# Boolean Logic
# Resolution Method

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# Course Outline

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- The aim of this course is to provide a background of discrete mathematics and computational complexity ideas useful for data science.
- Given a very limited time for the course, we have to choose a simple central topic to use as a running example.
- And this topic is going to be **Boolean logic.**
- Let us first remind the basics of it.

# Boolean Functions

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- A Boolean function is a *finite* object: it can be represented by a table (so-called *truth table*) of $2^n$ rows.

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- For example, we have 4 unary Boolean functions and $16 = 2^{2^2}$ binary ones.
- The only interesting unary Boolean function is *negation*, defined by the following truth table:

| $x$ | $\neg x$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

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- As for binary functions, among 16 possible there are several interesting ones: $\wedge$ (*conjunction*, “and”), $\vee$ (*disjunction*, “or”), $\to$ (*implication*, “if … then”).
- The truth tables for them are as follows:

| $x$ | $y$ | $x \wedge y$ | $x \vee y$ | $x \to y$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
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**Theorem**

*Any Boolean function can be represented as a composition of* $\neg$, $\wedge$, $\vee$, $\rightarrow$.

- For example, the **majority function** of three elements, which gives 1 iff at least two of its arguments are 1, has the following representation:

$$\mathrm{MAJ}_3(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

# Boolean Formulae

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- Such representations are formalized by **Boolean formulae**.
- The set $\mathrm{Fm}$ of Boolean formulae over a set of *variables* $\mathrm{Var}$ is defined as the minimal set obeying the following:
  - $\mathrm{Var} \subseteq \mathrm{Fm}$
  - $\bot, \top \in \mathrm{Fm}$ (these are *constants* for 0 and 1)
  - if $A, B \in \mathrm{Fm}$, then $(A \wedge B), (A \vee B), (A \to B), \neg A \in \mathrm{Fm}$

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  - $((r \to c) \land \neg c) \to \neg r$
- This formula is true for **any** values of $r$, $c$.
- Such formulae are called **tautologies.**

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- However, this requires **exponential time** (checking $2^n$ possible **assignments**).
- Is there a faster algorithm?..

# Satisfiability

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- A Boolean formula is satisfiable, if it is true for **at least one** assignment.
- Such an assignment is called a **satisfying assignment.**

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- Satisfiability is indeed dual to being a tautology:

  $$A \text{ is a tautology} \iff \neg A \text{ is not satisfiable.}$$

- And actually satisfiability is a very general model example of situations where we seek for existence of an object (here: satisfying assignment) with given properties (here: the given formula $A$).

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- The elementary dis- / conjunctions are called **clauses.**

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- The degenerate DNF with zero clauses is constant “false,” $\bot$.
- Dually, the empty CNF is $\top$, “true.”
- Indeed, DNF clauses **add possibilities,** while CNF ones **impose constraints.**

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| 0 | 1 | 0 | 0 | |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | $(x \wedge \overline{y} \wedge \overline{z})$ |
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| 1 | 1 | 0 | 0 | |
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(the clauses are combined with $\bigvee$)

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- The full DNF presented on the previous slide,

  $$(\overline{x} \wedge \overline{y} \wedge \overline{z}) \vee (x \wedge \overline{y} \wedge \overline{z}) \vee (x \wedge \overline{y} \wedge z) \vee (x \wedge y \wedge z),$$

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  is not the optimal (shortest) one for the given function.
- The following DNFs are equivalent to it and are shorter:

$$(\overline{x} \wedge \overline{y} \wedge \overline{z}) \vee (x \wedge \overline{y}) \vee (x \wedge y \wedge z)$$

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- This means that already $\neg, \vee$ and, dually, $\neg, \wedge$ are complete systems.
- In particular, $A \to B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$.

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| 0 | 1 | 0 | 0 | $(x \vee \overline{y} \vee z)$ |
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$\bigwedge$

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- This clause is satisfiable, and so is the whole DNF.
- For CNFs, satisfiability is a non-trivial question.
- Translating from CNF to DNF does not help: this could increase the size exponentially.

# Resolution Method

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# Resolution Method

- The fact that a formula is a tautology can be established (in contrast with checking all assignments) by **proving** it in the *classical propositional calculus.*
- In this course, we consider a dual situation: **disproving** satisfiability via **resolution method.**
- Recall that, by duality, proving that $A$ is a tautology is equivalent to disproving satisfiability of $\neg A$.

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- The one and only rule is **resolution,** which generates new clauses from already existing ones:

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- **Contradictive clause:** the empty one (obtained by resolution from $p$ and $\overline{p}$).

# Resolution Method

**Theorem (Soundness and Completeness)**

*A CNF is not satisfiable if and only if one can obtain the empty clause by applying resolutions, starting from the given CNF.*

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- The “only if” part (completeness) will be proved next time.

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# Saturation

- The soundness and completeness theorem validates the following **algorithm** for checking satisfiability of CNFs.
- Given a CNF (as a set of clause), let us **saturate** it by exhaustively applying resolutions until they stop generating new clauses.
- The CNF is satisfiable if and only if its saturation does not include the empty clause.

# Translating into CNF

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- The resolution method works only with CNFs.
- When checking a formula $A$ for being a tautology, it is convenient for $A$ to be in DNF, since then $\neg A$ is easily transformed into CNF by De Morgan.
- For implications, keep in mind the following equivalences:

$$A \to B \equiv \neg A \vee B \qquad \neg(A \to B) \equiv A \wedge \neg B$$

# Example

- Let us check whether the following formula is a tautology:

$$A = (p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

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- Let us negate $A$ and check whether $\neg A$ is satisfiable

$$\neg A = (\overline{p} \vee \overline{q} \vee r) \wedge (\overline{p} \vee q) \wedge p \wedge \overline{r}$$

# Example

$\overline{p} \vee \overline{q} \vee r$

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# Example

| | |
|---|---|
| $\overline{p} \vee \overline{q} \vee r$ | $\overline{q} \vee r$ |
| $\overline{p} \vee q$ | $\overline{p} \vee r$ |
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| $\overline{r}$ | $\bot$ |

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| | | |
|---|---|---|
| $\overline{p} \vee \overline{q} \vee r$ | $\overline{q} \vee r$ | |
| $\overline{p} \vee q$ | $\overline{p} \vee r$ | |
| $p$ | $r$ | |
| $\overline{r}$ | $\bot$ | $\Rightarrow$ NOT SATISFIABLE |

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- Indeed, applying resolution to 2-bounded clauses also yields a 2-bounded clause.
- And the total number of 2-bounded clauses is $\leq 4n^2 + 2n + 1$.
- Thus, checking satisfiability for 2-CNF **can be performed in polynomial time.**

# Polynomiality

- Traditionally, an algorithmic problem is considered "practically solvable," if there exists a polynomially bounded algorithm for it (that is, the number of steps, even in the worst case, is $\leq p(|x|)$, where $p$ is a fixed polynomial and $|x|$ is the input length).

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- This is, of course, a gross approximation: let, say, $p(n) = n^{100}$.

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  - A problem is polynomially solvable on a “real” computer iff it is polynomially solvable on a 1-tape Turing machine.
  - … but with a different degree of $p$.

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    - By now, it is unknown whether it is in P.
    - However, this is highly unlikely, because then a large class of similar problems, called NP, would be also in P.
    - These problems include, e.g., subgraph isomorphism, knapsack problem, subset sum problem, …

# Home Assignment # 1

- Satisfiability for 2-CNF will be your task for HW # 1.
- The **easy** version is to check satisfiability (using resolution method).
- The **full** task is to check satisfiability **and,** if the answer is “yes,” to return one of the satisfying assignments.

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- The program (in Python) should implement two functions:
  1. `is_satisfiable`, which takes a CNF and answers `True` or `False`, depending on whether it is satisfiable.
  2. `sat_assignment`, which takes a CNF and returns a satisfying assignment as an associative array:

     ```
     { 'x': True, 'y': False, 'z': True }
     ```

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- The CNF is a conjunction (`/\`) of clauses.

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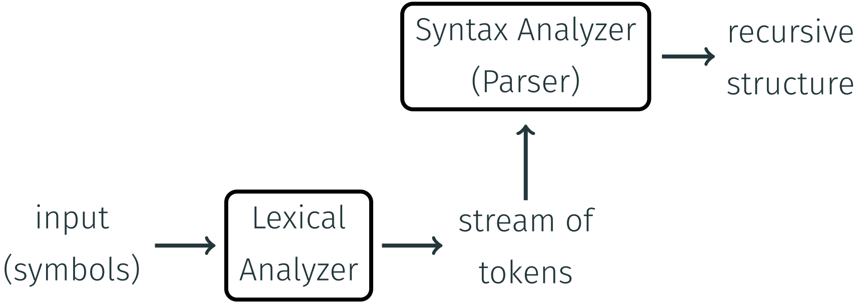
- We shall use specialized software, PLY (Python Lex & Yacc), in order to automatize the parsing process.

# The Parsing Workflow



# Lexical Analysis

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int main(void)
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    printf("Hello, World!\n");
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- Tokens are much more convenient to work with (in the grammar).

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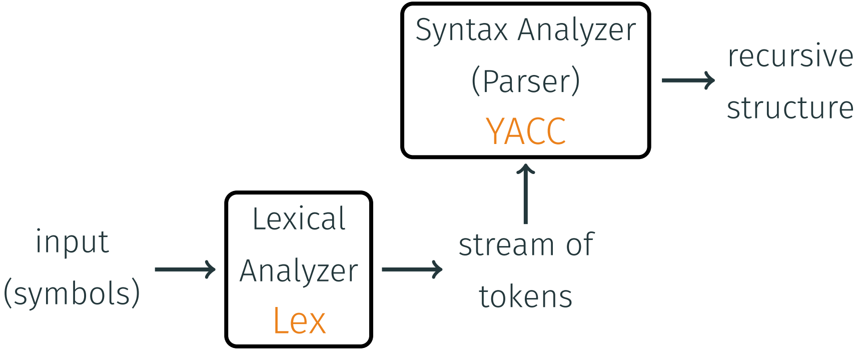
# Implementation: Lex & Yacc



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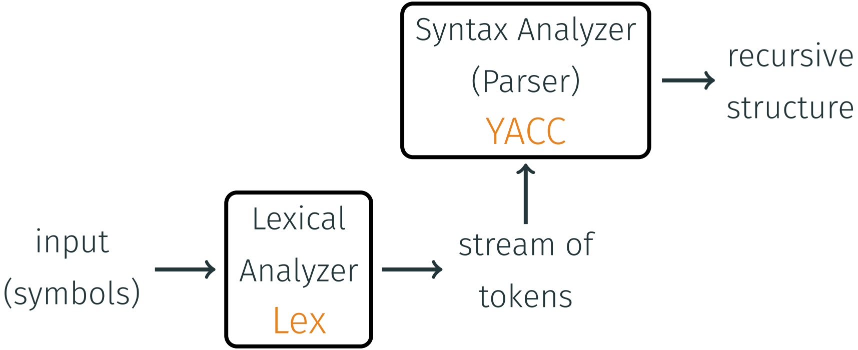


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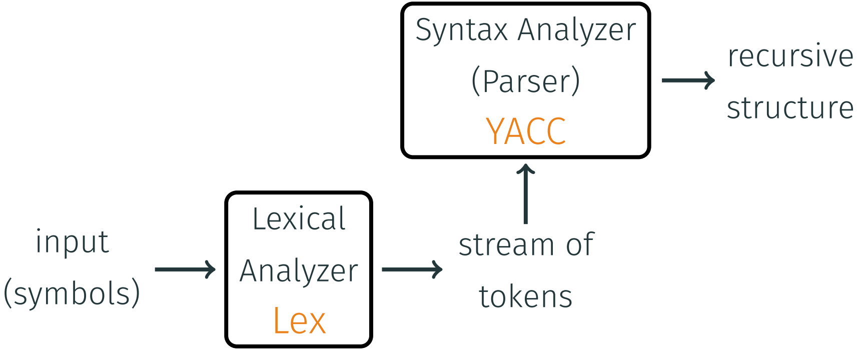


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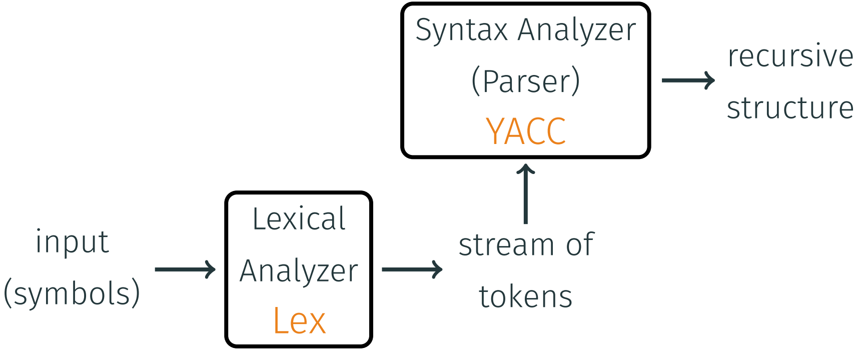


- YACC = Yet Another Compiler Compiler
- In Python, we use PLY (Python Lex & Yacc).

# PLY Code for Lexical Analysis

- Declare tokens and literals (one-symbol tokens):

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tokens = [ 'INT' ]
literals = ['+','-','(',')','^','x']
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- For each token, declare a “`t_`”-function:

```
def t_INT(t):
    r'\d+'
    try:
        t.value = int(t.value)
    except ValueError:
        print "Too large!", t.value
        t.value = 0
    return t
```

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- Another example: regular expression for **names** (identifiers)

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t_NAME    = r'[a-zA-Z_][a-zA-Z0-9_]*'
```

- Finally, build the lexer:

```
import ply.lex as lex
lex.lex()
```

# PLY Code for Parsing

- Each rule of the grammar is implemented as a "**p_**"-function:

```
def polymult(p,q) :
    r = []
    for i in xrange(len(p)) :
        for j in xrange(len(q)) :
            safeadd(r,i+j,p[i]*q[j])
    return r
```

...

```
def p_tm_mult(p):
    "tm : tm '(' expr ')'"
    p[0] = polymult(p[1],p[3])
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- A "`p_`"-function generates an object `p[0]`, using `p[1]`, `p[2]`, ..., which are obtained from the lexer or recursively from parsing.

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- For priorities of operations, see another example available on the webpage: calculator.

Good luck!

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- During the course, we'll highlight possible connections and applications in data analysis.