L-models and R-models for Lambek Calculus Enriched with Additives and the Multiplicative Unit*

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Abstract. Language and relational models, or L-models and R-models, are two natural classes of models for the Lambek calculus. Completeness w.r.t. L-models was proved by Pentus and completeness w.r.t. R-models by Andréka and Mikulás. It is well known that adding both additive conjunction and disjunction together yields incompleteness, because of the distributive law. The product-free Lambek calculus enriched with conjunction only, however, is complete w.r.t. L-models (Buszkowski) as well as R-models (Andréka and Mikulás). The situation with disjunction turns out to be the opposite: we prove that the product-free Lambek calculus enriched with disjunction only is incomplete w.r.t. L-models as well as R-models. If the empty premises are allowed, the product-free Lambek calculus enriched with conjunction only is still complete w.r.t. L-models but in which the empty word is allowed. Both versions are decidable (PSPACE-complete in fact). Adding the multiplicative unit to represent explicitly the empty word within the L-model paradigm changes the situation in a completely unexpected way. Namely, we prove undecidability for any L-sound extension of the Lambek calculus with conjunction and with the unit, whenever this extension includes certain L-sound rules for the multiplicative unit, to express the natural algebraic properties of the empty word. Moreover, we obtain undecidability for a small fragment with only one implication, conjunction, and the unit, obeying these natural rules. This proof proceeds by the encoding of two-counter Minsky machines.
Keywords: Lambek calculus, language models, relational models, distributive law, incompleteness, undecidability.

1 Introduction

By $\mathbf{L}{\vee}{\wedge}$ we denote the *Lambek calculus with additive connectives,* disjunction and conjunction. Formulae of $\mathbf{L}{\vee}{\wedge}$ are built from a countable set of variables

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(which we denote by $p, q, r, \ldots$) using five binary connectives: $\backslash$ (left implication), $/$ (right implication), $\cdot$ (product, or multiplicative conjunction), $\vee$ (additive disjunction), and $\wedge$ (additive conjunction). The Lambek calculus with additive connectives is formulated as a Gentzen-style sequent calculus of a two-sided (intuitionistic) format. Being a non-commutative substructural logic, however, it has an important difference from traditional sequent calculi. Namely, left-hand sides (antecedents) of $\mathbf{L}{\vee}{\wedge}$ sequents are finite *linearly ordered sequences* (not sets or multisets) of formulae. The right-hand side (succedent) of a sequent is a formula. Axioms and inference rules of $\mathbf{L}{\vee}{\wedge}$ are presented on Table 1.

$$\frac{}{A \vdash A}\ \mathrm{Id}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, \Phi, A \backslash B, \Sigma_2 \vdash C}\ \backslash\mathrm{L} \qquad \frac{A, \Sigma \vdash B}{\Sigma \vdash A \backslash B}\ \backslash\mathrm{R},\ \Sigma \text{ is not empty}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, B / A, \Phi, \Sigma_2 \vdash C}\ /\mathrm{L} \qquad \frac{\Sigma, A \vdash B}{\Sigma \vdash B / A}\ /\mathrm{R},\ \Sigma \text{ is not empty}$$

$$\frac{\Sigma_1, A, B, \Sigma_2 \vdash C}{\Sigma_1, A \cdot B, \Sigma_2 \vdash C}\ \cdot\mathrm{L} \qquad \frac{\Sigma_1 \vdash A \quad \Sigma_2 \vdash B}{\Sigma_1, \Sigma_2 \vdash A \cdot B}\ \cdot\mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \vee B, \Sigma_2 \vdash C}\ \vee\mathrm{L} \qquad \frac{\Sigma \vdash A}{\Sigma \vdash A \vee B} \quad \frac{\Sigma \vdash B}{\Sigma \vdash A \vee B}\ \vee\mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C} \quad \frac{\Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C}\ \wedge\mathrm{L} \qquad \frac{\Sigma \vdash A \quad \Sigma \vdash B}{\Sigma \vdash A \wedge B}\ \wedge\mathrm{R}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, \Phi, \Sigma_2 \vdash C}{\Sigma_1, \Phi, \Sigma_2 \vdash C}\ \mathrm{Cut}$$

Table 1. Lambek calculus with additive connectives

The first three connectives, namely, two implications, also called divisions (left and right, $\backslash$ and $/$) and product (multiplicative conjunction, $\cdot$), are due to Lambek [15]. These connectives are called *multiplicative.* Two *additive* connectives, $\vee$ and $\wedge$, are added to the Lambek calculus in the spirit of Girard's linear logic [6] (where they are denoted by $\oplus$ and $\&$, respectively). As noticed by Abrusci [1], the Lambek calculus can be considered as a non-commutative variant of linear logic. A specific feature of the Lambek calculus, however, is the so-called Lambek's non-emptiness restriction: as one can see from the form of the rules, left-hand sides of sequents are required to be non-empty. This restriction is motivated by linguistic applications of the Lambek calculus [17, Sect. 2.5].

The cut rule is eliminable by a standard argument. Cut elimination yields the subformula property and makes it easy to formulate elementary fragments. If one takes a subset of the set of connectives, and leaves only the corresponding rules of inference, the calculus obtained is a conservative fragment of $\mathbf{L}{\vee}{\wedge}$. The fragment without additive connectives ($\vee$ and $\wedge$) is the original Lambek calculus denoted by $\mathbf{L}$. Fragments with only one additive connective are denoted by $\mathbf{L}{\vee}$

and $\mathbf{L}\wedge$. We also consider product-free fragments with conjunction, $\mathbf{L}(\backslash, /, \wedge)$ and $\mathbf{L}(\backslash, \wedge)$, which include, respectively, only $\backslash$, $/$, $\wedge$ and only $\backslash$ and $\wedge$.

From the point of view of semantics, there exist many classes of models for the Lambek calculus. We consider two natural ones, language and relational ones. Language models, or *L-models,* are inspired by linguistic motivation and applications of the Lambek calculus. An L-model is defined on $\mathcal{P}(\Sigma^+)$, the set of all languages over an alphabet Σ without the empty word, by an interpretation function w which maps Lambek formulae to languages from $\mathcal{P}(\Sigma^+)$. The interpretation function is defined arbitrarily on variables, and should commute with Lambek connectives in the following way:

$$\begin{aligned}
&w(A \setminus B) = w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, vu \in w(B)\};\\
&w(B \,/\, A) = w(B) \,/\, w(A) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, uv \in w(B)\};\\
&w(A \cdot B) = w(A) \cdot w(B) = \{uv \mid u \in w(A), v \in w(B)\}.
\end{aligned}$$

A sequent $A_1, \ldots, A_n \vdash B$ is considered true in such a model, if and only if $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$.

Notice that the empty word, ε, is not allowed due to Lambek's restriction. The empty *set,* however, could appear as a result of division, and this is absolutely acceptable.

For a relational model, or *R-model,* the base set is the set of all subrelations of a fixed transitive binary relation $W \subseteq U \times U$, *i.e.,* $\mathcal{P}(W)$. The interpretation function now maps Lambek formulae to subsets of W, and should obey the following commutation rules:

$$\begin{aligned}
&w(A \setminus B) = w(A) \setminus w(B) = \{\langle y, z\rangle \in W \mid (\forall \langle x, y\rangle \in w(A))\, \langle x, z\rangle \in w(B)\};\\
&w(B \,/\, A) = w(B) \,/\, w(A) = \{\langle x, y\rangle \in W \mid (\forall \langle y, z\rangle \in w(A))\, \langle x, z\rangle \in w(B)\};\\
&w(A \cdot B) = w(A) \circ w(B) = \{\langle x, z\rangle \mid (\exists y \in U)\, \langle x, y\rangle \in w(A), \langle y, z\rangle \in w(B)\}.
\end{aligned}$$

Truth conditions for R-models are exactly the same as in L-models: $A_1, \ldots, A_n \vdash B$ is true, iff $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$.

Additive connectives, both in L-models and R-models, are interpreted as set-theoretical union and intersection:

$$\begin{aligned}
w(A \vee B) = w(A) \cup w(B);\\
w(A \wedge B) = w(A) \cap w(B).
\end{aligned}$$

Both L-models and R-models provide sound semantics for $\mathbf{L}\vee\wedge$ (and, therefore, all its elementary fragments): if a sequent is derivable, then it is true in all models. Completeness (the reverse implication), however, is a more subtle issue.

There is a folklore fact that $\mathbf{L}\vee\wedge$ is incomplete both w.r.t. both L-interpretation and R-interpretation, due to the distibutivity law

$$(A \vee C) \wedge (B \vee C) \vdash (A \wedge B) \vee C.$$

The distributivity law is true for set-theoretic interpretation of $\vee$ and $\wedge$—in particular, in all L-models and all R-models—but is not provable in $\mathbf{L}\vee\wedge$. The failure to derive distributivity is a common feature of several substructural logics, as noticed by Ono and Komori [19].

L-models and R-models are both specific subclasses of general algebraic models for $\mathbf{L}\vee\wedge$, *residuated lattices* [22, 5]. A residuated lattice is a lattice equipped with a monoidal structure (multiplication and the unit) and division operations, obeying the natural condition: $a \preceq c \,/\, b \iff a \cdot b \preceq c \iff b \preceq a \setminus c$ (where $\preceq$ is the lattice preorder). Residuated lattices in general, as opposed to lattices of formal languages or binary relations, are not required to be distributive. This removes the incompleteness issue mentioned above; in fact, $\mathbf{L}\vee\wedge$ is complete w.r.t. interpretations on arbitrary residuated lattices, which is proved by an argument in the style of Lindenbaum and Tarski. Moreover, there is a more specific completeness result for $\mathbf{L}\vee\wedge$ w.r.t. so-called *syntactic concept lattices,* introduced by Wurm [23] as a modification of L-models without the distributivity constraint.

By $\mathbf{L}\vee\wedge + \mathrm{distrib}$ we denote $\mathbf{L}\vee\wedge$ with the distributivity principle,

$$(A \vee C) \wedge (B \vee C) \vdash (A \wedge B) \vee C,$$

added as an extra axiom (Cut is kept as an official rule of the system, since it becomes non-eliminable after adding extra axioms). It looks natural to conjecture completeness of $\mathbf{L}\vee\wedge + \mathrm{distrib}$ w.r.t. L-models and/or R-models. However, these are both open questions.

Some fragments of $\mathbf{L}\vee\wedge$, however, are still complete w.r.t L-models and R-models. Algebraically this means that, in particular, distributivity cannot be expressed in the weaker languages of these fragments. Namely, the Lambek calculus extended with conjunction only, $\mathbf{L}\wedge$, is R-complete, as shown by Andréka and Mikulás [2]. For L-completeness, the question about $\mathbf{L}\wedge$, which includes both division, product, and conjunction, is still open. For the Lambek calculus without additives, however, L-completeness was shown by Pentus [21], and for $\mathbf{L}(\setminus, /, \wedge)$ L-completeness was shown by Buszkowski [3].

In this paper we emphasize $\mathbf{L}\vee$, the disjunction-only fragment of $\mathbf{L}\vee\wedge$. The situation with disjunction turns out to be the opposite: in Section 2 we prove that the product-free Lambek calculus enriched with disjunction only is incomplete w.r.t. L-models as well as R-models—in fact, w.r.t. any class of distributive residuated lattices.

If one abolishes Lambek's restriction, *i.e.* allows the use of empty premises, the product-free Lambek calculus enriched with conjunction only is still complete w.r.t. L-models in which the empty word is allowed. Both versions are decidable (PSPACE-complete in fact [11]).

Adding the multiplicative unit to represent explicitly the empty word within the L-model paradigm changes the situation in a completely unexpected way. Even the product-free fragment with only one implication, conjunction, and the unit cannot be extended to a decidable system complete with respect to L-models. This proof proceeds by the encoding of two-counter Minsky machines with the help of certain simple rules for the multiplicative unit, caused by the empty word.

Let us focus on L-models. The unit in L-models is necessarily interpreted as $\{\varepsilon\}$, where ε is the empty word. In particular, adding the unit forces us to allow the empty word in L-models.

An attempt to axiomatise the unit constant by the rules for multiplicative unit taken from linear logic [16] results in an L-sound, but not L-complete system [4, 12]. Unfortunately, no L-complete recursively enumerable axiomatisation for the Lambek calculus with the unit constant is known. In Section 3, we present an extension of the Lambek calculus that respects the most natural peculiarities of the empty word ε in L-models, such as: $\varepsilon \cdot \varepsilon = \varepsilon$ and $x \cdot \varepsilon = \varepsilon \cdot x$. Our main result is that this system, which we denote by $\mathbf{L}^{+\varepsilon}$, is undecidable. Moreover, we get undecidability for any L-sound calculus that includes $\mathbf{L}^{+\varepsilon}$.

2 Incompleteness of $\mathbf{L}\vee$ w.r.t. L-models and R-models

We show that $\mathbf{L}\vee$ is incomplete w.r.t. language and relational models by presenting a concrete example of a sequent true in all such models, but not derivable in $\mathbf{L}\vee$.

Theorem 1. *The sequent*

$$(((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x)) \cdot ((x\,/\,y) \vee x) \cdot (((x\,/\,y) \vee x)\,\backslash((x\,/\,z) \vee x)) \\ \vdash (x\,/(y \vee z)) \vee x$$

is not derivable in $\mathbf{L}\vee$*, but is derivable in* $\mathbf{L}\vee\wedge + \mathrm{distrib}$ *and, therefore, true in all L-models and all R-models.*

Before going into the detailed proof of this theorem, let us show the ideas behind it. The monstruous sequent which we use as our counter-example comes from the *diamond construction* originally due to Lambek [15]. For two formulae A and B let C be their *meeting* formula, if both $C \vdash A$ and $C \vdash B$ are derivable, and let D be their *joining* formula, if both $A \vdash D$ and $B \vdash D$. (Meeting and joining formulae are of course not unique.) In $\mathbf{L}\vee\wedge$, constructing meeting and joining formulae is trivial, since one just takes $C = A \wedge B$ and $D = A \vee B$. Moreover, this gives the maximum meeting and the minimum joining formula: for any other meeting formula C' and any other joining formula D' we have $C' \vdash A \wedge B$ and $A \vee B \vdash D'$.

In $\mathbf{L}\vee$, however, only the joining formula, $A \vee B$, is explicitly given. Wishing to encode distributivity, we need some meeting formula to use it *in lieu* of $A \wedge B$. Such a formula is given by the following lemma, which is a variation of the diamond constructions of Lambek [15] and Pentus [20].

Lemma 1. *For any calculus extending* $\mathbf{L}$*, if* D *is a joining formula for* A *and* B*, then* $(A\,/\,D) \cdot A \cdot (A \setminus B)$ *is a meeting formula for* A *and* B*. In particular, in* $\mathbf{L}\vee$ *for any two formulae* A *and* B *we have a meeting formula,* $(A\,/(A \vee B)) \cdot A \cdot (A \setminus B)$*.*

Proof. For $C = (A \,/\, D) \cdot A \cdot (A \setminus B)$ the necessary sequents $C \vdash A$ and $C \vdash B$ are derived as follows:

$$
\dfrac{\dfrac{A \vdash A \qquad \dfrac{B \vdash D \quad A \vdash A}{A \,/\, D, B \vdash A}}{A \,/\, D, A, A \setminus B \vdash A}}{(A \,/\, D) \cdot A \cdot (A \setminus B) \vdash A}
\qquad
\dfrac{\dfrac{A \vdash D \qquad \dfrac{A \vdash A \quad B \vdash B}{A, A \setminus B \vdash B}}{A \,/\, D, A, A \setminus B \vdash B}}{(A \,/\, D) \cdot A \cdot (A \setminus B) \vdash B}
$$

Now we are ready to explain the construction in Theorem 1 and prove the theorem. Take $A = (x \,/\, y) \vee x$ and $B = (x \,/\, z) \vee x$. Then $D = A \vee B$ is equivalent to $(x \,/\, y) \vee (x \,/\, z) \vee x$, and by Lemma 1 the left-hand side of the sequent in Theorem 1 is exactly the meeting formula for A and B, which we denote by C. Thus, $C \vdash A$ and $C \vdash B$ are derivable in $\mathbf{L}\vee$ and therefore in $\mathbf{L}\vee\wedge$, and so is $C \vdash A \wedge B$.

Recall that $A \wedge B = ((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x)$ and apply distributivity:

$$((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x) \vdash ((x \,/\, y) \wedge (x \,/\, z)) \vee x.$$

Finally, recall that $(x \,/\, y) \wedge (x \,/\, z)$ is equivalent to $x \,/ (y \vee z)$ in $\mathbf{L}\vee\wedge$,[5] which allows us to get rid of $\wedge$ in the right-hand side:

$$((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x) \vdash ((x \,/ (y \vee z)) \vee x.$$

Now cut with $C \vdash ((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x)$ (*i.e.,* $C \vdash A \wedge B$) yields the needed sequent in Theorem 1.

The second statement, that this sequent is not derivable in $\mathbf{L}\vee$ without distributivity, does not follow automatically from the fact that distributivity is not provable in $\mathbf{L}\vee\wedge$. This is because the formula C constructed using the diamond construction is a stronger meeting formula than $A \wedge B$: $C \vdash A \wedge B$, but not $A \wedge B \vdash C$. Thus, we still have to prove that the sequent in Theorem 1 is not derivable in $\mathbf{L}\vee$ or, equivalently, in $\mathbf{L}\vee\wedge$.

Such a non-derivability proof can be performed, as suggested by one of the anonymous reviewers, by presenting an algebraic counter-model, *i.e.,* an interpretation over a residuated lattice which falsifies the sequent in question. (This lattice should be necessarily non-distributive, thus, it is neither an L-model nor an R-model.) Another, purely syntactic strategy is to apply the proof search algorithm directly (recall that the derivability problem in $\mathbf{L}\vee\wedge$ is decidable). Following this line, we used an automatic theorem-prover for $\mathbf{L}\vee\wedge$,

[5] The derivations establishing equivalence are as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{y \vdash y \quad x \vdash x}{x \,/\, y, y \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), y \vdash x} \qquad \dfrac{\dfrac{z \vdash z \quad x \vdash x}{x \,/\, z, z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), y \vee z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z) \vdash x \,/ (y \vee z)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{y \vdash y}{y \vdash y \vee z} \quad x \vdash x}{x \,/ (y \vee z), y \vdash x}}{x \,/ (y \vee z) \vdash x \,/\, y} \qquad \dfrac{\dfrac{\dfrac{z \vdash z}{z \vdash y \vee z} \quad x \vdash x}{x \,/ (y \vee z), z \vdash x}}{x \,/ (y \vee z) \vdash x \,/\, z}}{x \,/ (y \vee z) \vdash (x \,/\, y) \wedge (x \,/\, z)}
$$

and its extension with Kleene star, implemented by Jipsen, based on [18, 19, 7, 8]. Jipsen's theorem-prover is avaliable online: `http://www1.chapman.edu/~jipsen/kleene/`. The algorithm performs exhaustive proof search and thus establishes non-derivability. In order to make this paper self-contained and independent from external derivability-checking software, in the Appendix we represent the execution of the proof search algorithm (and, thus, the proof of non-derivability), with some simplifications, in a human-readable form.

3 Undecidability of the Fragment $(\backslash, \wedge, \mathbf{1})$

In this section we consider the extension of the Lambek calculus with the multiplicative unit constant. In L-models, because of the principle $A \cdot \mathbf{1} \vdash A$, the constant $\mathbf{1}$ is necessarily interpreted as the singleton set $\{\varepsilon\}$, where ε is the empty word. In particular, introducing the unit constant requires modification of the definition of L-models by allowing the empty word to belong to our languages. For the same reason, we have to abolish Lambek's non-emptiness restriction. Because of this specific interpretation of the unit constant, we introduce principles connected with this particular interpretation of the unit. Such principles include $A \cdot \{\varepsilon\} = \{\varepsilon\} \cdot A$ and $\{\varepsilon\} \cdot \{\varepsilon\} = \{\varepsilon\}$. On Table 2, we present a calculus, denoted by $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, which reflects these two principles as sequential rules.

$$\frac{}{A \vdash A}\ \mathrm{Id} \qquad \frac{}{A, \mathbf{1} \vdash A}\ \mathbf{1}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, \Phi, A \backslash B, \Sigma_2 \vdash C}\ \backslash \mathrm{L} \qquad \frac{A, \Sigma \vdash B}{\Sigma \vdash A \backslash B}\ \backslash \mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C} \quad \frac{\Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C}\ \wedge\mathrm{L} \qquad \frac{\Sigma \vdash A \quad \Sigma \vdash B}{\Sigma \vdash A \wedge B}\ \wedge\mathrm{R}$$

$$\frac{\Sigma_1, A, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}{\Sigma_1, (\mathbf{1} \wedge G), A, \Sigma_2 \vdash C}\ \mathrm{L}\varepsilon \qquad \frac{\Sigma_1, (\mathbf{1} \wedge G), A, \Sigma_2 \vdash C}{\Sigma_1, A, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}\ \mathrm{R}\varepsilon$$

$$\frac{\Sigma_1, (\mathbf{1} \wedge G), (\mathbf{1} \wedge G), \Sigma_2 \vdash C}{\Sigma_1, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}\ \mathrm{D}\varepsilon$$

Table 2. Axioms and inference rules for a minimal $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$

The "commuting" rules Lε and Rε are caused by that, for any set X,

$$X \cdot \{\varepsilon\} = \{\varepsilon\} \cdot X, \quad \varnothing \cdot X = X \cdot \varnothing,$$

whereas the "doubling" rule Dε is caused by

$$\{\varepsilon\} \cdot \{\varepsilon\} = \{\varepsilon\}, \quad \varnothing \cdot \varnothing = \varnothing.$$

Thus, these rules express the natural algebraic properties of the empty word, ε. However, we do not claim that we get an L-complete system. Indeed, the L-complete extension happens to be quite involved (cf. [12]). In particular, it is still an open problem whether it is recursively enumerable.

We emphasize that our rules Lε, Rε, and Dε are *not* derivable in the multiplicative-additive Lambek calculus, that is, non-commutative multiplicative-additive linear logic (cf. [16, 10]).

The cut rule is not included in the system, so that all our derivations will be cut-free.

Theorem 2. *The derivability problem for $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$ is undecidable. Moreover, any L-sound system which includes $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, i.e., rules of Table 2, is undecidable.*

We prove undecidability by encoding of two-counter Minsky machines (cf. [9]).

In the forward encoding, from computations to derivations, we present explicit derivations in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

For the backwards direction, from derivations to computations, we use a semantic approach by constructing an appropriate L-model for the sequent in question (cf. [13, 14, 18], where phase semantics is used for similar purposes).

Definition 1. *In our encoding, we use the following construction.*
We fix an atomic proposition b, and define 'relative negation' A^b by:

$$A^b = (A \backslash b)$$

Our relative negation can be seen as a non-commutative analogue of the linear logic negation, which is defined by $A^{\perp} = A \multimap \perp$.
As for the relative "double negation," the novelty of our approach is that we are in favour of the "asymmetric"

$$A^{bb} = ((A \backslash b) \backslash b)$$

For the sake of readability of product-free formulas*,*

(a) *Here we will conceive of the formula $((A \cdot B) \backslash C)$ as abbreviation for $(B \backslash (A \backslash C))$. In particular, $(A \cdot B)^b$ is abbreviation for $(B \backslash (A \backslash b))$.*

(b) *Given a sequence of formulas α:*

$$\alpha = \alpha_1, \alpha_2, \dots \alpha_{m-1}, \alpha_m$$

we will conceive of the expression $(\alpha \backslash C)$ as abbreviation for the following product-free formula

$$(\alpha \backslash C) = (\alpha_m \backslash (\alpha_{m-1} \backslash (\dots \backslash (\dots \backslash (\alpha_2 \backslash (\alpha_1 \backslash C))))))$$

In particular,

$$\alpha^b = (\alpha_m \backslash (\alpha_{m-1} \backslash (\dots \backslash (\dots \backslash (\alpha_2 \backslash (\alpha_1 \backslash b))))))$$

Lemma 2. *Given a sequence of formulas α and a sequence of formulas β, let the following sequent be derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, i.e., by the rules from Table 2:*

$$(1 \wedge G),\ \alpha,\ \Delta \vdash\ b \tag{1}$$

Let $g_{\alpha,\beta}$ be defined as:

$$g_{\alpha,\beta} = (\beta \backslash \alpha^{bb}) = (\beta \backslash ((\alpha \backslash b) \backslash b)) \tag{2}$$

Then the sequent

$$(1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b \tag{3}$$

is also (cut-free) derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

Proof. We develop a chain of derivable sequents:

$$\begin{array}{rl}
(1 \wedge G),\ \alpha,\ \Delta \vdash b & \\
\alpha,\ (1 \wedge G),\ \Delta \vdash b & \text{“}\varepsilon \cdot \alpha \Rightarrow \alpha \cdot \varepsilon\text{”} \\
(1 \wedge G),\ \Delta \vdash (\alpha \backslash b) & \\
(1 \wedge G),\ \Delta,\ ((\alpha \backslash b) \backslash b) \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ (\beta \backslash ((\alpha \backslash b) \backslash b)) \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ g_{\alpha,\beta} \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ (1 \wedge g_{\alpha,\beta}) \vdash b & \\
(1 \wedge G),\ (1 \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \text{“}\delta \cdot \varepsilon \Rightarrow \varepsilon \cdot \delta\text{”} \\
(1 \wedge G \wedge g_{\alpha,\beta}),\ (1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \qquad (4) \\
(1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \text{“}\varepsilon \cdot \varepsilon = \varepsilon\text{”}
\end{array}$$

which concludes the proof. ■

Corollary 1 (“Post-ish productions”). *Let $\xi_1,\ \xi_2, \ldots, \xi_n$ be a list of all atomic propositions in question. Let Δ_1 and Δ_2 be sequences made from the above atomic propositions (repetitions are allowed).*
Let G be of the form

$$G\ \equiv\ G' \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i}\ \equiv\ G' \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash \xi_i^{bb}) \tag{5}$$

Then a sequent of the form

$$(1 \wedge G),\ \Delta_1,\ \Delta_2\ \vdash\ b \tag{6}$$

is cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$ *(i.e., by the rules from Table 2) if and only if the following sequent is cut-free derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\ \Delta_2,\ \Delta_1\ \vdash\ b \tag{7}$$

Proof. By induction with the help of $g_{\xi,\xi}$ of the “trivial” form, $g_{\xi,\xi} = (\xi \backslash \xi^{bb})$

3.1 From Computations to Derivations

Definition 2 (Machine encoding). *Here e_1, e_2, p_1, p_2, l_0, l_1, l_2, ... are distinct atomic propositions: e_1 and e_2 serve as "end markers," p_1 and p_2 are used to represent the counters c_1 and c_2, respectively, l_0, l_1, l_2, ... represent "states."*

Taking advantage of the fact that the number of counters is no more than 2, so that one and the same l_i is able of controlling the "left part" and the "right part" simultaneously, we represent a configuration (L_i, k_1, k_2) of our Minsky machine in the state L_i, in which the value of c_1 is k_1, and the value of c_2 is k_2, as the following sequence of atomic propositions:

$$e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \ times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \ times}, e_2 \tag{8}$$

The final configuration $(L_0, 0, 0)$ is represented as

$$e_1, l_0, e_2 \tag{9}$$

Definition 3. *The Minsky instructions are encoded as follows*

(a) *An instruction I of the form: "$L_i\colon inc(c_1);\ goto\ L_j;$" will be encoded in the "reverse" form as the product-free formula (see Definition 1)*

$$A_I = (l_i \backslash (p_1 \cdot l_j)^{bb}) \tag{10}$$

It is worth noting that $A_I = g_{\alpha,\beta}$, where $\alpha = p_1, l_j$, and $\beta = l_i$.

(b) *An instruction I of the form "$L_i\colon inc(c_2);\ goto\ L_j;$" will be encoded in the "reverse" form as:*

$$A_I = (l_i \backslash (l_j \cdot p_2)^{bb}) \tag{11}$$

(c) *An instruction I of the form "$L_i\colon dec(c_1);\ goto\ L_j;$" will be encoded in the "reverse" form as:*

$$A_I = ((p_1 \cdot l_i) \backslash l_j^{bb}) \tag{12}$$

(d) *An instruction I of the form "$L_i\colon dec(c_2);\ goto\ L_j;$" will be encoded in the "reverse" form as:*

$$A_I = ((l_i \cdot p_2) \backslash l_j^{bb}) \tag{13}$$

(e) The most challenging issues *to be addressed to is our encoding of the zero-tests.*

A zero-test with the c_1 counter of the form "$L_i\colon if\ (c_1 = 0)\ goto\ L_j;$" will be encoded by

$$A_I = ((e_1 \cdot l_i) \backslash (e_1 \cdot l_j)^{bb}) \tag{14}$$

(f) *A zero-test with the c_2 counter of the form "$L_i\colon if\ (c_2 = 0)\ goto\ L_j;$" will be encoded by*

$$A_I = ((l_i \cdot e_2) \backslash (l_j \cdot e_2)^{bb}) \tag{15}$$

Lemma 3. *A move by instruction of Case (a) from a configuration with L_i to the configuration with L_j is simulated as follows. Taking $\alpha = p_1, l_j$, and $\beta = l_i$, let G be of the form*

$$G \;\equiv\; G' \wedge A_I \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i} \;\equiv\; G' \wedge g_{\alpha,\beta} \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash \xi_i^{bb}) \tag{16}$$

Let a sequent (representing a Minsky configuration) be cut-free derivable

$$(1 \wedge G),\; e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1+1 \;times}, l_j, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2 \;\vdash\; b \tag{17}$$

Then the following sequent is also cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\; e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2 \;\vdash\; b \tag{18}$$

Proof. According to Corollary 1, the sequent (17) can be transformed into a cut-free derivable sequent of the form

$$(1 \wedge G),\; p_1, l_j,\; \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2, e_1,\; \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;times} \;\vdash\; b \tag{19}$$

By Lemma 2, we get the following

$$(1 \wedge G),\; \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2, e_1,\; \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;times},\; l_i \;\vdash\; b \tag{20}$$

and, applying Corollary 1 once more, we conclude with (18).

Lemma 4. *A move by instruction of Case (e) from a configuration with L_i to the configuration with L_j is simulated as follows. (Here we have to answer to the challenge of the zero-tests.) Taking $\alpha = e_1, l_j$, and $\beta = e_1, l_i$, let G be of the form*

$$G \;\equiv\; G' \wedge A_I \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i} \;\equiv\; G' \wedge g_{\alpha,\beta} \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash \xi_i^{bb}) \tag{21}$$

Let a sequent (representing a Minsky configuration) be cut-free derivable

$$(1 \wedge G),\; e_1, l_j, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2 \;\vdash\; b \tag{22}$$

Then the following sequent is also cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\; e_1, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;times}, e_2 \;\vdash\; b \tag{23}$$

Proof. By Lemma 2, applied to (22), we get the following

$$(1 \wedge G),\ \underbrace{p_2, p_2, \ldots, p_2}_{k_2\ times}, e_2,\ e_1, l_i\ \vdash\ b \tag{24}$$

and, applying Corollary 1, we conclude with (23). ∎

The other cases are considered in a similar fashion.

Corollary 2. *With a configuration* (L_i, k_1, k_2)*, let* M *terminate in* $(L_0, 0, 0)$*. Then the following sequent is cut-free derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*, i.e., by the rules from Table 2:*

$$(1 \wedge G),\ e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1\ times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2\ times}, e_2\ \vdash\ b \tag{25}$$

where G *is of the form:*

$$G = ((e_1 \cdot l_0 \cdot e_2) \backslash b) \wedge \bigwedge_{i=1}^{n} g_{\xi_i, \xi_i} \wedge \bigwedge_{over\ instructions\ I} A_I \tag{26}$$

Proof. By induction on the length of a terminating sequence of configurations. ∎

3.2 From Derivations to Computations

We prove that our encoding is faithful:

Lemma 5. *Let the sequent (25) be derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*. Then, with the configuration* (L_i, k_1, k_2)*,* M *terminates in* $(L_0, 0, 0)$*.*

Proof. By interpretation with the help of L-models.

Each of the atomic propositions a, save b, is interpreted by "itself":

$$w(a) = \{a\} \tag{27}$$

Our specific b is interpreted as

$$w(b) = \{\, xy \mid x \text{ and } y \text{ are words such that } yx \in B_M \,\} \tag{28}$$

where the set of "terminating strings," B_M, is defined as

$$B_M = \{\, e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1\ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2\ times} e_2 \mid \text{from } (L_i, k_1, k_2),\ M \text{ goes to } (L_0, 0, 0) \,\} \tag{29}$$

Lemma 6. $w(1 \wedge G) = \{\varepsilon\}$.

Proof. Assume A_I be of the form (see Definition 3)

$$A_I = (l_i \backslash (p_1 \cdot l_j)^{bb})$$

To show that $\varepsilon \in w(A_I)$, we prove that for any word x, the following holds:

$$p_1 l_j \cdot x \in w(b) \implies x \cdot l_i \in w(b) \tag{30}$$

If $p_1 l_j \cdot x \in w(b)$ then the word x is of the form

$$x = \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} \tag{31}$$

with M going from $(L_j, k_1 + 1, k_2)$ to $(L_0, 0, 0)$. Then, by applying this instruction I, with (L_i, k_1, k_2), M terminates in $(L_0, 0, 0)$. Hence

$$e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 \in B_M$$

which results in the desired $x \cdot l_i \in w(b)$.

The other cases should be considered in a similar fashion. ■

If the sequent (25) is derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, then

$$w(1 \wedge G) \cdot e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 \in w(b)$$

and, hence, with the configuration (L_i, k_1, k_2), M terminates in $(L_0, 0, 0)$.

Now Theorem 2 follows from Corollary 1 and Lemma 5.

4 Concluding Remarks

In the present paper we have proved two main results.

First, the Lambek calculus extended with additive disjunction is not complete w.r.t. L-models and R-models.

Second, any extension of the Lambek calculus with one implication, conjunction, and the multiplicative unit turns out to be undecidable, if we enrich this calculus with the natural rules, representing the basic properties of the empty word, ε, in L-models.

Namely, the "commuting" rules Lε and Rε are caused by that, for any word x and set X, $\varepsilon \cdot x = x \cdot \varepsilon$, $\varnothing \cdot X = X \cdot \varnothing$, whereas the "doubling" rule Dε is caused by $\varepsilon \cdot \varepsilon = \varepsilon$, $\varnothing \cdot \varnothing = \varnothing$.

There are several questions left open. One open question is, whether the Lambek calculus with product and both implications enriched with additive conjunction is L-complete. Another open question is whether there is a recursively enumerable extension of the Lambek calculus with the unit, which is L-complete; the same question for R-completeness. Notice that some of our rules motivated by the L-sound behaviour of ε are not valid in R-models, where the unit is interpreted as the diagonal relation. More precisely, the "doubling" rule is valid in R-models, while the "commuting" rule is not.

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Appendix

In this Appendix, we give a complete proof of the fact that the sequent from Theorem 1, namely

$$(((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x)) \cdot ((x\,/\,y) \vee x) \cdot (((x\,/\,y) \vee x)\,\backslash((x\,/\,z) \vee x)) \\ \vdash (x\,/(y \vee z)) \vee x,$$

is not derivable in $\mathbf{L}\vee$.

Our argument is based on brute-force proof search; however, some of the sequents proven to be non-derivable get marked (numbered) and then referred to, if they appear in the proof search again. This makes our proof a bit shorter.

Due to cut elimination, we seek only for a cut-free proof. We start with the well-known fact that rules $\cdot$L, $/\,$R, $\backslash\,$R, and $\vee$L are invertible: derivability of their conclusion yields derivability of their premise(s). Thus, in our proof search, if such a rule is applicable, we can always suppose that it was applied immediately, as the last (lowermost) rule in the derivation. Moreover, $\vee$L has two premises, and they should be *both* derivable if so is the goal sequent. Hence, when trying to prove non-derivability of the goal sequent we can *choose* one of the premises of $\vee$L and prove that it is not derivable.

Now, we are ready to prove that the sequent of Theorem 1 is not derivable in **L**$\vee$. First, by invertibility of $\cdot$L we replace all $\cdot$'s by commas in the left-hand side of the sequent:

$$((x \,/\, y) \vee x) \,/((x \,/\, y) \vee (x \,/\, z) \vee x), (x \,/\, y) \vee x, ((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x) \\ \vdash (x \,/(y \vee z)) \vee x$$

Next, we apply invertibility of $\vee$L to $(x \,/\, y) \vee x$. The sequent should be derivable with both $x \,/\, y$ and x in this place; we choose $x \,/\, y$:

$$((x \,/\, y) \vee x) \,/((x \,/\, y) \vee (x \,/\, z) \vee x), x \,/\, y, ((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x) \\ \vdash (x \,/(y \vee z)) \vee x$$

The lowermost rule in the definition introduces one of the main connectives. There are now 4 of them:

$$((x \,/\, y) \vee x) \overset{1}{/} ((x \,/\, y) \vee (x \,/\, z) \vee x), x \overset{2}{/} y, ((x \,/\, y) \vee x) \overset{3}{\setminus} ((x \,/\, z) \vee x) \\ \vdash (x \,/(y \vee z)) \overset{4}{\vee} x$$

Now we consider all possible cases. The enumeration of cases is as follows: for $/$ and $\setminus$ connectives, the case number is of the form n–m, where n is the number of the connective (as shown above) and m is the number of formulae that are sent to Φ by the $/\,$L or $\setminus\,$L rule.[6] For the 4th connective, $\vee$, we have cases 4a and 4b, for choosing $x \,/(y \vee z)$ or x, respectively.

Case 1–1. In this case we have $x \,/\, y \vdash (x \,/\, y) \vee (x \,/\, z) \vee x$ (fine) and

$$(x \,/\, y) \vee x, ((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x) \vdash (x \,/(y \vee z)) \vee x.$$

Invert $\vee$L and choose $x \,/\, y$. Now we have 3 options:

$$x \overset{1}{/} y, ((x \,/\, y) \vee x) \overset{2}{\setminus} ((x \,/\, z) \vee x) \vdash (x \,/(y \vee z)) \overset{3}{\vee} x.$$

Notice that here Φ in $/\,$L or $\setminus\,$L is determined in a unique way.

Subcase 1. We get

$$((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x) \vdash y \tag{32}$$

as the left premise. This sequent is not derivable ($\setminus$ cannot be decomposed, since there is nothing to the left of the formula).

Subcase 2. Here we have $x \,/\, y \vdash (x \,/\, y) \vee x$ (fine) as the left premise and

$$(x \,/\, z) \vee x \vdash (x \,/(y \vee z)) \vee x \tag{33}$$

as the right one. We show that (33) is not derivable. Inverting $\vee$L and choosing $x \,/\, z$ yields $x \,/\, z \vdash (x \,/(y \vee z)) \vee x$, and now either $x \,/\, z \vdash x \,/(y \vee z)$ or $x \,/\, z \vdash x$

[6] Due to Lambek's restriction, Φ should be non-empty, *i.e.*, $m > 0$.

should be derivable. The latter is trivially not. For the former, inverting $/\,$R and $\vee$L, choosing y, gives $x\,/\,z, y \vdash x$, which is also not derivable.

Subcase 3a. Here we get

$$x\,/\,y, ((x\,/\,y) \vee x) \setminus ((x\,/\,z) \vee x) \vdash x\,/(y \vee z).$$

Inverting $/\,$R and $\vee$L, choosing y, gives

$$x/y, ((x\,/\,y) \vee x)\setminus((x\,/\,z) \vee x), y \vdash x.$$

Decomposing $/$ with $\Phi = ((x\,/\,y) \vee x) \setminus ((x\,/\,z) \vee x)$ gives the left premise $((x\,/\,y) \vee x) \setminus ((x\,/\,z) \vee x) \vdash y$, which is already shown to be non-derivable (32). Decomposing $/$ with $\Phi = ((x\,/\,y) \vee x) \setminus ((x\,/\,z) \vee x), y$ gives a non-derivable right premise $x\,/\,y \vdash x$. Finally, decomposing $\setminus$ gives $(x\,/\,z) \vee x, y \vdash x$, which is also not derivable: inverting $\vee$L and choosing x gives $x, y \vdash x$.

Subcase 3b. In this case we have

$$x/y, ((x\,/\,y) \vee x)\setminus((x\,/\,z) \vee x) \vdash x.$$

Decomposing $/$ yields (32), which is not derivable. Decomposing $\setminus$ yields $((x\,/\,z) \vee x \vdash x$, which is shown to be non-derivable by inverting $\vee$L and choosing $x\,/\,z$.

Case 1–2. The right premise now is

$$(x\,/\,y) \vee x \vdash (x\,/(y \vee z)) \vee x, \tag{34}$$

which is shown to be non-derivable exactly as (33).

Case 2–1. The left premise here is (32), which is not derivable.

Case 3–1. Here the left premise is fine, and the right one is

$$((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x), (x\,/\,z) \vee x \vdash (x\,/(y \vee z)) \vee x.$$

Invert $\vee$L and choose $x\,/\,z$:

$$((x\,/\,y) \vee x)/((x\,/\,y) \vee (x\,/\,z) \vee x), x/z \vdash (x\,/(y \vee z)) \vee x.$$

Decomposing the left $/$ yields (34), which is not derivable, as the right premise. Decomposing the right $/$ is impossible, since Φ should be non-empty. Finally, decomposing $\vee$ on the right yields two subcases.

Subcase a.

$$((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x), x\,/\,z \vdash x\,/(y \vee z).$$

Inverting $/\,$R and $\vee$L, choosing y, yields

$$((x\,/\,y) \vee x)/((x\,/\,y) \vee (x\,/\,z) \vee x), x/z, y \vdash x. \tag{35}$$

Decomposing the right $/$ would yield $y \vdash z$, which is not derivable. So the only option is decomposing the left $/$. This gives two possible situations, depending on how many formulae go to Φ. If Φ takes one formula, then the right premise of $/\,\mathrm{L}$ is

$$(x\,/\,y) \vee x, y \vdash x.$$

The choice of x in inverting $\vee\mathrm{L}$ gives $x, y \vdash x$, which is not derivable. If Φ takes two formulae, then we have the right premise of the form

$$(x\,/\,y) \vee x \vdash x,$$

which is also not derivable, now by choosing $x\,/\,y$.

Subcase b.

$$((x\,/\,y) \vee x) \boldsymbol{/} ((x\,/\,y) \vee (x\,/\,z) \vee x), x\,/\,z \vdash x.$$

Now we can only decompose the left $/$, which yields

$$(x\,/\,y) \vee x, x\,/\,z \vdash x$$

as the right premise. Both choices in inverting $\vee\mathrm{L}$ fail: neither $x\,/\,y, x\,/\,z \vdash x$, nor $x, x\,/\,z \vdash x$ is derivable.

Case 3–2. Here the right premise is (33), which is not derivable.

Case 4a. Here we again invert $/\,\mathrm{R}$ and $\vee\mathrm{L}$, choosing y:

$$((x\,/\,y) \vee x) \overset{1}{/} ((x\,/\,y) \vee (x\,/\,z) \vee x), x \overset{2}{/} y, ((x\,/\,y) \vee x) \overset{3}{\backslash} ((x\,/\,z) \vee x), y \vdash x$$

Again, as in the top-level analysis, we consider several cases.

Subcase 1–1. The right premise is of the form

$$(x\,/\,y) \vee x, x\,/\,y, ((x\,/\,y) \vee x) \,\backslash ((x\,/\,z) \vee x), y \vdash x$$

Invert $\vee\mathrm{L}$ choosing x:

$$x, x \overset{1}{/} y, ((x\,/\,y) \vee x) \overset{2}{\backslash} ((x\,/\,z) \vee x), y \vdash x$$

Here decomposition 1–1 yields non-derivable (32); 1–2 yields

$$((x\,/\,y) \vee x) \,\backslash ((x\,/\,z) \vee x), y \vdash y, \tag{36}$$

which is also not derivable (no decomposition possible). Decomposition 2–1 gives right premise

$$x, (x\,/\,z) \vee x, y \vdash x,$$

and choosing x in the inversion of $\vee\mathrm{L}$ gives non-derivable $x, x, y \vdash x$. Finally, decomposition 2–2 gives

$$(x\,/\,z) \vee x, y \vdash x \tag{37}$$

as the right premise, and choosing x also makes it non-derivable: $x, y \vdash x$.

Subcase 1–2. This gives a non-derivable right premise (37).

Subcase 1–3. Here the right premise is $(x \,/\, y) \vee x \vdash x$, which is invalidated by choosing $x \,/\, y$ in the inversion of $\vee$L.

Subcases 2–1 and 2–2 give non-derivable left premises: (32) and (36) respectively.

Subcase 3–1. Here we get

$$((x \,/\, y) \vee x) \,/((x \,/\, y) \vee (x \,/\, z) \vee x), (x \,/\, z) \vee x, y \vdash x.$$

Inverting $\vee$L, choosing $x \,/\, z$, yields (35), which is not derivable.

Subcase 3–2. Here the right premise is

$$(x \,/\, z) \vee x, y \vdash x,$$

which is not derivable (37).

Case 4b. In this case we have

$$((x \,/\, y) \vee x) \overset{1}{/} ((x \,/\, y) \vee (x \,/\, z) \vee x), x \overset{2}{/} y, ((x \,/\, y) \vee x) \overset{3}{\backslash} ((x \,/\, z) \vee x) \vdash x.$$

Here we return to the beginning of the proof and consider the same cases 1–1, 1–2, 2–1, 3–1, and 3–2. Each of these cases decomposes / or \, with the same left premise. The right premises are here are of the form $\Gamma \vdash x$. We suppose that such a sequent is derivable. Then, by application of $\vee$L, we get $\Gamma \vdash (x \,/(y \vee z)) \vee x$. Now we are *exactly* in the situation of one of the cases from 1–1 to 3–2, and can use the argumentation above "as is."

This finishes our case analysis and thus the proof of Theorem 1.