Cut Elimination in Intuitionistic Predicate Logic

Logic II, University of Pennsylvania, Spring 2017

The Calculus

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- Propositional rules:

$$\frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge L) \qquad \frac{\Gamma \Rightarrow A_1 \quad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \wedge A_2}\ (\wedge R)$$

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$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow C}{\Delta, \Gamma, A \rightarrow B \Rightarrow C}\ (\rightarrow L) \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\rightarrow R)$$

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- Quantifier rules:

$$\frac{\Gamma, A(t) \Rightarrow C}{\Gamma, \forall x\, A(x) \Rightarrow C}\ (\forall L) \qquad \frac{\Gamma \Rightarrow A(y)}{\Gamma \Rightarrow \forall x\, A(x)}\ (\forall R)$$

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- Cut:

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ (cut)$$

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If $\Gamma \Rightarrow A$ and $A, \Delta \Rightarrow C$ have cut-free proofs, so does $\Gamma, \Delta \Rightarrow C$.

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- Induction on the sum of derivation depths.
- Cut with an axiom is trivial.

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Both new cuts have smaller depth.

Cut Elimination: Principal Case (Implication)

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,(\to R) \qquad \dfrac{\Pi \Rightarrow A \quad \Delta, B \Rightarrow C}{\Delta, \Pi, A \to B \Rightarrow C}\,(\to L)}{\Delta, \Pi, \Gamma \Rightarrow C}\,(cut)$$

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  Problem! For the lower cut, the induction parameter cannot be controlled.

Overcoming Issues with Contraction

Possible ways:

- Reformulate the calculus to avoid contraction (let Γ be a set rather than multiset, include contraction into the rules, ...)
- Prove elimination of a more general *mix* rule (a combination of cut and contraction):

$$\frac{\Gamma \Rightarrow A \quad A^n, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C} \ (mix)$$

Mix Elimination: Principal Case (Implication)

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,(\to R) \qquad \dfrac{\Pi \Rightarrow A \quad \Delta, B, (A \to B)^{n-1} \Rightarrow C}{\Delta, \Pi, (A \to B)^{n} \Rightarrow C}\,(\to L)}{\Delta, \Pi, \Gamma \Rightarrow C}\,(mix)$$

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Case 1: $n = 1$ (mix is cut)

$$\dfrac{\Pi \Rightarrow A \quad \dfrac{\Gamma, A \Rightarrow B \quad \Delta, B \Rightarrow C}{\Delta, A, \Gamma \Rightarrow C}\,(cut)}{\Delta, \Pi, \Gamma \Rightarrow C}\,(cut)$$

Both new cuts have smaller cut formulae.

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Case 2: $n > 1$

$$\dfrac{\dfrac{\Pi \Rightarrow A \quad \dfrac{\Gamma, A \Rightarrow B \quad \dfrac{\Gamma \Rightarrow A \to B \quad \Delta, B, (A \to B)^{n-1} \Rightarrow C}{\Delta, B, \Gamma \Rightarrow C}\,(mix)}{\Delta, A, \Gamma, \Gamma \Rightarrow C}\,(cut)}{\Delta, \Pi, \Gamma, \Gamma \Rightarrow C}\,(cut)}{\Delta, \Pi, \Gamma \Rightarrow C}\,(contr)$$

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Both cuts have smaller cut formulae; mix has smaller depth.

Mix Elimination: Principal Case (Conjunction)

$$\dfrac{\dfrac{\Gamma \Rightarrow A_1 \quad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \wedge A_2}\ (\wedge R) \qquad \dfrac{A_i, (A_1 \wedge A_2)^{n-1}, \Delta \Rightarrow C}{(A_1 \wedge A_2)^n, \Delta \Rightarrow C}\ (\wedge L)}{\Gamma, \Delta \Rightarrow C}\ (mix)$$

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Mix Elimination: Principal Case (Disjunction)

$$\dfrac{\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\,(\vee R) \quad \dfrac{A_1, (A_1 \vee A_2)^{n-1}, \Delta \Rightarrow C \quad A_2, (A_1 \vee A_2)^{n-1}, \Delta \Rightarrow C}{(A_1 \vee A_2)^n, \Delta \Rightarrow C}\,(\vee L)}{\Gamma, \Delta \Rightarrow C}\,(mix)$$

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Case 2: $n > 1$

$$\dfrac{\dfrac{\Gamma \Rightarrow A_i \quad \dfrac{\Gamma \Rightarrow A_1 \vee A_2 \quad A_i, (A_1 \vee A_2)^{n-1}, \Delta \Rightarrow C}{A_i, \Gamma, \Delta \Rightarrow C}\,(mix)}{\Gamma, \Gamma, \Delta \Rightarrow C}\,(cut)}{\Gamma, \Delta \Rightarrow C}\,(contr)$$

Mix Elimination: Principal Case (Quantifier)

$$
\dfrac{\dfrac{\Gamma \Rightarrow A(t)}{\Gamma \Rightarrow \exists x\, A(x)}\ (\exists R) \qquad \dfrac{\dfrac{\mathcal{D}(y)}{A(y), (\exists x\, A(x))^{n-1}, \Delta \Rightarrow C}}{(\exists x\, A(x))^{n}, \Delta \Rightarrow C}\ (\exists L)}{\Gamma, \Delta \Rightarrow C}\ (mix)
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Mix Elimination: Other Cases

- Axiom case: mix trivializes.
- Non-principal cases: the same as cut.
- Weakening principal case: do weakening for Γ.
- Contraction principal case: contraction includes into mix.