More Semantics for Modal Logics

Logic II, University of Pennsylvania, Spring 2017

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Corollary. A normal modal logic axiomatised by Sahlqvist formulae and/or formulae without variables is Kripke complete.

Sahlqvist Formulae: Examples

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The **GL** axiom, $\Box(\Box p \rightarrow p) \rightarrow \Box p$, is **not** a Sahlqvist formula. It is also not canonical and doesn't correspond to a first-order condition on frames.

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- Then $v_{\mathrm{Kripke}}(\Box A) = \mathrm{Int}(v(A)) = v_{\mathrm{topological}}(\Box A)$.

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- Completeness follows from Kripke completeness by considering the upset topology ($x \in U$ and $xRy \Rightarrow y \in U$) on a finite irreflexive Kripke frame (isolated points are maximal elements):

$$x \Vdash \Diamond A \iff \exists y \in R(x)\, y \Vdash A \iff x \in d(v(A)).$$