leq p(|x|)$ steps (where p is a fixed polynomial).

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- Special case: **decision problem**, where the output is just one bit ($f\colon \{0,1\}^* \to \{0,1\}$).
- Another way is to express a decision problem as a subset (predicate) $A \subseteq \{0,1\}^*$, where $A = \{x \mid f(x) = 1\}$.

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- The next higher complexity class is NP ("non-deterministically polynomial").
- Def. 1: the problem is in NP, if there is a non-deterministic polynomial-time algorithm, such that $x \in A$ iff the algorithm yields 1 on at least one path ("angelic choice").

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- Any problem from NP is algorithmically solvable (brute-force search for witness).
- The model case is **satisfiability of Boolean formulae** (SAT).

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- **Checking** this condition is easy (polynomial); **finding** a satisfying assignment for a given φ could be hard.

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  - It is possible that P $=$ NP, then SAT is polynomial.
- We show "hardness" of SAT by a conditional argument, comparing decision problems by their complexity.
- For two problems $A, B \subseteq \{0,1\}^*$, we say

$$A \leq_m^P B \iff \forall x \, (x \in A \iff f(x) \in B)$$

for some polynomially computable function f **(polynomial Carp reduction)**.

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    - There are also two degenerate problems, which are even simpler: $A = \emptyset$ and $A = \{0,1\}^*$ ("always no" or "always yes").
- On top of NP, there are **NP-complete** problems—the hardest ones, w.r.t. $\leq_m^P$.

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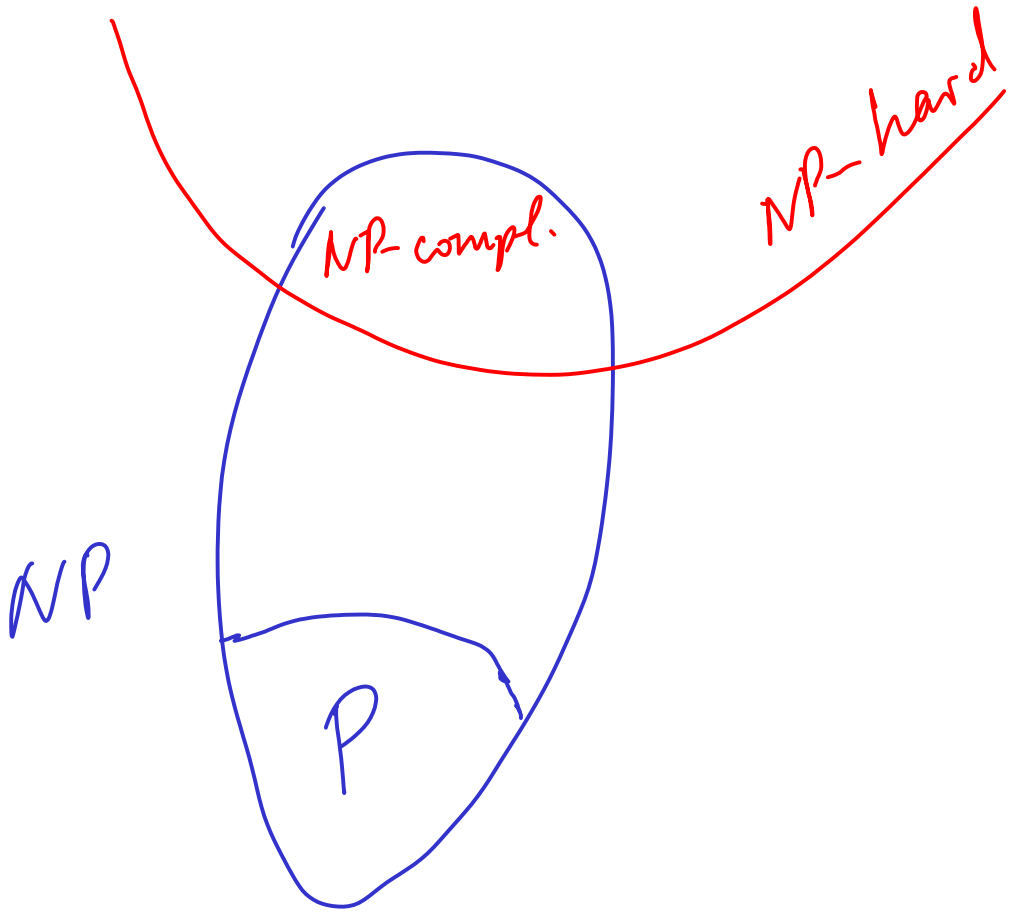
- A problem is **NP-complete** if it is NP-hard and belongs to NP.
  - For example, the validity problem for **first-order** formulae (with quantifiers) is NP-hard, but not NP-complete (being undecidable).

Course Overview

Complexity landscape, if P $\neq$ NP:



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  - $x \in A \iff \varphi_{A,x}$ is satisfiable.

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- A **DNF** (disjunctive normal form) is a big disjunction of clauses of the form $\neg x \wedge y \wedge \neg z$.

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- DNF-SAT is polynomially solvable: one just has to find at least one non-contradictory clause.
- For CNF-SAT, there is the **resolution method**, which is a bit more advanced than brute-force.

Course Overview

- In the resolution method, the CNF is **saturated** by applying the resolution rule:

$$\frac{A \vee p \quad \neg p \vee B}{A \vee B}$$

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- **Completeness theorem:** the CNF is satisfiable iff it does not include the empty clause ("false") after saturation.
- Unfortunately, for $k \geq 3$ saturation can lead to exponential blowup.

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- In contrast, **2-SAT belongs to P**, since applying resolutions to 2-clauses yields also 2-clauses, and there are a polynomial number of such.
- Also, resolution allows to solve the **search problem** for 2-SAT (yield at least one satisfying assignment), or yield all satisfying assignments with **polynomial delay**.

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- Graphs and their generalisations (e.g., hypergraphs) are a generic representation of **structured data** in numerous applications: social networks, bioinformatics, GIS, network topology, etc.
- Graph theory provides many examples of problems from the NP class: "given a graph G, determine whether it includes some specific substructure".

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- For some of these problems, we proved that they belong to P: finding an Euler path / cycle; 2-colorability (via reduction to 2-SAT).
- For others, we proved NP-completeness. These include 3-COLOR (and k-COLOR for $k \geq 3$); Hamitonian path / cycle search (both directed and undirected); subgraph isomorphism and its special cases (CLIQUE, INDSET, VERTEXCOVER).

Course Overview

- Finally, the graph isomorphism problem is neither known to belong to P, neither known to be NP-complete.
- For proving NP-completeness, we usually construct a reduction **from** 3-SAT, e.g.:

$$\text{3-SAT} \leq_m^P \text{3-COLOR},$$

while for proving polynomial solvability the **opposite** reduction can be used:

$$\text{2-COLOR} \leq_m^P \text{2-SAT}.$$