First-Order Predicate Logic (exercises)

Throughout this exercise sheet, variables (x, y, z, …) range over elements of a non-empty *domain* M. Predicate symbols' (P, Q, R, …) interpretations range over *predicates* of M, i.e., functions of the form $\bar{P}\colon \underbrace{M \times \ldots \times M}_{k} \to \{0,1\}$, where k is the number of arguments of P.

1. Which of the following formulae are generally true (i.e., true on any M and under any interpretations of predicate symbols)?
   (a) $\forall x\,(P(x) \lor Q(x)) \to (\forall x\,P(x)) \lor (\forall x\,Q(x))$
   (b) $\forall x\,(P(x) \lor Q(x)) \to (\forall x\,P(x)) \lor (\exists x\,Q(x))$
   (c) $(\forall x\,(P(x) \to Q(x)) \land \neg\exists x\,Q(x)) \to \forall y\,\neg P(y)$
   (d) $\forall x\,\exists y\,R(x,y) \to \exists y\,\forall x\,R(x,y)$
   (e) $\exists x(D(x) \to \forall y\,D(y))$

2. Which of the following formulae are satisfiable (i.e., true on some M for some interpretation of predicate symbols)?
   (a) $\exists x\,\forall y\,(Q(x,x) \land \neg Q(x,y))$
   (b) $\exists x\,\exists y\,(P(x) \land \neg P(y))$
   (c) $\exists x\,\forall y\,(Q(x,y) \to \forall z\,R(x,y,z))$.

3. Show that the following formula could be true only on an infinite M:
$$(\forall x\,\exists y\,Q(x,y)) \land \forall x\,\forall y\,\forall z\,(\neg Q(x,x) \land (Q(x,y) \to (Q(y,z) \to Q(x,z)))).$$

4. Let $M = \mathbb{N}$ be the set of natural numbers, and let $R(a,b)$ be true if and only if $a < b$. Write a formula $\varphi(u,v)$ with two parameters, u and v, which is true if and only if $v = u + 1$.

5. Write a formula using a binary predicate symbol R which expresses the fact that the R relation is:
   (a) reflexive
   (b) transitive
   (c) symmetric
   (d) antisymmetric

6. Show that the following formula is true on $M = \{a,b,c\}$ for any interpretation of R:
$$(\forall x\,R(x,x)) \land \forall x\,\forall y\,\forall z\,(R(x,z) \to (R(x,y) \lor R(y,z))) \to \exists u\,\forall v\,R(u,v).$$