NP-completeness of the Hamiltonian Path Problem

Definition. A *Hamiltonian path* is a path in a graph which visits each vertex exactly once.

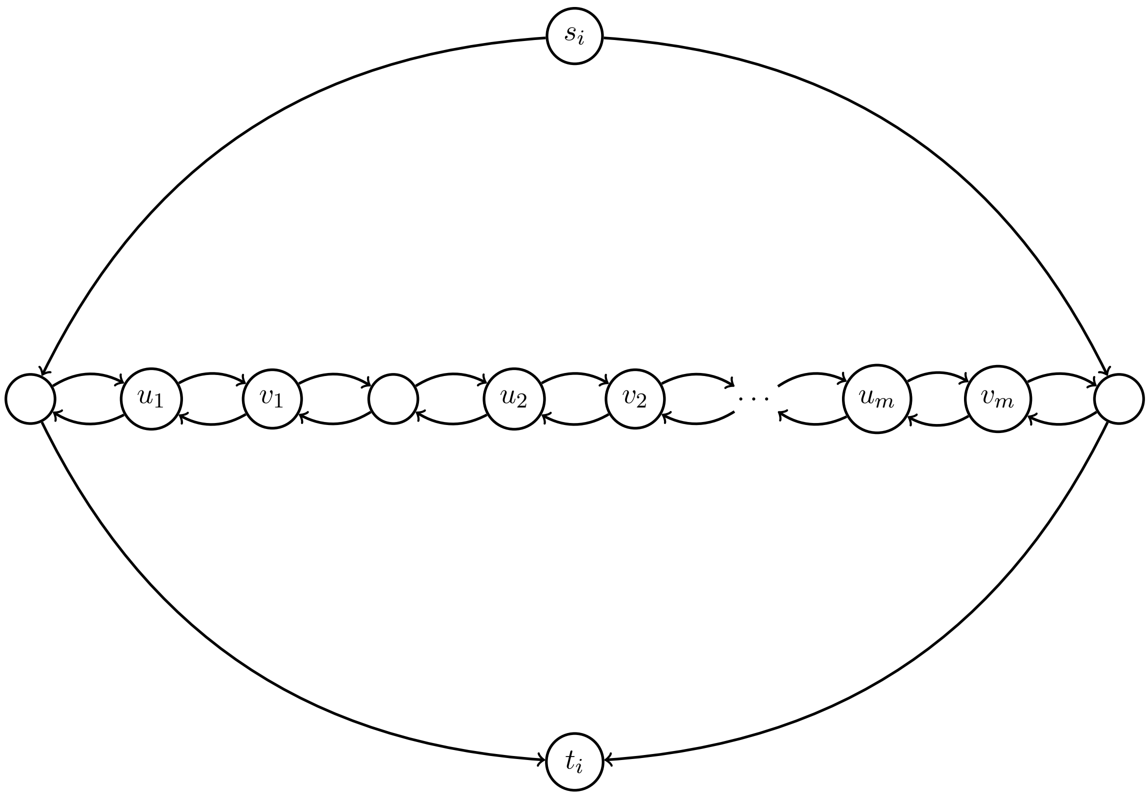
By HAMPATH we denote the following algorithmic problem: given a *directed graph* and two its vertices, s and t, find out whether there exists a Hamiltonian path from s to t.

Theorem 1. HAMPATH *is* NP-*complete.*

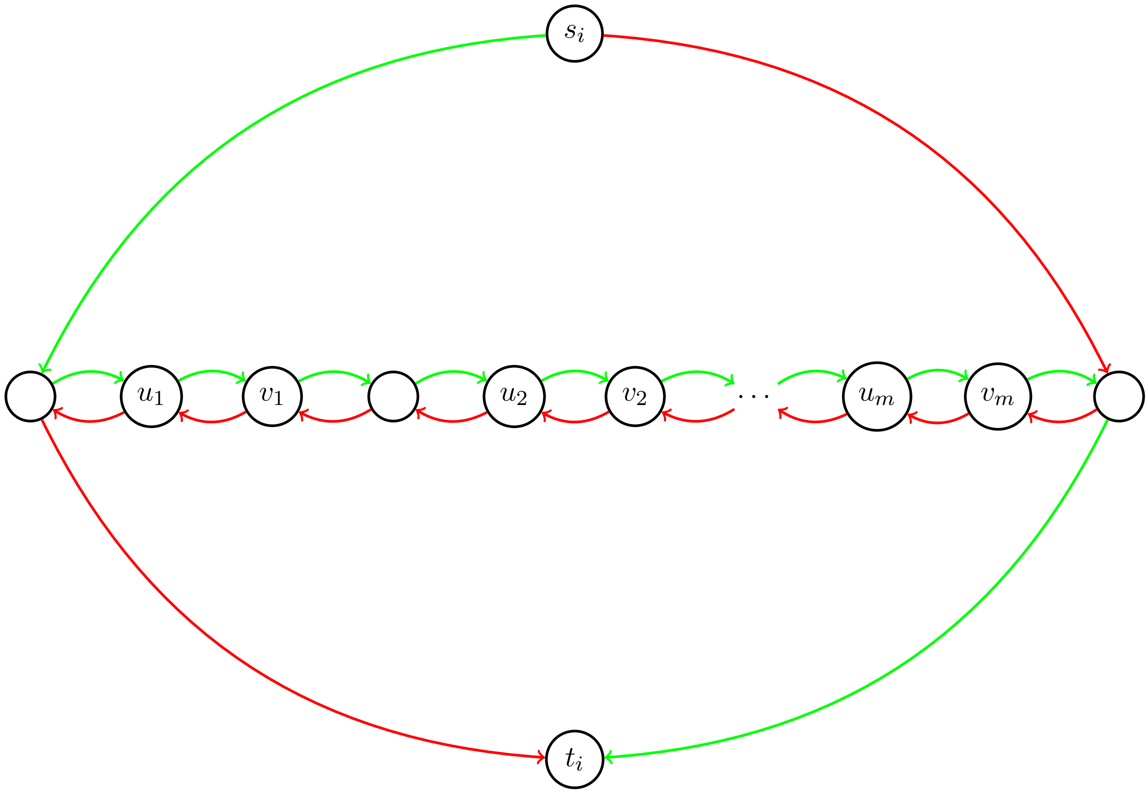
It is easy to see that HAMPATH belongs to the NP class: if the necessary Hamiltonian path exists, one can just non-deterministically guess it. In order to establish NP-hardness of HAMPATH, we prove that 3-SAT $\leq_m^P$ HAMPATH.

In other words, we're going to construct a polynomially computable function f which maps Boolean formulae in 3-CNF to directed graphs with designated vertices s and t, such that φ is satisfiable if and only if there is a Hamiltonian path from s to t in the graph $f(\varphi)$.

Let φ include m clauses. For each *variable* x_i of φ we construct the following subgraph called *gadget:*

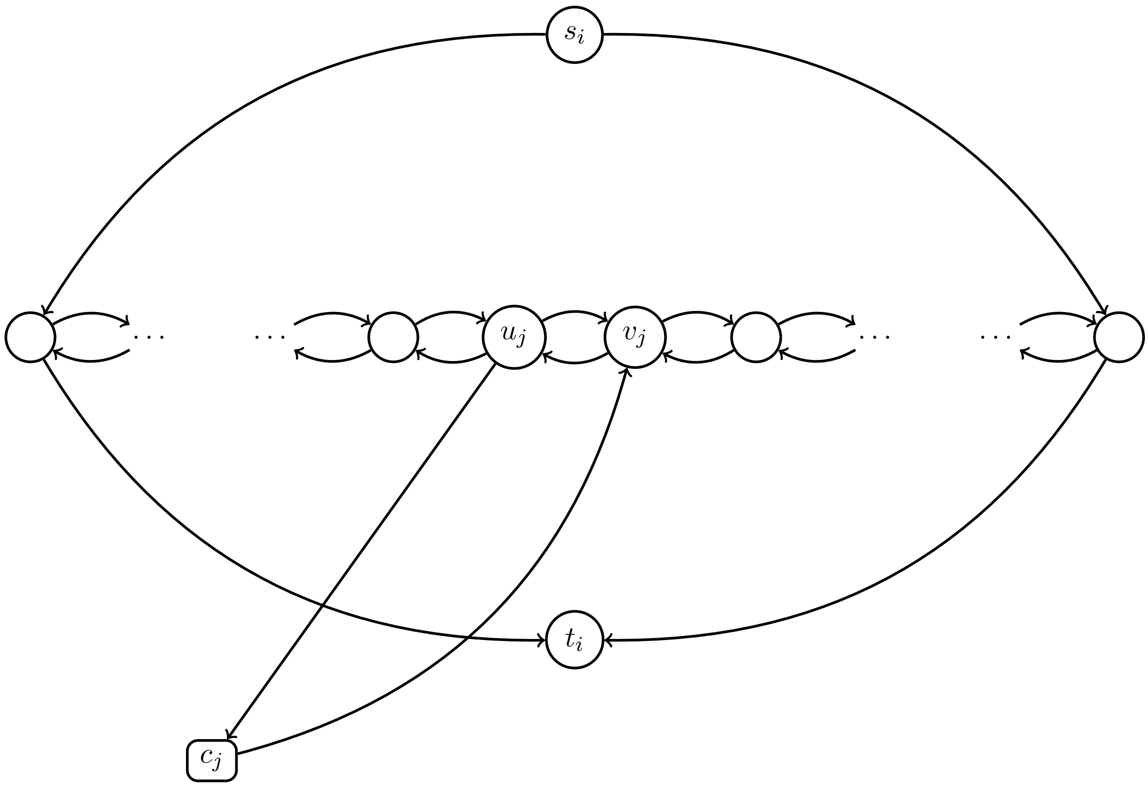


In a Hamiltonian path, this gadget can be traversed, from s_i to t_i, only in the following two ways, called *green* and *red* paths:

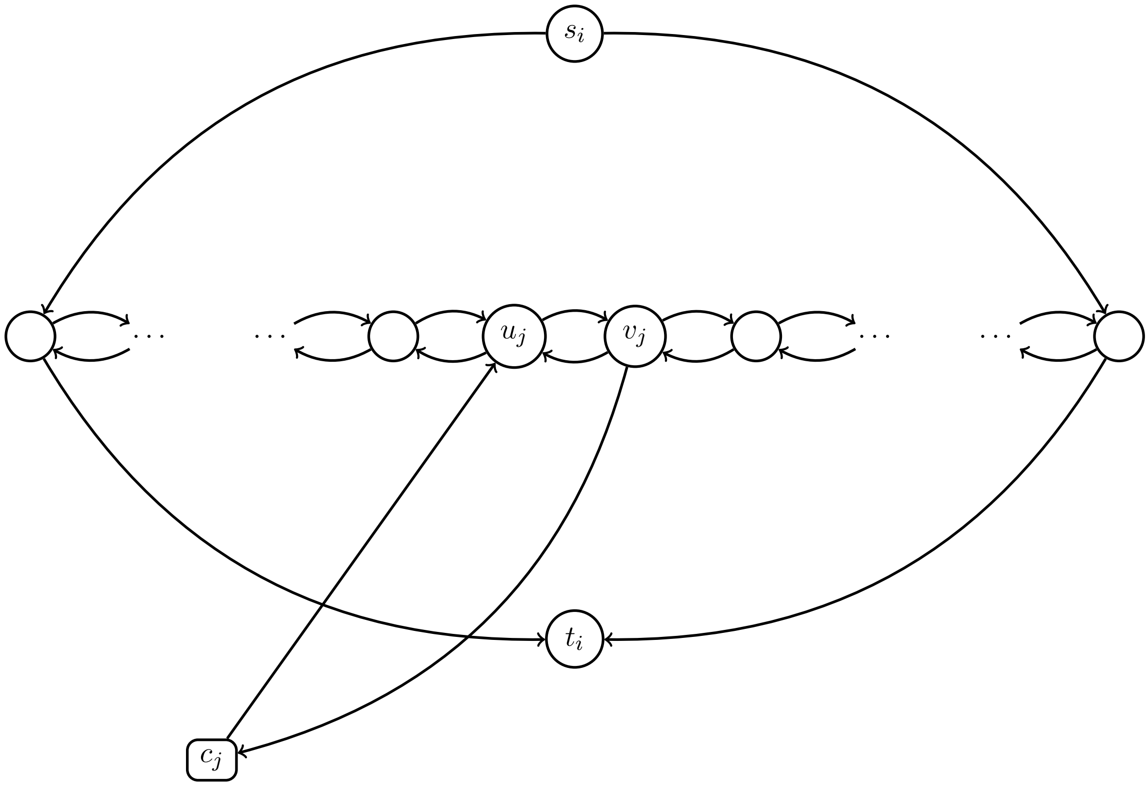


The green path will reflect x_i being *true;* red stands for $x_i =$ *false.*

Next, for each clause C_j we add a designated vertex c_j. If C_j includes x_i, this vertex is connected to the i-th gadget in the following way, so that it can be visited when traversing the i-th gadget by the green path:



Symmetrically, if C_j includes $\neg x_i$, we connect it to the i-th gadget in such a way that c_j can be visited on the red traversing path of the gadget:



Finally, we connect the gadgets in a line, by identifying vertices: $t_1 = s_2$, $t_2 = s_3$, $\ldots$, $t_{n-1} = s_n$, and let $s = s_1$ and $t = t_n$.

Now the graph constructed has a Hamiltonian path from s to t if and only if φ is satisfiable. Indeed, if φ has a satisfying assignment, we traverse each gadget by green or red path, depending on whether x_i is true or false under this assignment. Since in each clause at least one literal is true, the corresponding c_j can be visited on one of these gadget traversing paths.

Conversely, if we have a Hamiltonian path from s to t, this path should traverse each gadget by either green or red path, possibly with detours for visiting c_j's. The choice of green or red path on i-th gadget dictates the truth value of x_i. Since all c_j's were correctly visited, each C_j is true under this assignment.

This finishes the proof of 3-SAT $\leq_m^P$ HAMPATH and thus Theorem 1.