# Craig's Trick and a Non-Sequential System for the Lambek Calculus and Its Fragments

Stepan Kuznetsov*[1], Valentina Lugovaya[2], and Anastasiia Ryzhova[2]

[1]Steklov Mathematical Institute (Moscow) of the RAS, 8 Gubkina St., Moscow, Russia

[2]Lomonosov Moscow State University, Leninskie Gory, Moscow, Russia

May 8, 2018

**Abstract**

We show that Craig's trick is not valid for the Lambek calculus, i.e., there exists such a recursively enumerable theory (set of sequents) over the Lambek calculus, which does not have a decidable axiomatisation. We show that Lambek's non-emptiness restriction (the constraint that left-hand sides of all sequents should be non-empty) and an infinite set of variables are crucial for the failure of Craig's trick. We also present a non-sequential formulation of the product-free fragment of the Lambek calculus and show its equivalence to the sequential one.

**Keywords:** Craig's trick, Lambek calculus, non-sequential calculus, substructural logic, Lambek's restriction

*corresponding author; e-mail: `skuzn@inbox.ru`

## 1 Introduction

A well-known result by Craig [12] states that for a well-behaved logic (classical or intuitionstic, for example) any theory with a recursively enumerable set of theorems has a decidable, even a primitively recursive axiomatisation. Indeed, if the set of theorems $\mathscr{A} = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ of a theory is enumerated by a total computable function $f$, $\varphi_k = f(k)$, $k = 0, 1, 2, \ldots$, then the set $\mathscr{A}' = \{\varphi_0, \varphi_1 \wedge \varphi_1, \varphi_2 \wedge \varphi_2 \wedge \varphi_2, \ldots\}$ is a valid axiomatisation for the theory and is decidable. The former is obvious and the latter is proved by the following decision procedure. Given a formula $\psi$, try to find it within the first $n$ elements of $\mathscr{A}'$, where $n$ is the length of $\psi$. Elements with greater numbers are *a priori* longer than $\psi$ and therefore cannot coincide with it.

Throughout this paper, we denote this construction by a bit informal term *"Craig's trick"* instead of a more official *"Craig's lemma"* in order to avoid naming collision with a more sophisticated and famous Craig's interpolation lemma [13].

Craig's trick is indeed very general. The only thing we need from the logic in question is the following property: for any formula $\psi$ there exists, and can be effectively constructed, an equivalent formula $\psi'$ longer than $\psi$. Then we can take $\mathscr{A}' = \{\varphi_0, \varphi_1', \varphi_2'', \ldots\}$. Since $'$ increases the length of formula, the $n$-th formula in this sequence has size at least $n$. This can work, for example, for substructural systems which do not enjoy $\psi \leftrightarrow \psi \wedge \psi$: once we have an operation $\circ$ that has a unit $\mathbf{1}$, we can take $\psi' = \mathbf{1} \circ \psi$ and enjoy Craig's trick.

Thus, it looks interesting to find a logic for which Craig's trick fails. Of course, we are not interested in degenerate examples like a "logic" without any rules of inference—in this case any theory has a unique "axiomatisation" (all "theorems" should be "axioms"), which can be recursively enumerable, but not decidable. There exists, however, a non-trivial example among useful and interesting logical systems. Namely, we show that Craig's trick fails for the Lambek calculus.

Besides that, we consider alternative, non-sequential formulations for fragments of the Lambek calculus obtained by restricting the set of connectives, and prove their equivalence to the Gentzen-style formulation.

## 2 The Lambek Calculus

The Lambek calculus, **L**, was initially introduced by J. Lambek [17] as a basis for categorial, or type-logical, grammars, a formalism for describing syntactic validity in natural language in terms of logical derivability. Nowadays, we consider Lambek calculus as a variant of non-commutative intuitionistic linear logic (see Abrusci [1]).

In this section we present **L** as a Gentzen-style sequent calculus. For non-sequential formulations of **L** and its elementary fragments, see Section 4. Formulae of **L**, also called *types,* are built from a countable set of variables (primitive types) $\mathrm{Var} = \{p_0, p_1, p_2, \ldots\}$ using three binary connectives, $/$, $\backslash$, and $\cdot$; sequents are expressions of the form $\Gamma \to A$, where $\Gamma$ is a linearly ordered sequence of formulae and $A$ is a formula. For all derivable sequents, $\Gamma$ will be non-empty.

Being originally introduced by Lambek as operations on syntactic categories ($\cdot$ for concatenation and $/$, $\backslash$ for division operations), nowadays connectives of the Lambek calculus can be viewed from the logical point of view as (multiplicative) conjunction and implications (in the non-commutative situation we distinguish left and right implication).

Axioms of **L** are sequents of the form $A \to A$; the rules of inference are as follows:

$$\frac{A, \Pi \to B}{\Pi \to A \mathbin{\backslash} B}\ (\to \backslash), \text{ where } \Pi \text{ is non-empty} \qquad \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \mathbin{\backslash} B, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi, A \to B}{\Pi \to B \mathbin{/} A}\ (\to /), \text{ where } \Pi \text{ is non-empty} \qquad \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \mathbin{/} A, \Pi, \Delta \to C}\ (/ \to)$$

$$\frac{\Pi \to A \quad \Delta \to B}{\Pi, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ (\cdot \to)$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

Notice that left-hand sides *(antecedents)* of Lambek sequents are explicitly required to be non-empty. This constraint, called *Lambek's restriction,* which looks a bit strange from the logical point of view, is motivated by linguistic application. The standard example is as follows. Let $n$ denote the syntactic type of nouns and noun groups, then $n \,/\, n$ describes adjectives (something that lacks a noun phrase to the right to become a bigger noun phrase). For example, derivability of sequents $n \,/\, n, n \to n$ and $n \,/\, n, n \,/\, n, n \to n$ corresponds to the fact that "interesting book" and "big red ball" are valid noun groups. Going further, we can assign $(n \,/\, n) \,/ (n \,/\, n)$ to the adverb "very," and derivability of $(n \,/\, n) \,/ (n \,/\, n), n \,/\, n, n \to n$ justifies "very interesting book" being a noun group. Without Lambek's restriction, however, the sequent $(n \,/\, n) \,/ (n \,/\, n), n \to n$ would be also derivable, assigning type $n$ to "very book," which is not a correct noun group.

For further information on applications of Lambek calculus in the study of natural language, we refer to textbooks on type-logical grammar [10, 21, 20].

In the view of Craig's trick we talk about derivability from *theories.* A theory $\mathscr{A}$ is just an arbitrary set of sequents. (Here and further "theory" actually means "axiomatisation of a theory." As opposed to finite axiomatic extensions of the Lambek calculus studied by Buszkowski [8], here we allow $\mathscr{A}$ to be infinite.)

By $\mathscr{A} \vdash_{\mathbf{L}} \Pi \to A$ we denote the fact that $\Pi \to A$ is derivable from $\mathscr{A}$ in $\mathbf{L}$, *i.e.,* there exists a derivation tree with axioms or sequents from $\mathscr{A}$ at its leafs and applications of $\mathbf{L}$ rules in the inner nodes. Notice that these derivations can use cut: in general, it is not eliminable for non-empty $\mathscr{A}$.

If $\mathscr{A}$ is empty, we write just $\vdash_{\mathbf{L}} \Pi \to A$.

Two theories are *equivalent,* $\mathscr{A}_1 \approx \mathscr{A}_2$, if they derive the same set of theorems: $\mathscr{A}_1 \vdash_{\mathbf{L}} \Pi \to A \iff \mathscr{A}_2 \vdash_{\mathbf{L}} \Pi \to A$.

In this paper, we are particularly interested in the product-free fragment of $\mathbf{L}$, obtained by restricting its language, denoted by $\mathbf{L}(/, \backslash)$. In $\mathbf{L}(/, \backslash)$, we consider only sequents without $\cdot$ (when talking about derivability from theories, both in $\Pi \to A$ and $\mathscr{A}$), in other words, consider only sequents in the restricted product-free $(/, \backslash)$ language. The set of types (formulae) without $\cdot$ is denoted by $\mathrm{Tp}(/, \backslash)$.

Axioms of $\mathbf{L}(/, \backslash)$ are, again, of the form $A \to A$, and rules of inference are $(\to \backslash)$, $(\backslash \to)$, $(\to /)$, $(/ \to)$, and (cut).

The desired property of $\mathbf{L}(/, \backslash)$ is its conservativity as a fragment of $\mathbf{L}$ w.r.t. derivations from theories: if $\mathscr{A}$ and $\Pi \to A$ are in the product-free language, then $\mathscr{A} \vdash_{\mathbf{L}} \Pi \to A$ iff $\mathscr{A} \vdash_{\mathbf{L}(\backslash,/)} \Pi \to A$. The "if" (right-to-left) part is obvious. For the "only if" part, one usually applies cut-elimination and subformula property; however, for derivation from theories ($\mathscr{A}$), cut elimination does not hold. This could be fixed by switching to *cut normalisation,* or propagating the cut towards axioms and gaining a form of subformula property. In this paper, we follow a different strategy and prove conservativity by a semantic argument (see Proposition 4 in Section 3).

One of the main results of this paper states that Craig's trick fails for $\mathbf{L}$:

**Theorem 1.** *There exists such a recursively enumerable $\mathscr{A}$ that there is no decidable $\mathscr{A}'$ equivalent to $\mathscr{A}$.*

We prove this theorem in Section 5, after developing the necessary semantic machinery (Section 3). We also introduce a new non-sequential axiomatisation for the $\mathbf{L}(/, \backslash)$ fragment and prove that it is equivalent to the sequential one (Section 4).

## 3 Language and Relational Models

Linguistic applications of the Lambek calculus suggest its natural interpretation on the algebra of formal languages, or *L-interpretation* for short. An L-intepretation consists of an alphabet $\Sigma$ and an interpretation function $w \colon \mathrm{Tp} \to \mathcal{P}(\Sigma^+)$, obeying the following conditions:

$$w(A \setminus B) = w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\ vu \in w(B)\}$$

$$w(B \,/\, A) = w(B) \,/\, w(A) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\ uv \in w(B)\}$$

$$w(A \cdot B) = w(A) \cdot w(B) = \{uv \mid u \in w(A), v \in w(B)\}$$

As usually, $w$ can be defined arbitrarily on variables, and then it is uniquely propagated to all formulae.

We allow alphabet $\Sigma$ to be an arbitrary finite or countable set. However, as noticed by Pentus, any L-model can be reduced to an L-model over a two-letter alphabet $\Sigma_2 = \{b, c\}$ by substituting $bc^ib$ for each letter $a_i$ of the original alphabet $\Sigma = \{a_1, a_2, \dots\}$.

A sequent $A_1, \dots, A_n \to B$ is called *true* under interpretation function $w$, if $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$. An L-interpretation is called an *L-model* for a theory $\mathscr{A}$ (notation: $\langle \Sigma, w \rangle \vDash \mathscr{A}$), if all sequents from $\mathscr{A}$ are true under this interpretation. Finally, a sequent $\Pi \to C$ is *L-entailed* by $\mathscr{A}$ (notation: $\mathscr{A} \vDash_L \Pi \to C$), if it is true under any L-model of $\mathscr{A}$. (For the special case of an empty $\mathscr{A}$ we get the notion of $\Pi \to C$ being *generally L-true*, $\vDash_L \Pi \to C$.)

The Lambek calculus is sound w.r.t. this interpretation:

**Proposition 2.** *If* $\mathscr{A} \vdash_{\mathbf{L}} \Pi \to B$, *then* $\mathscr{A} \vDash_L \Pi \to B$.

Soundness is proved by straightforward induction on the derivation. We call this proposition *strong soundness,* to distinguish from weak soundness: if $\vdash_{\mathbf{L}} \Pi \to B$, then $\vDash_L \Pi \to B$.

Completeness is a more subtle question. As shown by Pentus [24], $\mathbf{L}$ enjoys weak completeness w.r.t. L-interpretations: if $\vDash_L \Pi \to B$, then $\vdash_{\mathbf{L}} \Pi \to B$. On the other hand, as noticed by Buszkowski [9], *strong* completeness fails.[1] Namely, $p \to p \cdot p \vDash_L p \to q$, but $p \to p \cdot p \nvdash_{\mathbf{L}} p \to q$ ($p$ and $q$ are distinct variables).

However, the product-free fragment $\mathbf{L}(/, \backslash)$ enjoys strong completeness:

**Proposition 3.** *If* $\mathscr{A}$ *and* $\Pi \to B$ *are in the product-free language and* $\mathscr{A} \vDash_L \Pi \to B$, *then* $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Pi \to B$.

This strong completeness result is established by a canonical model construction sketched by Buszkowski [6]. We redisplay the proof here in detail, since we shall need it for further modifications.

[1] Usually weak and strong completeness theorems coincide due to deduction theorem. For substructural logics, such as Lambek calculus, this does not work.

*Proof of Proposition 3.* Define the canonical L-model for $\mathscr{A}$, denoted by $\mathcal{M}_{\mathscr{A}} = \langle \Sigma_{\mathscr{A}}, w_{\mathscr{A}} \rangle$, as follows:

$$\Sigma_{\mathscr{A}} = \mathrm{Tp}(/, \backslash) \quad \text{and} \quad w_{\mathscr{A}}(A) = \{\Gamma \mid \mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Gamma \to A\}.$$

Now it is sufficient to establish the following:

1. $\mathcal{M}_{\mathscr{A}}$ is an L-interpretation (*i.e.,* $w_{\mathscr{A}}(B \setminus C) = w_{\mathscr{A}}(B) \setminus w_{\mathscr{A}}(C)$ and the same for $/$);

2. all sequents from $\mathscr{A}$ are true under $\mathcal{M}_{\mathscr{A}}$;

3. if $\mathscr{A} \not\vdash_{\mathbf{L}(\backslash,/)} \Pi \to B$, then $\Pi \to B$ is false under $\mathcal{M}_{\mathscr{A}}$.

These statements are proved as follows:

1. Let $\Gamma \in w_{\mathscr{A}}(B \setminus C)$, i.e., $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Gamma \to B \setminus C$, and let $\Delta$ be an arbitrary element of $w_{\mathscr{A}}(B)$, i.e., $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Delta \to B$. Then the following derivation establishes $\Delta, \Gamma \to C$:

$$\dfrac{\Gamma \to B \setminus C \qquad \dfrac{\Delta \to B \quad C \to C}{\Delta, B \setminus C \to C}\,(\setminus \to)}{\Delta, \Gamma \to C}\,(\mathrm{cut})$$

Thus, $\Delta\Gamma \in w_{\mathscr{A}}(C)$, and therefore $\Gamma \in w_{\mathscr{A}}(B) \setminus w_{\mathscr{A}}(C)$.

For the opposite direction, let $\Gamma \in w_{\mathscr{A}}(B) \setminus w_{\mathscr{A}}(C)$, i.e., $\Delta\Gamma \in w_{\mathscr{A}}(C)$ for any $\Delta \in w_{\mathscr{A}}(B)$. Take $\Delta = B$. Then $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} B, \Gamma \to C$, and by $(\to \setminus)$ we get $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Gamma \to B \setminus C$, whence $\Gamma \in w_{\mathscr{A}}(B \setminus C)$.

The $/$ case is symmetric.

2. If $(A_1 \ldots A_n \to B) \in \mathscr{A}$, then $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} A_1, \ldots, A_n \to B$ and for any $\Gamma_1 \in w_{\mathscr{A}}(A_1)$, $\ldots$, $\Gamma_n \in w_{\mathscr{A}}(A_n)$ we get $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Gamma_1, \ldots, \Gamma_n \to B$ by cut. Hence, $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$.

3. Let $\Pi = A_1 \ldots A_n$. Since $A_i \in w_{\mathscr{A}}(A_i)$, $\Pi$ belongs to $w_{\mathscr{A}}(A_1) \cdot \ldots \cdot w_{\mathscr{A}}(A_n)$, but $\Pi \notin w_{\mathscr{A}}(B)$. Hence, $w_{\mathscr{A}}(A_1) \cdot \ldots \cdot w_{\mathscr{A}}(A_n) \not\subseteq w(B)$, and therefore $\Pi \to B$ is false in $\mathcal{M}_{\mathscr{A}}$. □

Strong completeness of $\mathbf{L}(/, \backslash)$ yields conservativity of $\mathbf{L}$ over $\mathbf{L}(/, \backslash)$:

**Proposition 4.** *If $\mathscr{A}$ and $\Pi \to A$ are in the product-free language, then $\mathscr{A} \vdash_{\mathbf{L}} \Pi \to A$ iff $\mathscr{A} \vdash_{\mathbf{L}(\backslash,/)} \Pi \to A$.*

*Proof.* Let $\mathscr{A}$ and $\Gamma \to A$ be in the product-free language. Then $\mathscr{A} \vdash_{\mathbf{L}} \Gamma \to A$ entails $\mathscr{A} \vDash_L \Gamma \to A$ (by Proposition 2), which in its turn implies $\mathscr{A} \vdash_{\mathbf{L}(/,\backslash)} \Gamma \to A$ (by Proposition 3). $\square$

Another natural class of interpretations for the Lambek calculus is the class of relational interpretations, or *R-interpretations* for short, proposed by van Benthem [3]. An R-interpretation $\mathcal{M} = \langle W, U, v \rangle$ consists of a binary relation $U \subseteq W \times W$, where $W$ is an arbitrary non-empty set, and an interpretation function $v \colon \mathrm{Tp} \to \mathcal{P}(U)$ (that is, interpretation of Lambek types are also binary relations on $W$, included in $U$).[2] The conditions for $v$ are as follows:

$$v(A \cdot B) = v(A) \circ v(B) = \{\langle x, z \rangle \mid (\exists y \in W)\, \langle x, y \rangle \in v(A), \langle y, z \rangle \in v(B)\}$$

$$v(A \setminus B) = \{\langle y, z \rangle \in U \mid v(A) \circ \{\langle y, z \rangle\} \subseteq v(B)\}$$

$$v(B \,/\, A) = \{\langle x, y \rangle \in U \mid \{\langle x, y \rangle\} \circ v(A) \subseteq v(B)\}$$

(In other words, multiplication is defined as composition of relations, and divisions are its residuals w.r.t. the $\subseteq$ preorder on $\mathcal{P}(U)$.)

A sequent $A_1, \ldots, A_n \to B$ is true in $\mathcal{M}$ (notation: $\mathcal{M} \vDash A_1, \ldots, A_n \to B$), if $v(A_1) \circ \ldots \circ v(A_n) \subseteq v(B)$. The notions of an R-model of a theory ($\mathcal{M} \vDash \mathscr{A}$) and R-entailment ($\mathscr{A} \vDash_R \Pi \to B$) are defined exactly as for L-models.

In contrast to L-models, R-models enjoy strong completeness:

**Theorem 5.** *$\mathscr{A} \vdash_{\mathbf{L}} \Pi \to B$ if and only if $\mathscr{A} \vDash_R \Pi \to B$.*

Soundness (the "only if" part) is due to van Benthem [4], and completeness ("if") was proved by Andréka and Mikulás [2].

We shall use relational semantics in Section 6 for falsifying Craig's trick for $\mathbf{L}$, and language semantics in Section 4 in order to prove equivalence of the non-sequential and sequential formulations of $\mathbf{L}(/, \backslash)$.

[2] Andréka and Mikulás [2] call these structures representable relativised relational structures (RRS) and use a close, but more Kripke-style definition for R-models. Here we follow the terminology of Pentus [24].

## 4 Non-Sequential Calculi for Fragments of L

In this section we consider alternative, non-sequential axiomatisations of the Lambek calculus and its fragments. These axiomatisations are sometimes called "Hilbert-style" ones, which is a bit inaccurate, since they still are based mostly on inference rules rather than axioms. Derivable objects in the non-sequential systems for the Lambek calculus are of the form $A \to B$, where $A$ and $B$ are Lambek formulae (types).

The non-sequential counterpart of the whole Lambek calculus, which we denote by $\mathbf{L}_{\mathrm{H}}$, was introduced already in the first paper of Lambek [17]. This calculus is specified by axioms of the form

$$A \to A; \quad (A \cdot B) \cdot C \to A \cdot (B \cdot C); \quad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

and the following derivation rules:

$$\frac{A \cdot C \to B}{C \to A \setminus B} \quad \frac{C \cdot A \to B}{C \to B \mathbin{/} A} \quad \frac{C \to A \setminus B}{A \cdot C \to B} \quad \frac{C \to B \mathbin{/} A}{C \cdot A \to B} \quad \frac{A \to B \quad B \to C}{A \to C}$$

A sequent of $\mathbf{L}$, $A_1, \dots, A_n \to B$, is translated to the form acceptable in $\mathbf{L}_{\mathrm{H}}$ by replacing the commas with multiplication: $A_1 \cdot \ldots \cdot A_n \to B$ (due to associativity, we can omit parentheses). If $\mathscr{A}$ is a set of $\mathbf{L}$ sequents, then by $\mathscr{A}_{\mathrm{H}}$ we denote the set of their translations into the language of $\mathbf{L}_{\mathrm{H}}$. In the sense of this translation, $\mathbf{L}_{\mathrm{H}}$ is strongly equivalent to $\mathbf{L}$:

**Proposition 6.** $\mathscr{A} \vdash_{\mathbf{L}} A_1, \dots, A_n \to B \iff \mathscr{A}_{\mathrm{H}} \vdash_{\mathbf{L}_{\mathrm{H}}} A_1 \cdot \ldots \cdot A_n \to B$.

This equivalence result is essentially due to Lambek [17], though formally Lambek proves only weak equivalence (for an empty $\mathscr{A}$).

*Proof.* $\boxed{\Leftarrow}$ First, notice that the $(\cdot \to)$ rule in $\mathbf{L}$ is invertible:

$$\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A \cdot B}\,(\to \cdot) \qquad \Gamma, A \cdot B, \Delta \to C}{\Gamma, A, B, \Delta \to C}\,(\mathrm{cut})$$

Therefore $A_1, \dots, A_n \to B$ is equiderivable (in $\mathbf{L}$) with $A_1 \cdot \ldots \cdot A_n \to B$, and the same for $\mathscr{A}$ and $\mathscr{A}_{\mathrm{H}}$. Thus, it is sufficient to establish the following:

$$\mathscr{A}_{\mathrm{H}} \vdash_{\mathbf{L}} A_1 \cdot \ldots \cdot A_n \to B \quad \Longleftarrow \quad \mathscr{A}_{\mathrm{H}} \vdash_{\mathbf{L}_{\mathrm{H}}} A_1 \cdot \ldots \cdot A_n \to B.$$

Axioms of $\mathbf{L}_{\mathrm{H}}$ are valid in $\mathbf{L}$. The first and the third rules of $\mathbf{L}_{\mathrm{H}}$ are derivable in $\mathbf{L}$ using cut:

$$\dfrac{\dfrac{\dfrac{A \to A \quad C \to C}{A, C \to A \cdot C}\,(\to \cdot) \quad A \cdot C \to B}{A, C \to B}\,(\mathrm{cut})}{C \to A \backslash B}\,(\to \backslash) \qquad \dfrac{\dfrac{C \to A \backslash B \quad A, A \backslash B \to B}{A, C \to B}\,(\mathrm{cut})}{A \cdot C \to B}\,(\cdot \to)$$

Derivations of the second and the fourth rule are symmetric; the fifth rule is a particular case of cut.

$\boxed{\Rightarrow}$ Here we follow Lambek [17] and first establish the following *monotonicity rules* in $\mathbf{L}_{\mathrm{H}}$:

$$\dfrac{A_1 \to A_2 \quad B_1 \to B_2}{A_1 \cdot B_1 \to A_2 \cdot B_2}\,(\cdot\mathrm{mon}) \qquad \dfrac{A_1 \to A_2 \quad B_1 \to B_2}{A_2 \backslash B_1 \to A_1 \backslash B_2}\,(\backslash\,\mathrm{mon}) \qquad \dfrac{A_1 \to A_2 \quad B_1 \to B_2}{B_1 \,/\, A_2 \to B_2 \,/\, A_1}\,(/\,\mathrm{mon})$$

These rules are justified by the following derivations, due to Lambek [17]:

$$\dfrac{\dfrac{\dfrac{A_1 \to A_2 \quad \dfrac{A_2 \cdot B_1 \to A_2 \cdot B_1}{A_2 \to (A_2 \cdot B_1) \,/\, B_1}}{A_1 \to (A_2 \cdot B_1) \,/\, B_1}}{A_1 \cdot B_1 \to A_2 \cdot B_1} \quad \dfrac{\dfrac{B_1 \to B_2 \quad \dfrac{A_2 \cdot B_2 \to A_2 \cdot B_2}{B_2 \to A_2 \backslash (A_2 \cdot B_2)}}{B_1 \to A_2 \backslash (A_2 \cdot B_2)}}{A_2 \cdot B_1 \to A_2 \cdot B_2}}{A_1 \cdot B_1 \to A_2 \cdot B_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A_1 \to A_2 \quad \dfrac{\dfrac{A_2 \backslash B_1 \to A_2 \backslash B_1}{A_2 \cdot (A_2 \backslash B_1) \to B_1}}{A_2 \to B_1 \,/ (A_2 \backslash B_1)}}{A_1 \to B_1 \,/ (A_2 \backslash B_1)}}{A_1 \cdot (A_2 \backslash B_1) \to B_1}}{A_2 \backslash B_1 \to A_1 \backslash B_1} \quad \dfrac{\dfrac{\dfrac{\dfrac{A_1 \backslash B_1 \to A_1 \backslash B_1}{A_1 \cdot (A_1 \backslash B_1) \to B_1} \quad B_1 \to B_2}{A_1 \cdot (A_1 \backslash B_1) \to B_2}}{A_1 \backslash B_1 \to A_1 \backslash B_2}}{A_2 \backslash B_1 \to A_1 \backslash B_2}$$

(The derivation for $(/\,\mathrm{mon})$ is symmetric.)

Notice that these derivations establish not only the fact that these rules are *admissible* in $\mathbf{L}_{\mathrm{H}}$ (i.e., if the premises are derivable in the pure calculus, with $\mathscr{A}_{\mathrm{H}} = \varnothing$, then so is the conclusion), but a stronger fact that they are *derivable.* Therefore, they can be also used for deriving from theories, which we actually need. Thus, we can use it in building derivations from $\mathscr{A}_{\mathrm{H}}$ in $\mathbf{L}_{\mathrm{H}}$.

The only axiom of $\mathbf{L}$, $A \to A$, is also an axiom of $\mathbf{L}_{\mathrm{H}}$.

Translations of $\mathbf{L}$ rules into the $\mathbf{L}_{\mathrm{H}}$ format are as follows:

$$\dfrac{A \cdot P \to B}{P \to A \backslash B}\,(\to \backslash) \qquad \dfrac{P \to A \quad G \cdot B \cdot D \to C}{G \cdot P \cdot (A \backslash B) \cdot D \to C}\,(\backslash \to)$$

$$\frac{P\cdot A\to B}{P\to B\,/\,A}\ (\to /) \qquad \frac{P\to A \quad G\cdot B\cdot D\to C}{G\cdot (B\,/\,A)\cdot P\cdot D\to C}\ (/\to)$$

$$\frac{P\to A \quad D\to B}{P\cdot D\to A\cdot B}\ (\to \cdot) \qquad \frac{G\cdot A\cdot B\cdot D\to C}{G\cdot A\cdot B\cdot D\to C}\ (\cdot\to)$$

$$\frac{P\to A \quad G\cdot A\cdot D\to C}{G\cdot P\cdot D\to C}\ (\mathrm{cut})$$

(Here $G$, $D$, and $P$ stand for products of $\Gamma$, $\Delta$, and $\Pi$ respectively; due to associativity of $\mathbf{L}_{\mathrm{H}}$, the bracketing does not matter.)

We notice that rules $(\to \backslash)$ and $(\to /)$ are actually rules of $\mathbf{L}_{\mathrm{H}}$, rule $(\to \cdot)$ is one of the monotonicity rules, $(\cdot\mathrm{mon})$, and $(\cdot \to)$ became trivial. The cut rule follows from monotonicity and transitivity as follows:

$$\dfrac{\dfrac{\dfrac{G\to G \quad P\to A}{G\cdot P\to G\cdot A} \quad D\to D}{G\cdot P\cdot D\to G\cdot A\cdot D} \quad G\cdot A\cdot D\to C}{G\cdot P\cdot D\to C}$$

Finally, $(\backslash \to)$ is obtained by the following derivation (also using monotonicity)

$$\dfrac{\dfrac{G\to G \quad \dfrac{\dfrac{P\to A \quad A\,\backslash\, B\to A\,\backslash\, B}{P\cdot (A\,\backslash\, B)\to A\cdot (A\,\backslash\, B)} \quad \dfrac{A\,\backslash\, B\to A\,\backslash\, B}{A\cdot (A\,\backslash\, B)\to B}}{P\cdot (A\,\backslash\, B)\to B} \quad D\to D}{G\cdot P\cdot (A\,\backslash\, B)\cdot D\to G\cdot B\cdot D} \quad G\cdot B\cdot D\to C}{G\cdot P\cdot (A\,\backslash\, B)\cdot D\to C}$$

and $(/ \to)$ is symmetric. Thus, each derivation in $\mathbf{L}$ from a theory $\mathscr{A}_{\mathrm{H}}$ is mapped to a derivation in $\mathbf{L}_{\mathrm{H}}$.

□

The strongly conservative fragment of $\mathbf{L}$ without product, $\mathbf{L}(/,\backslash)$, results naturally due to Proposition 4: it is just necessary to exclude the rules containing the product connective. It is not the case, however, for the non-sequential variant $\mathbf{L}_{\mathrm{H}}$, since all rules of $\mathbf{L}_{\mathrm{H}}$, except the last one, use the product connective. Thus, the construction of the fragments of $\mathbf{L}_{\mathrm{H}}$ with restricted sets of connectives is nontrivial. For the one-division fragment $\mathbf{L}(\backslash)$ this was done by Yu. Savateev [22] (unfortunately, this paper has never been published). Our goal is to construct $\mathbf{L}_{\mathrm{H}}(/,\backslash)$ and prove its strong equivalence to $\mathbf{L}(/,\backslash)$.

The calculus $\mathbf{L}_{\mathrm{H}}(/,\backslash)$ is specified by the following axioms:

$(A1)$ $A\to A$; $\quad$ $(A2)$ $B\,\backslash\, C\to (A\,\backslash\, B)\,\backslash(A\,\backslash\, C)$; $\quad$ $(A3)$ $A\,\backslash(B\,/\,C)\leftrightarrow (A\,\backslash\, B)\,/\,C$

and the following derivation rules:

$$\frac{A \to B \qquad B \to C}{A \to C}\ (R1) \qquad\qquad \frac{A \to B \qquad C \to D}{B \backslash C \to A \backslash D}\ (R2)$$

$$\frac{A \to B \,/\, C}{C \to A \backslash B}\ (R3) \qquad\qquad \frac{A \to B \backslash C}{B \to C \,/\, A}\ (R4)$$

Axiom (2) is actually a well-known principle called *Geach rule*. It was introduced by Geach in [14] (but in a slightly different form) and many times used for alternative axiomatizations of the Lambek calculus in papers on categorial grammar, e.g., [22], [11], [25], [23], [15] etc.

It is significant that all the equivalence proofs in [14], [22], [11], and [25] were purely syntactic, whereas our proof is based on semantic methods.

The main result of this section is equivalence of $\mathbf{L}_{\mathrm{H}}(/, \backslash)$ and $\mathbf{L}(/, \backslash)$. For simplicity, we consider only sequents of the form $A \to B$ (both in $\mathscr{A}$ and as the goal sequent). In this case $\mathscr{A}_{\mathrm{H}} = \mathscr{A}$. Sequents of the form $A_1, \ldots, A_n \to B$, $n \geq 2$, are represented (equivalently in $\mathbf{L}(/, \backslash)$) as $A_n \to A_{n-1} \backslash \ldots \backslash (A_2 \backslash (A_1 \backslash B))$.

**Theorem 7.** $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/, \backslash)} A \to B \iff \mathscr{A} \vdash_{\mathbf{L}(/, \backslash)} A \to B.$

*Proof.* 1) $\boxed{\Rightarrow}$ This is an easy, syntactic part of the proof.

It is necessary to show that axioms and rules of $\mathbf{L}_{\mathrm{H}}(/, \backslash)$ are derivable in the sequential product-free Lambek calculus $\mathbf{L}(/, \backslash)$. Deriving axioms is routine. Rule $(R1)$ is just a particular case of cut. Rule $(R2)$ is one of the monotonicity rules, derived as follows (due to Lambek [17]):

$$\dfrac{\dfrac{A \to B \qquad \dfrac{B \to B \quad C \to D}{B, B \backslash C \to D}\,(\backslash \to)}{A, B \backslash C \to D}\,(\mathrm{cut})}{B \backslash C \to A \backslash D}\,(\to \backslash)$$

Adduce the derivations of rules $(R3)$ and $(R4)$ in L$(/, \backslash)$:

$$\dfrac{\dfrac{A \to B \,/\, C \qquad \dfrac{C \to C \quad B \to B}{B \,/\, C \quad C \to B}\,(/ \to)}{A \quad C \to B}\,(\mathrm{cut})}{C \to A \backslash B}\,(\to \backslash) \qquad\qquad \dfrac{\dfrac{A \to B \backslash C \qquad \dfrac{B \to B \quad C \to C}{B \quad B \backslash C \to C}\,(\backslash \to)}{B \quad A \to C}\,(\mathrm{cut})}{B \to C \,/\, A}\,(\to /)$$

Therefore, $\mathbf{L}_{\rm H}(/,\backslash)$ is correct with respect to $\mathbf{L}(/,\backslash)$. Moreover, this correctness result is strong, i.e., also valid for derivability from theories.

2) $\boxed{\Leftarrow}$ In this part we utilise a semantic proof method, based on L-models (see Section 3), i.e., follow Buszkowski's canonical model construction (Proposition 3). We shall use the model $\mathcal{M}_{\mathscr{A}} = \langle \Sigma_{\mathscr{A}}, w_{\mathscr{A}} \rangle$, where $\Sigma_{\mathscr{A}} = \mathrm{Tp}(/,\backslash)$ and

$$w_{\mathscr{A}}(A) = \{\, B_1 B_2 B_3 \dots B_n \mid \mathscr{A} \vdash_{\mathbf{L}_{\rm H}(/,\backslash)} B_n \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus A)) \,\}$$

for all $A \in \mathrm{Tp}(/,\backslash)$.

By strong soundness (Proposition 2), if all formulas from a set of formulas $\mathscr{A}$ are true in an L-model $\mathcal{M}$, then every formula derivable from $\mathscr{A}$ in L is true in $\mathcal{M}$. So, first, let us check that all formulas from $\mathscr{A}$ are true in our model $\mathcal{M}_{\mathscr{A}}$.

Let $(C \to D) \in \mathscr{A}$. We are to show that $w_{\mathscr{A}}(C) \subseteq w_{\mathscr{A}}(D)$. We take $B_1 \dots B_n \in w_{\mathscr{A}}(C)$ and have to prove that $B_1 \dots B_n \in w_{\mathscr{A}}(D)$. By definition of $\mathcal{M}_{\mathscr{A}}$, $\mathscr{A} \vdash_{\mathbf{L}_{\rm H}(/,\backslash)} B_n \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus C))$.

Or goal is to show that $\mathscr{A} \vdash_{\mathbf{L}_{\rm H}(/,\backslash)} B_n \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus D))$. It can be done as follows. First we apply $(R2)$ several times to $C \to D$ (which is in $\mathscr{A}$), in order to get $B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus C)) \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus D))$. Second, we apply $(R1)$ with the left premise belonging to $\mathscr{A}$:

$$\frac{B_n \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus C)) \quad B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus C)) \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus D))}{B_n \to B_{n-1} \setminus \dots \setminus (B_2 \setminus (B_1 \setminus D))}\ (R1)$$

The second step is to check that $\langle \Sigma_{\mathscr{A}}, w_{\mathscr{A}} \rangle$ is really an L-model, that is:

1. $w_{\mathscr{A}}(C \,/\, D) = w_{\mathscr{A}}(C) \,/\, w_{\mathscr{A}}(D)$;

2. $w_{\mathscr{A}}(C \setminus D) = w_{\mathscr{A}}(C) \setminus w_{\mathscr{A}}(D)$.

Notice that the axioms of $\mathbf{L}_{\rm H}(/,\backslash)$ are asymmetric, so the proofs are different. $\boxed{1, \subseteq}$ Let $B_1 \dots B_n \in w_{\mathscr{A}}(C \,/\, D)$, $E_1 \dots E_k \in w_{\mathscr{A}}(D)$, and let us prove that $B_1 \dots B_n E_1 \dots E_k \in w_{\mathscr{A}}(C)$.

By definition of $\mathcal{M}_{\mathscr{A}}$, we have $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} B_n \to B_{n-1} \backslash \dots \backslash (B_1 \backslash (C / D))$ and $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash D$, and our goal is to show that $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash B_n \backslash \dots \backslash B_1 \backslash C$.

First we derive $B_{n-1} \backslash \dots \backslash (B_1 \backslash (C / D)) \to (B_{n-1} \backslash \dots \backslash (B_1 \backslash C)) / D$ by induction on $n$, applying axiom $(A3)$ and rule $(R2)$. Next,

$$\dfrac{\dfrac{\dfrac{\mathscr{A}}{B_n \to B_{n-1} \backslash \dots \backslash (B_1 \backslash (C / D))} \qquad B_{n-1} \backslash \dots \backslash (B_1 \backslash (C / D)) \to (B_{n-1} \backslash \dots \backslash (B_1 \backslash C)) / D}{B_n \to (B_{n-1} \backslash \dots \backslash (B_1 \backslash C)) / D}\,(R1)}{D \to B_n \backslash \dots \backslash B_1 \backslash C}\,(R3)$$

$$\dfrac{\dfrac{\mathscr{A}}{E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash D} \qquad \dfrac{E_{k-1} \backslash \dots \backslash E_1 \to E_{k-1} \backslash \dots \backslash E_1 \qquad D \to B_n \backslash \dots \backslash B_1 \backslash C}{E_{k-1} \backslash \dots \backslash E_1 \backslash D \to E_{k-1} \backslash \dots \backslash E_1 \backslash B_n \backslash \dots \backslash B_1 \backslash C}\,(R2)}{E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash B_n \backslash \dots \backslash B_1 \backslash C}\,(R1)$$

$\boxed{1, \supseteq}$ Let $B_1 \dots B_n \in w_{\mathscr{A}}(C) / w_{\mathscr{A}}(D)$ and let us prove that $B_1 \dots B_n \in w_{\mathscr{A}}(C / D)$, i.e. $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} B_n \to B_{n-1} \backslash \dots \backslash B_1 \backslash (C / D)$.

We have $B_1 \dots B_n A \in w_{\mathscr{A}}(C)$ for all $A \in w_{\mathscr{A}}(D)$. Let $A = D$, then $B_1 \dots B_n D \in w_{\mathscr{A}}(C)$, i.e., $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} D \to B_n \backslash \dots \backslash B_1 \backslash C$. Let us derive the required formula:

$$\dfrac{\dfrac{\dfrac{\mathscr{A}}{D \to B_n \backslash \dots \backslash B_1 \backslash C}}{B_n \to (B_{n-1} \backslash \dots \backslash (B_1 \backslash C)) / D}\,(R4) \qquad (B_{n-1} \backslash \dots \backslash (B_1 \backslash C)) / D \to B_{n-1} \backslash \dots \backslash B_1 \backslash (C / D)}{B_n \to B_{n-1} \backslash \dots \backslash B_1 \backslash (C / D)}\,(R1)$$

The right premise was already derived above.

$\boxed{2, \subseteq}$ Let $B_1 \dots B_n \in w_{\mathscr{A}}(C \backslash D)$ and $E_1 \dots E_k \in w_{\mathscr{A}}(C)$, and let us prove that $E_1 \dots E_k B_1 \dots B_n \in w_{\mathscr{A}}(D)$.

By definition of $\mathcal{M}_{\mathscr{A}}$, $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} B_n \to B_{n-1} \backslash \dots \backslash (B_1 \backslash (C \backslash D))$ and $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash C$.

We need to obtain that $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\backslash)} B_n \to B_{n-1} \backslash \dots \backslash B_1 \backslash E_k \backslash \dots \backslash E_1 \backslash D$. First we apply $(R2)$:

$$\dfrac{\dfrac{\mathscr{A}}{E_k \to E_{k-1} \backslash \dots \backslash E_1 \backslash C} \qquad E_{k-1} \backslash \dots \backslash E_1 \backslash D \to E_{k-1} \backslash \dots \backslash E_1 \backslash D}{(E_{k-1} \backslash \dots \backslash E_1 \backslash C) \backslash (E_{k-1} \backslash \dots \backslash E_1 \backslash D) \to E_k \backslash (E_{k-1} \backslash \dots \backslash E_1 \backslash D)}\,(R2)$$

Next, using axiom $(A2)$ several times: $C \backslash D \to (E_{k-1} \backslash \dots \backslash E_1 \backslash C) \backslash (E_{k-1} \backslash \dots \backslash E_1 \backslash D)$, and rule $(R1)$ we obtain: $C \backslash D \to E_k \backslash \dots \backslash E_1 \backslash D$.

Further, $C \setminus D \to E_k \setminus \ldots \setminus E_1 \setminus D$, by several applications of $(R2)$, yields $B_{n-1} \setminus \ldots \setminus B_1 \setminus (C \setminus D) \to B_{n-1} \setminus \ldots \setminus B_1 \setminus E_k \setminus \ldots \setminus E_1 \setminus D$, and finally

$$\dfrac{\dfrac{\mathscr{A}}{B_n \to B_{n-1} \setminus \ldots \setminus B_1 \setminus (C \setminus D)} \qquad B_{n-1} \setminus \ldots \setminus B_1 \setminus (C \setminus D) \to B_{n-1} \setminus \ldots \setminus B_1 \setminus E_k \setminus \ldots \setminus E_1 \setminus D}{B_n \to B_{n-1} \setminus \ldots \setminus B_1 \setminus E_k \setminus \ldots \setminus E_1 \setminus D} \ (R1)$$

$\boxed{2, \supseteq}$ Let $B_1 \ldots B_n \in w_{\mathscr{A}}(C) \setminus w_{\mathscr{A}}(D)$. Let us prove that $B_1 \ldots B_n \in w_{\mathscr{A}}(C \setminus D)$, i.e., $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\setminus)} B_n \to B_{n-1} \setminus \ldots \setminus B_1 \setminus (C \setminus D)$. We have $AB_1 \ldots B_n \in w_{\mathscr{A}}(D)$ for all $A \in w_{\mathscr{A}}(C)$. Let $A = C$, then $CB_1 \ldots B_n \in w_{\mathscr{A}}(D)$, i.e. $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\setminus)} B_n \to B_{n-1} \setminus \ldots \setminus B_1 \setminus C \setminus D$.

Finally, since $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\setminus)} A \to B$ and $\langle \Sigma_{\mathscr{A}}, w_{\mathscr{A}} \rangle$ is an L-model, by strong soundness we get $w_{\mathscr{A}}(A) \subseteq w_{\mathscr{A}}(B)$. Thus, it remains to show that if $w_{\mathscr{A}}(A) \subseteq w_{\mathscr{A}}(B)$, then $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(/,\setminus)} A \to B$ (i.e., $\mathcal{M}_{\mathscr{A}}$ is a *universal L-model* for $\mathscr{A}$). This follows from the fact that $A \in w_{\mathscr{A}}(A) \subseteq w_{\mathscr{A}}(B)$. $\square$

This argument also works for $\mathbf{L}(\setminus)$ and $\mathbf{L}_{\mathrm{H}}(\setminus)$. We can describe Savateev's calculus $\mathbf{L}_{\mathrm{H}}(\setminus)$ just as a fragment of our calculus: $\mathbf{L}_{\mathrm{H}}(\setminus)$ is obtained from $\mathbf{L}_{\mathrm{H}}(/,\setminus)$ presented above by removing axiom $(A3)$ and rules $(R3)$ and $(R4)$. The following theorem (a strong version of Savateev's theorem [22]) now follows from the proof of Theorem 7:

**Theorem 8.** $\mathscr{A} \vdash_{\mathbf{L}_{\mathrm{H}}(\setminus)} A \to B \iff \mathscr{A} \vdash_{\mathbf{L}(\setminus)} A \to B$.

For $\mathbf{L}_{\mathrm{H}}(/)$, one first needs to consider $\mathbf{L}_{\mathrm{H}}(\setminus, /)$, obtained from $\mathbf{L}_{\mathrm{H}}(/,\setminus)$ by swapping $\setminus$ and $/$. (Being both equivalent to $\mathbf{L}(/,\setminus)$, the calculi $\mathbf{L}_{\mathrm{H}}(\setminus, /)$ and $\mathbf{L}(/,\setminus)$ derive the same set of theorems from a theory $\mathscr{A}$; however, their axioms and rules differ.) Next, one removes $(A3)$, $(R3)$, and $(R4)$ from the new calculus, and obtains a non-sequential formulation for $\mathbf{L}_{\mathrm{H}}(/)$.

# 5 Craig's Trick for L

In this section we show that Craig's trick fails for $\mathbf{L}$, *i.e.,* prove Theorem 1. We construct a concrete example of a recursively enumerable theory $\mathscr{A}_0$ over $\mathbf{L}$, such that there is no decidable $\mathscr{A}_0'$ equivalent to $\mathscr{A}_0$. Our construction is based on the following technical lemma:

**Lemma 5.1.** *Let* $\mathrm{Var} = \{q, p_1, p_2, \ldots\}$ *($q = p_0$) and let* $\mathscr{E} = \{p_i \to q \mid i \geq 1\}$. *If* $\mathscr{A} \subseteq \mathscr{E}$ *and* $\mathscr{A}' \approx \mathscr{A}$, *then* $\mathscr{A}' \cap \mathscr{E} = \mathscr{A}$.

*Proof.* Let $D = \{i \mid (p_i \to q) \in \mathscr{A}\} \subseteq \{1, 2, 3, \ldots\}$ and consider the following R-interpretation $\mathcal{M} = \langle W, U, v\rangle$ (depicted on Table 1):

$$W = \{a_i \mid i \geq 1\} \sqcup \{b_i \mid i \in \mathbb{N}\} \sqcup \{a', a'', b', b''\};$$

$$U = \{\langle a_i, b_i\rangle \mid i \geq 1\} \cup \{\langle a', b'\rangle, \langle a'', b''\rangle\};$$

$$v(p_i) = \{\langle a_i, b_i\rangle\};$$

$$v(q) = \{\langle a_j, b_j\rangle \mid j \in D\} \cup \{\langle a', b'\rangle\}.$$

Here each pair $\langle a_i, b_i\rangle$ encodes the corresponding variable $p_i$. Two additional pairs, $\langle a', b'\rangle$ and $\langle a'', b''\rangle$, are added to ensure that $v(q)$ is neither $\varnothing$, nor $v(p_i)$ for some $i$, nor $U$.

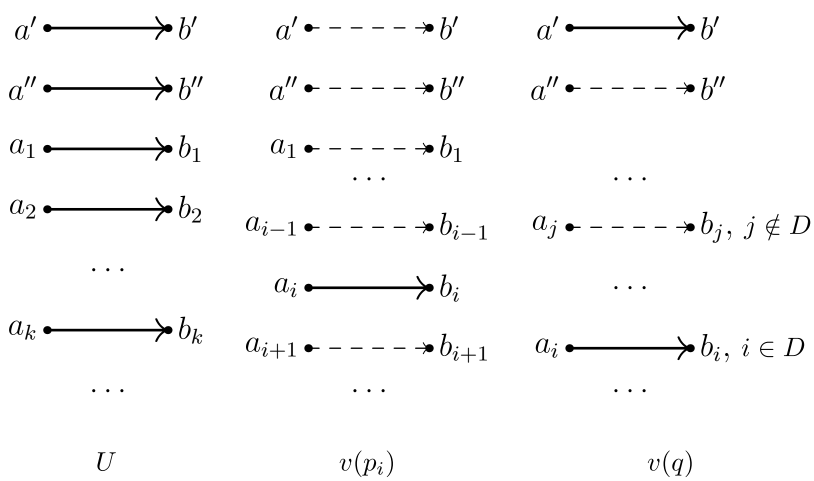


Table 1: R-model $\mathcal{M}$ for $\mathscr{A}$

Obviously, $\mathcal{M}$ is an R-model for $\mathscr{A}$ (for any $(p_i \to q) \in \mathscr{A}$ we have $v(p_i) \subseteq v(q)$).

Notice that in $U$ there exist no two pairs of the form $\langle x, y\rangle$ and $\langle y, z\rangle$, *i.e.*, no compositions. This trivialises interpretation of formulae other than variables.

First, $v(E \cdot F) = \varnothing$ for any $E$ and $F$. Second, since $v(E \,/\, F) = \{\langle x, y\rangle \in U \mid \{\langle x, y\rangle\} \circ v(F) \subseteq v(E)\}$ and, as noticed above, any composition in $U$ is empty, the condition for $\langle x, y\rangle$ is always true. Therefore, $v(E \,/\, F) = U$ for any $E$ and $F$, and the same for $v(F \setminus E)$.

The information on whether a sequent $A \to B$ is true or false in $\mathcal{M}$, depending on the form of $A$ and $B$, is gathered in the *truth table* for $\mathcal{M}$, shown on Table 2.

| $A$ \ $B$ | $p_j$ | $q$ | $E \cdot F$ | $E \,/\, F$ or $F \setminus E$ |
|---|---|---|---|---|
| $p_i$ | true for $i = j$,<br>false for $i \neq j$ | true for $i \in D$,<br>false for $i \notin D$ | false | true |
| $q$ | false | true | false | true |
| $E \cdot F$ | true | true | true | true |
| $E \,/\, F$ or $F \setminus E$ | false | false | false | true |

Table 2: Truth table for $\mathcal{M}$

Sequents with longer antecedents, $A_1, \ldots, A_n \to B$, are replaced with $A_1 \cdot \ldots \cdot A_n \to B$, and are true if $n \geq 2$, according to the truth table.

Sequents for which this table says "false" are not entailed by $\mathscr{A}$ and therefore cannot belong to $\mathscr{A}'$. Thus, if $(p_i \to q) \in \mathscr{A}'$, then $i \in D$, whence $(p_i \to q) \in \mathscr{A}$. This establishes the inclusion $\mathscr{A}' \cap \mathscr{E} \subseteq \mathscr{A}$.

For the opposite inclusion, let $D' = \{i \mid (p_i \to q) \in \mathscr{A}'\}$ and consider the corresponding modified model $\mathcal{M}'$, obtained from $\mathcal{M}$ by replacing $D$ with $D'$ in the definition. The truth table for $\mathcal{M}'$ differs from the one for $\mathcal{M}$ only in the case of $p_i \to q$: there one has to replace $D$ with $D'$. All other cells of the table remain the same.

Let us show that $\mathcal{M}' \models \mathscr{A}'$. Indeed, for sequents of the form $p_i \to q$ it is postulated in the definition of $\mathcal{M}'$. In other cells the behaviour of $\mathcal{M}'$ coincides with the behaviour of $\mathcal{M}$. Sequents for which the truth table says "false" cannot belong to $\mathscr{A}'$, since $\mathscr{A}' \approx \mathscr{A}$ and $\mathcal{M} \models \mathscr{A}$. For all other sequents the truth table says "true," thus they are true both in $\mathcal{M}$ and $\mathcal{M}'$.

Since $\mathscr{A} \approx \mathscr{A}'$, we also have $\mathcal{M}' \vDash \mathscr{A}$. If $(p_i \to q) \in \mathscr{A}$, then $\mathscr{A} \vdash p_i \to q$ and due to strong correctness $\mathscr{A} \vDash_R p_i \to q$, whence $\mathcal{M}' \vDash p_i \to q$, which yields $(p_i \to q) \in \mathscr{A}'$. This establishes the second inclusion, $\mathscr{A} \subseteq \mathscr{A}' \cap \mathscr{E}$. □

Lemma 5.1 immediately yields the failure of Craig's trick for **L** (Theorem 1). Indeed, let $D_0$ be a recursively enumerable, but undecidable subset of $\{1, 2, 3, \ldots\}$. Take theory $\mathscr{A}_0 = \{p_i \to q \mid i \in D_0\}$. Then for any alternative axiomatisation $\mathscr{A}' \approx \mathscr{A}_0$ its intersection with $\mathscr{E}$, namely $\mathscr{A}' \cap \mathscr{E} = \mathscr{A}_0$, is undecidable, and therefore so is $\mathscr{A}'$.

The proof of Theorem 1 can be also used to establish the failure of Craig's trick for $\mathbf{L}(/, \backslash)$ and $\mathbf{L}(\backslash)$ (one just omits the cases for connectives not in use). In [16], we prove this using L-models instead of R-models.

Strong equivalence results (Section 4) propagate the failure of Craig's trick to the non-sequential versions of the Lambek calculus and its fragments ($\mathbf{L}_{\mathrm{H}}$ by Lambek [17], $\mathbf{L}_{\mathrm{H}}(\backslash)$ by Savateev [22], and $\mathbf{L}_{\mathrm{H}}(/, \backslash)$ presented in this paper).

# 6 Craig's Trick in Variants of the Lambek Calculus

In this section we show that Theorem 1 is not at all robust: slight modifications of the calculus restore Craig's trick.

First we consider the Lambek calculus allowing empty antecedents, $\mathbf{L}^*$ [18]. This calculus is obtained from **L** (see Section 2) by dropping Lambek's restriction (the constraint "where $\Pi$ is non-empty" on $(\to /)$ and $(\to \backslash)$). While Lambek's restriction and the corresponding calculus **L** are motivated linguistically, the calculus allowing empty antecedents, $\mathbf{L}^*$, is more convenient from the algebraic and logical point of view. Notice that $\mathbf{L}^*$ is not a conservative extension of **L**, since there exist sequents with a non-empty antecedent which yet require empty antecedents in their derivations ($(p \backslash p) \backslash q \to q$, for example).

**Theorem 9.** *The Lambek calculus allowing empty antecedents enjoys Craig's trick, i.e., for any recursively enumerable theory $\mathscr{A}$ there exists a decidable theory $\mathscr{A}'$ such that $\mathscr{A} \vdash_{\mathbf{L}^*} \Pi \to B$ iff $\mathscr{A}' \vdash_{\mathbf{L}^*} \Pi \to B$ for any sequent $\Pi \to B$.*

*Proof.* Let us call the total number of variable occurrences of a formula the *size* of this formula (resp., sequent). We show that for each formula has an equivalent one of bigger size and then use the standard reasoning for Craig's trick (see Introduction). Namely, if for any $B$ there exists a formula $B'$ whose size is greater than the size of $B$, then the size of $B^{\overbrace{\prime\prime\cdots\prime}^{k}}$ has size at least $k$. Thus, if $\mathscr{A}$ is enumerated as $\Pi_0 \to B_0$, $\Pi_1 \to B_1$, ... (more formally, $\mathscr{A}$ is the image of a total computable function $f \colon n \mapsto (\Pi_n \to B_n)$), then $\mathscr{A}' = \{\Pi_0 \to B_0, \Pi_1 \to B_1', \Pi_2 \to B_2'', \dots\}$ is equivalent to $\mathscr{A}$ and decidable, since for checking a sequent of size $k$ one needs to test only the first $k$ elements of $\mathscr{A}'$.

In $\mathbf{L}^*$, we have $B \leftrightarrow (B \,/\, B) \setminus B$, thus one can take $B' = (B \,/\, B) \setminus B$. $\square$

Another variation is connected to the number of variables we are allowed to use. In our construction for Theorem 1, we essentially used an infinite (countable) set of variables. The following result shows that it is inevitable:

**Theorem 10.** *If there exists such $m$ that formulae from $\mathscr{A}$ include only variables from $\{p_0, p_1, \dots, p_{m-1}\}$ and $\mathscr{A}$ is recursively enumerable, then there exists a decidable theory $\mathscr{A}'$ equivalent to $\mathscr{A}$ over $\mathbf{L}$ (i.e., $\mathscr{A} \vdash_{\mathbf{L}} \Pi \to B$ iff $\mathscr{A}' \vdash_{\mathbf{L}} \Pi \to B$ for any sequent $\Pi \to B$).*

*Proof.* The proof of this theorem is just a bit trickier than the proof of the previous one. First, replace every sequent with a long antecedent, $A_1, \dots, A_n \to B$, with $A_1 \cdot \ldots \cdot A_n \to B$. Next we show that every formula, *except variables,* can be increased in size using the following equivalences (provable in $\mathbf{L}$):

$$E \,/\, F \leftrightarrow E \,/((E \,/\, F) \setminus E)$$

$$F \setminus E \leftrightarrow (E \,/(F \setminus E)) \setminus E$$

$$E \cdot F \leftrightarrow ((E \cdot F) \,/\, F) \cdot F$$

Now we notice that $\mathscr{A}$ contains only a finite number (not more than $m^2$) of sequents of the form $p_i \to p_j$. Thus, $\mathscr{A}$ is a union of two parts: (1) a finite set $\mathscr{A}_1$ and (2) a recursively enumerable set $\mathscr{A}_2 = \mathscr{A} - \mathscr{A}_1$ of axioms of the form $A \to B$ where either $A$ or $B$ is not a variable. For such a sequent, we can construct $(A \to B)' = (A' \to B')$ such that $A' \leftrightarrow A$, $B' \leftrightarrow B$, and at least one of $A'$ and $B'$ is greater in size than $A$ or $B$ respectively. This is done using the equivalences above. Thus, we can equivalently replace $\mathscr{A}_2 = \{A_0 \to B_0, A_1 \to B_1, A_2 \to B_2, \ldots\}$ with a decidable set $\mathscr{A}_2' = \{A_0 \to B_0, A_1' \to B_1', A_2'' \to B_2'', \ldots\}$. Finally, $\mathscr{A}' = \mathscr{A}_1 \cup \mathscr{A}_2'$ is decidable and equivalent to $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$, as requested. $\square$

Analogs of Theorems 9 and 10 also hold for the product-free calculi, $\mathbf{L}^{*}(/, \backslash)$ (with a possibly infinite set of variables) and $\mathbf{L}(/, \backslash)$ restricted to a finite set of variables, respectively. For Theorem 9, the proof remains exactly the same. For Theorem 10, only one fix is needed: a long sequent, $A_1, \ldots, A_n \to B$, should be translated as $A_1 \to (B \,/\, A_n) \,/ \ldots /\, A_2$ instead of $A_1 \cdot \ldots \cdot A_n \to B$, to avoid using the product connective.

Moreover, $\mathbf{L}^{*}(\backslash)$ restricted to a finite set of variables also enjoys Craig's trick, due to the following equivalence (provable in $\mathbf{L}^{*}(\backslash)$):

$$(F \backslash F) \backslash (F \backslash E) \leftrightarrow F \backslash E.$$

Using this equivalence, we apply the same reasoning as in Theorem 10: any sequent not of the form $p_i \to p_j$ can be increased in size, and the set $\mathscr{A}_1$ of sequents of the form $p_i \to p_j$ in $\mathscr{A}$ is finite due to the restriction on the set of variables.

As usually, $/$ is dual to $\backslash$.

Several cases, however, are still open for future research (see the next section).

# 7 Conclusions and Future Work

In this paper we have shown that Craig's trick is invalid for the product-free fragment of the Lambek calculus, $\mathbf{L}(\backslash, /)$, both for its standard sequent version and for a newly developed non-sequential calculus $\mathbf{L}_{\mathrm{H}}(\backslash, /)$.

There are two questions left open for further research. First, since with product strong L-completeness fails, the question of Craig's trick for the $\vDash_L$ entailment relation, as opposed to $\vdash_{\mathbf{L}}$, is still open. Second, in our construction we essentially use an infinite set of variables: $\mathbf{L}(/, \backslash)$ with a finite set of variables enjoys Craig's trick (see above). If we restrict our language even further, however, and consider the Lambek calculus with only one division, $\mathbf{L}(\backslash)$, the question whether the finite-variable fragment of it enjoys Craig's trick is still open[3]. Notice that Craig's trick for $\mathbf{L}(\backslash)$ with a finite set of variables, even if it is valid, cannot be proved by standard means, since, as shown in an unpublished paper of I. Bolgova [5], in $\mathbf{L}(\backslash)$ there are no non-trivial equivalences (if $A \leftrightarrow B$, then $A$ just coincides with $B$).

## Acknowledgments

The authors are grateful to Ilya Shapirovsky for asking the question whether Craig's trick is valid for the Lambek calculus. They would also like to thank the participants of ESSLLI 2016 in Bozen, TbiLLC 2017 in Lagodekhi, and the seminar 'Algorithmic Problems in Algebra and Logic' ran by Prof. S. I. Adian and Prof. L. D. Beklemishev at Moscow State University, for fruitful discussions.

The first author, Stepan Kuznetsov, is one of the Young Russian Mathematics award winner and would like to thank the sponsors and jury of the contest for this high honour. (The work of Stepan Kuznetsov on this particular paper was funded from another source, as mentioned in the 'Funding' section below.)

## Funding

The work of Stepan Kuznetsov towards this paper was supported by the Russian Science Foundation under grant 14-50-00005 and performed at Steklov Mathematical Institute of the Russian Academy of Sciences in Moscow.

[3] This question was suggested by Anastasia Onoprienko.

Sections 5 and 7 were contributed by Stepan Kuznetsov. Sections 4 and 6 were contributed by Valentina Lugovaya and Anastasiia Ryzhova. (Sections 1–3 are introductory and do not contain new results or ideas.)

## References


[1] V. M. Abrusci. A comparison between Lambek syntactic calculus and intuitionistic linear logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **36**, 11–15, 1990.

[2] H. Andréka, S. Mikulás. Lambek calculus and its relational semantics: completeness and incompleteness. *Journal of Logic, Language, and Information*, **3**, 1–37, 1994.

[3] J. van Benthem. Language in action. *Journal of Philosophical Logic,* **20**, 225–263, 1989.

[4] J. van Benthem. Language in action. North-Holland, Amsterdam, 1991.

[5] I. Bolgova. Equivalence of types in the Lambek calculus with one division (in Russian). Unpublished manuscript. Moscow State University, 2003.

[6] W. Buszkowski. Compatibility of a categorial grammar with an associated category system. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **28**, 229–238, 1982.

[7] W. Buszkowski. Completeness results for Lambek syntactic calculus. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **32**, 13–28, 1986.

[8] W. Buszkowski. Lambek calculus with nonlogical axioms. In: Language and Grammar, ed. by C. Casadio, P.J. Scott, R.A.G. Seedy.

Studies in Mathematical Linguistics and Natural Language, *CSLI Lecture Notes* vol. 168, 77–93, 2005.

[9] W. Buszkowski. Lambek calculus and substructural logics. *Linguistic Analysis*, **36**, 15–48, 2010.

[10] B. Carpenter. Type-logical semantics. The MIT Press, Cambridge, MA, 1997.

[11] J.M. Cohen. The equivalence of two concepts of categorial grammar. *Information and Control,* **10**, 475–484, 1967

[12] W. Craig. On axiomatizability within a system. *Journal of Symbolic Logic*, **18**(1), 30–32, 1953.

[13] W. Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *Journal of Symbolic Logic*, **22**(3), 269–285, 1957.

[14] P. T. Geach. A program for syntax. *Synthese* **22**, 3–17, 1970.

[15] L. Humberstone. Geach's categorial grammar. *Linguistics and Philosophy,* **28**(3), 281–317, 2005.

[16] S. Kuznetsov, V. Lugovaya, A. Ryzhova. Recursive enumerability doesn't always give a decidable axiomatization. Extended abstract. *XII Tbilisi Symposium of Language, Logic, and Computation (TbiLLC 2017),* pp. 109–111. Lagodekhi, 2017.

[17] J. Lambek. The mathematics of sentence structure. *The American Mathematical Monthly*, **65**, 154–170, 1958.

[18] J. Lambek. On the calculus of syntactic types. In: Structure of language and its mathematical aspects (ed. by R. Jakobson), pp. 166–178. American Mathematical Society, Providence, RI, 1991.

[19] V. Lugovaya, A. Ryzhova. Hilbert-style Lambek calculus with two divisions. *Proceedings of the ESSLLI 2016 Student Session,* pp. 179–183. Bozen/Bolzano, 2016.

[20] M. Moot, C. Retoré. The logic of categorial grammars. A deductive account of natural language syntax and semantics. *Lecture Notes in Computer Science* vol. 6850, Springer, 2012.

[21] G.V. Morrill. Categorial grammar. Logical syntax, semantics, and processing. Oxford University Press, 2011.

[22] Yu. Savateev. Variants of the Lambek calculus with one division. Unpublished manuscript, in Russian. Moscow State University, 2004.

[23] M. Steedman. Categorial grammar. University of Pennsylvania Department of Computer and Information Science Technical Report No. MS-CIS-92-52, 1992.

[24] M. Pentus. Models for the Lambek calculus. *Annals of Pure and Applied Logic*, **75**, 179–213, 1995.

[25] W. Zielonka. Axiomatisability of the Ajdukiewicz-Lambek calculus by means of cancellation schemes. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **27**, 215–224, 1981.