NP-Completeness of HAMPATH

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The Hamiltonian Path Problem

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Theorem

HAMPATH is NP-complete.

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- NP-hardness is proved by backwards reduction:

$$\text{3-SAT} \leq_m^P \text{HAMPATH}.$$

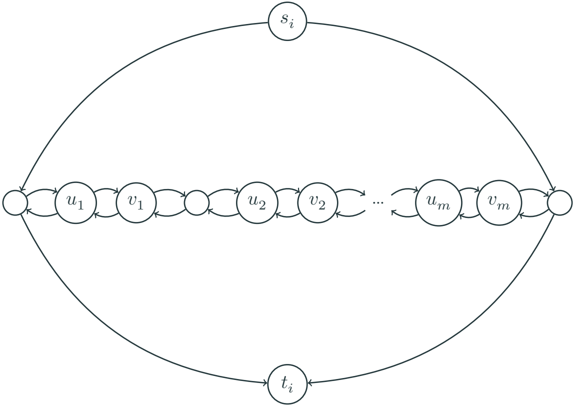
Backwards Reduction

Backwards reduction means that we need to construct a polynomial algorithm which takes a 3-CNF φ and constructs a directed graph G_φ with the following property:

G_φ has a Hamiltonian path from s to t
if and only if
φ is satisfiable.

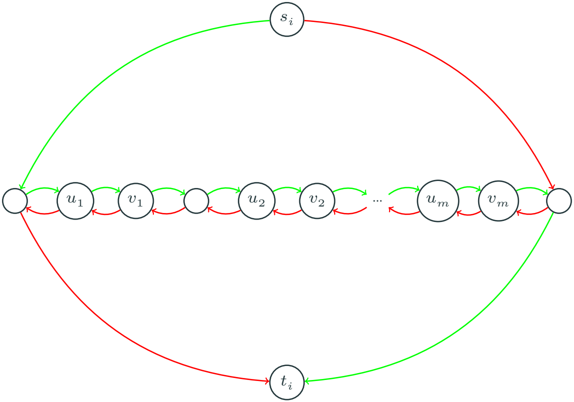
Gadget

Let φ include m clauses. For each *variable* x_i of φ we construct the following subgraph called *gadget:*



Gadget

In a Hamiltonian path, this gadget can be traversed, from s_i to t_i, only in the following two ways:



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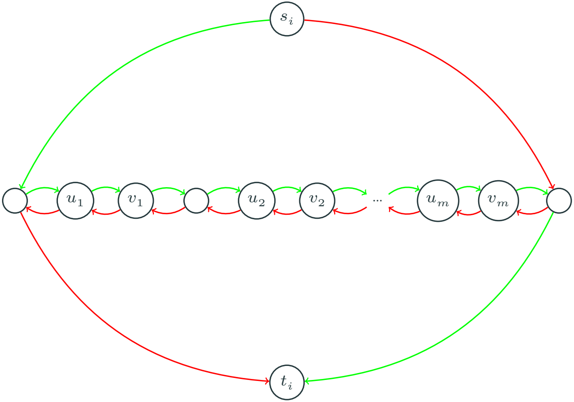
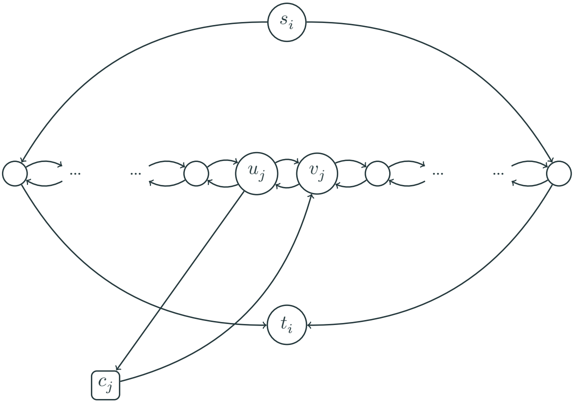


Green means true, red means false.

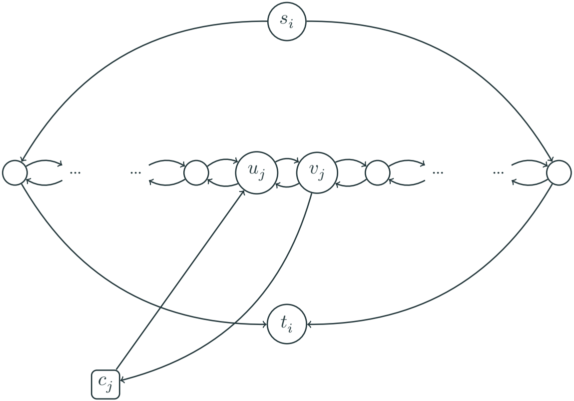
Modelling Clauses

Next, for each clause C_j we add a designated vertex c_j. If C_j includes x_i:



Modelling Clauses

If C_j includes $\neg x_i$:



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- If C_j includes x_i, then it can be visited, while traversing the i-th gadget, only on the green path ($x_i = 1$).

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- Finally, we connect the gadgets: $s_1 = s$, $s_2 = t_1$, ..., $s_n = t_{n-1}$, $t_n = t$.
- There is a Hamiltonian path from s to t in G_φ iff φ is satisfiable.