More on Hamiltonian Paths

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- A Hamiltonian path is a path which traverses each **vertex** of a given graph exactly once.
- Such a path can be either a **cycle** or have distinct start and end.

The Hamiltonian Path Problem

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- On the previous lecture, we have sketched a proof of NP-completeness for the problem of existence of a Hamiltonian path in a **directed** graph, with fixed start s and end t.
- This problem is called HAMPATH.
- In our construction s had no incoming edges and t had no outgoing ones.
- Thus, (1) we do not have to fix s and t in advance; (2) by identifying s and t, we get NP-completeness for HAMCYCLE.

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- NP-hardness is proved by backwards reduction:

$$\text{3-SAT} \leq_m^P \text{HAMPATH}.$$

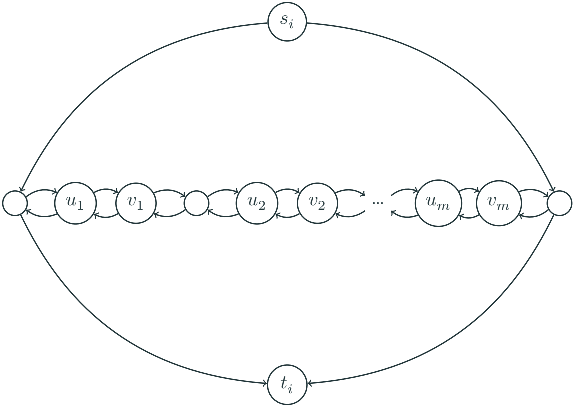
Backwards Reduction

Backwards reduction means that we need to construct a polynomial algorithm which takes a 3-CNF φ and constructs a directed graph G_φ with the following property:

G_φ has a Hamiltonian path from s to t
if and only if
φ is satisfiable.

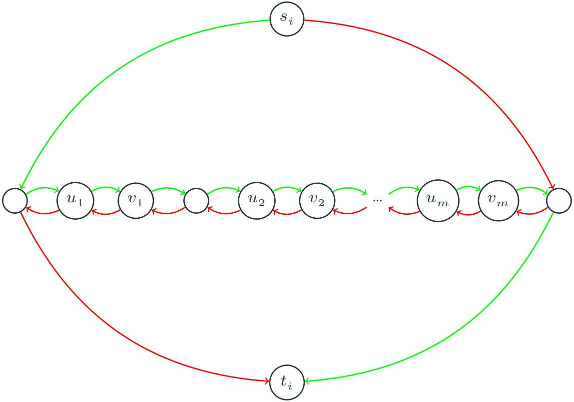
Gadget

Let φ include m clauses. For each *variable* x_i of φ we construct the following subgraph called *gadget:*



Gadget

In a Hamiltonian path, this gadget can be traversed, from s_i to t_i, only in the following two ways:



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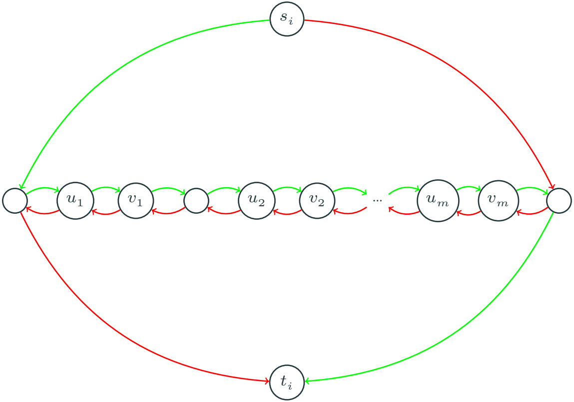

Green means true, red means false.

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- The 3-CNF φ is modelled in our graph G_φ as follows:

$$\begin{array}{r}
\text{variables} \Rightarrow \text{gadgets} \\
\text{clauses } C_j \Rightarrow \text{extra vertices } c_j \\
\hline
\text{satisfying assignment} \Rightarrow \text{Hamiltonian path}
\end{array}$$

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- The i-th gadget can be traversed either by the green, or by the red path.
- This encodes the value of x_i in our assignment: $x_i = 1$ or $x_i = 0$.

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- For $\neg x_i$ in C_j, we can visit it on the red path: $v_j \to c_j \to u_j$.
- Thus, if (and only if) the assignment satisfies φ, the path traversing the gadgets can be augmented by visiting all c_j's, thus becoming a Hamiltonian path.

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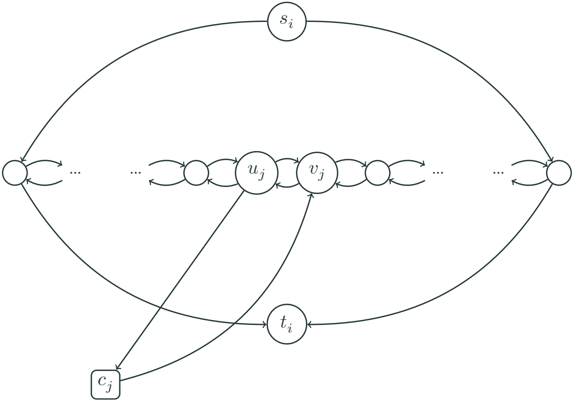
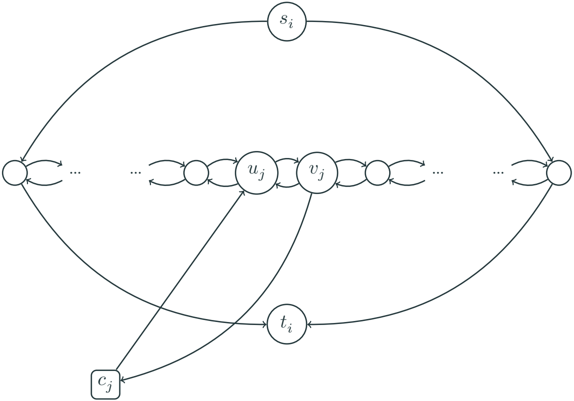


Modelling Clauses

If C_j includes $\neg x_i$:



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- Finally, we connect the gadgets: $s_1 = s$, $s_2 = t_1$, ..., $s_n = t_{n-1}$, $t_n = t$.
- There is a Hamiltonian path from s to t in G_φ iff φ is satisfiable.

Undirected Graphs

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- However, it is crucial for our construction that the graph is **directed**.
- If we make edges undirected, each c_j would be traversable on **both** red and green paths, thus, the graph will **always** have a Hamiltonian path.

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- The idea is to model direction by replacing each vertex with a more complicated subgraph.

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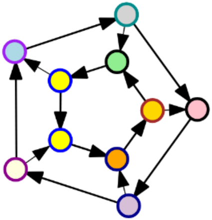
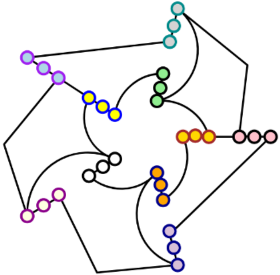
- Each edge $u \to v$ of the original graph is replaced by $u_{\text{out}} - v_{\text{in}}$.
- The Hamiltonian path can traverse v_{mid} only between v_{in} and v_{out}.

Undirected Graphs



$\Rightarrow$



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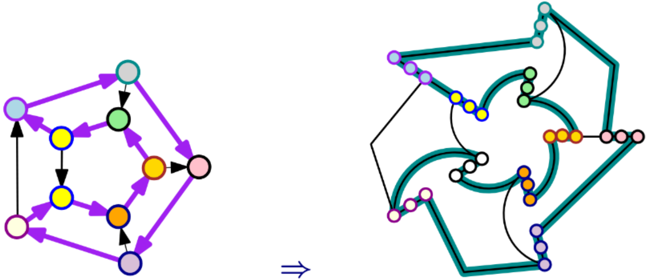

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- Each Hamiltonian (directed) path in G maps onto a Hamiltonian (undirected) path in G'.
- Conversely, from a Hamiltonian path in G' (from s_{mid} to t_{mid}) we can reconstruct a **directed** Hamiltonian path in the original graph G, taking only the 'mid'-vertices.

Undirected Graphs

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- On the next lecture, we shall discuss NP-completeness of yet another problem: 3-COLOR.