# L-completeness of the Lambek Calculus with the Reversal Operation Allowing Empty Antecedents

Stepan Kuznetsov

Moscow State University

**Abstract**

In this paper we prove that the Lambek calculus allowing empty antecedents and enriched with a unary connective corresponding to language reversal is complete with respect to the class of models on subsets of free monoids (L-models).

## 1 The Lambek Calculus with the Reversal Operation

We consider the calculus L, introduced in [4]. The set $\mathrm{Pr} = \{p_1, p_2, p_3, \dots\}$ is called the set of *primitive types*. *Types* of L are built from primitive types using three binary connectives: $\backslash$ *(left division)*, $/$ *(right division)*, and $\cdot$ *(multiplication)*; we shall denote the set of all types by Tp. Capital letters $(A, B, \dots)$ range over types. Capital Greek letters (except $\Sigma$) range over finite (possibly empty) sequences of types; $\Lambda$ stands for the empty sequence. Expressions of the form $\Gamma \to C$, where $\Gamma \neq \Lambda$, are called *sequents* of L.

Axioms: $A \to A$.

Rules:

$$\frac{A\Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma \Pi (A \setminus B) \Delta \to C}\ (\setminus \to)$$

$$\frac{\Pi A \to B}{\Pi \to B \,/\, A}\ (\to /),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma (B \,/\, A) \Pi \Delta \to C}\ (/ \to)$$

$$\frac{\Pi \to A \quad \Delta \to B}{\Pi \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma A B \Delta \to C}{\Gamma (A \cdot B) \Delta \to C}\ (\cdot \to)$$

Published in *Categories and types in logic, language, and physics: Essays dedicated to Jim Lambek on the occasion of his 90th birthday,* LNCS vol. 8222, 2014, pp. 268–278.
The final publication is available at link.springer.com:
http://link.springer.com/chapter/10.1007/978-3-642-54789-8_15

$$\frac{\Pi \to A \quad \Gamma A \Delta \to C}{\Gamma \Pi \Delta \to C} \text{ (cut)}$$

The (cut) rule is eliminable [4].

We also consider an extra unary connective $^{\mathrm{R}}$ (written in the postfix form, $A^{\mathrm{R}}$). The extended set of types is denoted by $\mathrm{Tp}^{\mathrm{R}}$. For a sequence of types $\Gamma = A_1 A_2 \dots A_n$ let $\Gamma^{\mathrm{R}} \leftrightharpoons A_n^{\mathrm{R}} \dots A_2^{\mathrm{R}} A_1^{\mathrm{R}}$ ("$\leftrightharpoons$" here and further means "equal by definition").

The calculus $\mathrm{L}^{\mathrm{R}}$ is obtained from L by adding three rules for $^{\mathrm{R}}$:

$$\frac{\Gamma \to C}{\Gamma^{\mathrm{R}} \to C^{\mathrm{R}}} \ (^{\mathrm{R}} \to {}^{\mathrm{R}}) \qquad \frac{\Gamma A^{\mathrm{RR}} \Delta \to C}{\Gamma A \Delta \to C} \ (^{\mathrm{RR}} \to)_{\mathrm{E}} \qquad \frac{\Gamma \to C^{\mathrm{RR}}}{\Gamma \to C} \ (\to {}^{\mathrm{RR}})_{\mathrm{E}}$$

Dropping the $\Pi \neq \Lambda$ restriction on the $(\to \backslash)$ and $(\to /)$ rules of L leads to *the Lambek calculus allowing empty antecedents* called $\mathrm{L}^*$. The calculus $\mathrm{L}^{*\mathrm{R}}$ is obtained from $\mathrm{L}^*$ by changing the type set from Tp to $\mathrm{Tp}^{\mathrm{R}}$ and adding the $(^{\mathrm{R}} \to {}^{\mathrm{R}})$, $(^{\mathrm{RR}} \to)_{\mathrm{E}}$, and $(\to {}^{\mathrm{RR}})_{\mathrm{E}}$ rules.

Unfortunately, there is no subformula property known for $\mathrm{L}^{\mathrm{R}}$ and $\mathrm{L}^{*\mathrm{R}}$. Nevertheless, $\mathrm{L}^{\mathrm{R}}$ is a conservative extension of L, and $\mathrm{L}^{*\mathrm{R}}$ is a conservative extension of $\mathrm{L}^*$:

**Lemma 1.** *A sequent formed of types from* Tp *is provable in* $\mathrm{L}^{\mathrm{R}}$ *(*$\mathrm{L}^{*\mathrm{R}}$*) if and only if it is provable in* L *(resp.,* $\mathrm{L}^*$*).*

This lemma will be proved later via a semantic argument.

## 2 Normal Form for Types

The $^{\mathrm{R}}$ connective in the Lambek calculus and linear logic was first considered in [5] (there it is denoted by $\breve{\ }$). In [5], this connective is axiomatised using Hilbert-style axioms:

$$A^{\mathrm{RR}} \leftrightarrow A \qquad \text{and} \qquad (A \cdot B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \cdot A^{\mathrm{R}}.$$

Here $F \leftrightarrow G$ ("$F$ is *equivalent* to $G$") is a shortcut for two sequents: $F \to G$ and $G \to F$. The relation $\leftrightarrow$ is reflexive, symmetric, and transitive (due to the rule (cut)). Using (cut) one can prove that if $\mathrm{L}^{\mathrm{R}} \vdash F_1 \to G_1$, $F_1 \leftrightarrow F_2$, and $G_1 \leftrightarrow G_2$, then $\mathrm{L}^{\mathrm{R}} \vdash F_2 \to G_2$. Also, $\leftrightarrow$ is a *congruence relation,* in the following sense: if $A_1 \leftrightarrow A_2$ and $B_1 \leftrightarrow B_2$, then $A_1 \cdot B_1 \leftrightarrow A_2 \cdot B_2$, $A_1 \backslash B_1 \leftrightarrow A_2 \backslash B_2$, $B_1 \,/\, A_1 \leftrightarrow B_2 \,/\, A_2$, $A_1^{\mathrm{R}} \leftrightarrow A_2^{\mathrm{R}}$.

These axioms are provable in $\mathrm{L}^{\mathrm{R}}$ and, vice versa, adding them to L yields a calculus equivalent to $\mathrm{L}^{\mathrm{R}}$. The same is true for $\mathrm{L}^{*\mathrm{R}}$ and $\mathrm{L}^*$ respectively.

Furthermore, the following two equivalences hold in $\mathrm{L}^{\mathrm{R}}$ and $\mathrm{L}^{*\mathrm{R}}$:

$$(A \backslash B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \,/\, A^{\mathrm{R}} \qquad \text{and} \qquad (B \,/\, A)^{\mathrm{R}} \leftrightarrow A^{\mathrm{R}} \backslash B^{\mathrm{R}}.$$

Using the four equivalences above one can prove by induction that any type $A \in \mathrm{Tp}^{\mathrm{R}}$ is equivalent to its *normal form* $tr(A)$, defined as follows:

1. $tr(p_i) \leftrightharpoons p_i$;
2. $tr(p_i^{\mathrm{R}}) \leftrightharpoons p_i^{\mathrm{R}}$;
3. $tr(A \cdot B) \leftrightharpoons tr(A) \cdot tr(B)$;
4. $tr(A \setminus B) \leftrightharpoons tr(A) \setminus tr(B)$;
5. $tr(B \,/\, A) \leftrightharpoons tr(B) \,/\, tr(A)$;
6. $tr((A \cdot B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \cdot tr(A^{\mathrm{R}})$;
7. $tr((A \setminus B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \,/\, tr(A^{\mathrm{R}})$;
8. $tr((B \,/\, A)^{\mathrm{R}}) \leftrightharpoons tr(A^{\mathrm{R}}) \setminus tr(B^{\mathrm{R}})$;
9. $tr(A^{\mathrm{RR}}) \leftrightharpoons tr(A)$.

In the normal form, the $^{\mathrm{R}}$ connective can appear only on occurrences of primitive types. Obviously, $tr(tr(A)) = tr(A)$ for every type $A$.

We also consider variants of L and $\mathrm{L}^*$ with $\mathrm{Tp} \cup \{p^{\mathrm{R}} \mid p \in \mathrm{Tp}\}$ instead of Tp as the set of primitive types. These calculi will be called $\mathrm{L}'$ and $\mathrm{L}^{*\prime}$ respectively. Obviously, if a sequent is provable in $\mathrm{L}'$, then all its types are in normal form and this sequent is provable in $\mathrm{L}^{\mathrm{R}}$ (and the same for $\mathrm{L}^{*\prime}$ and $\mathrm{L}^{*\mathrm{R}}$). Later we shall prove the converse statement:

**Lemma 2.** *A sequent $F_1 \dots F_n \to G$ is provable in $\mathrm{L}^{\mathrm{R}}$ (resp., $\mathrm{L}^{*\mathrm{R}}$) if and only if the sequent $tr(F_1) \dots tr(F_n) \to tr(G)$ is provable in $\mathrm{L}'$ (resp., $\mathrm{L}^{*\prime}$).*

## 3 L-models

Now let $\Sigma$ be an alphabet (an arbitrary nonempty set, finite or countable). By $\Sigma^+$ we denote the set of all nonempty words over $\Sigma$; the set of all words over $\Sigma$, including the empty word, is denoted by $\Sigma^*$. The set $\Sigma^*$ with the operation of word concatenation is the *free monoid* generated by $\Sigma$; the empty word $\epsilon$ is the unit of this monoid. Subsets of $\Sigma^*$ are called *languages* over $\Sigma$. The set $\Sigma^+$ with the same operation is the *free semigroup* generated by $\Sigma$. Its subsets are *languages without the empty word*.

The set $\mathcal{P}(\Sigma^*)$ of all languages is also a monoid: if $M, N \subseteq \Sigma^*$, then let $M \cdot N$ be $\{uv \mid u \in M, v \in N\}$; the singleton $\{\epsilon\}$ is the unit. Likewise, the set $\mathcal{P}(\Sigma^+)$ is a semigroup with the same multiplication operation.

On these two structures one can also define two *division* operations: $M \setminus N \leftrightharpoons \{u \in \Sigma^* \mid (\forall v \in M)\, vu \in N\}$, $N \,/\, M \leftrightharpoons \{u \in \Sigma^* \mid (\forall v \in M)\, uv \in N\}$ for $\mathcal{P}(\Sigma^*)$, and $M \setminus N \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, vu \in N\}$, $N \,/\, M \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, uv \in N\}$ for $\mathcal{P}(\Sigma^+)$. Note that, unlike multiplication, the $\mathcal{P}(\Sigma^*)$ version of division operations does not coincide with the $\mathcal{P}(\Sigma^+)$ one even for languages without the empty word. For example, if $M = N = \{a\}$ $(a \in \Sigma)$, then $M \setminus N$ is $\{\epsilon\}$ in $\mathcal{P}(\Sigma^*)$ and empty in $\mathcal{P}(\Sigma^+)$.

These three operations on languages naturally correspond to three connectives of the Lambek calculus, thus giving an interpretation for Lambek types and sequents. An *L-model* is a pair $\mathcal{M} = \langle \Sigma, w \rangle$, where $\Sigma$ is an alphabet and $w$ is a function that maps Lambek calculus types to languages over $\Sigma$, such that $w(A \cdot B) = w(A) \cdot w(B)$, $w(A \setminus B) = w(A) \setminus w(B)$, and $w(B \,/\, A) = w(B) \,/\, w(A)$ for all $A, B \in \mathrm{Tp}$. One can consider models either with or without the empty word, depending on what set of languages ($\mathcal{P}(\Sigma^*)$ or $\mathcal{P}(\Sigma^+)$), and, more importantly, what version of the division operations is used. Models with and without the empty word are similar but different (in particular, models with the empty word are not a generalisation of models without it). Obviously, $w$ can be defined on primitive types in an arbitrary way, and then it is uniquely propagated to all types.

A sequent $F_1 \dots F_n \to G$ is considered *true* in a model $\mathcal{M}$ ($\mathcal{M} \vDash F_1 \dots F_n \to G$) if $w(F_1) \cdot \ldots \cdot w(F_n) \subseteq w(G)$. If the sequent has an empty antecedent ($n = 0$), i. e., is of the form $\to G$, then it is considered true if $\epsilon \in w(G)$. This implies that such sequents are never true in L-models without the empty word. L-models give sound and complete semantics for L and $\mathrm{L}^*$, due to the following theorem:

**Theorem 1.** *A sequent is provable in* L *if and only if it is true in all L-models without the empty word. A sequent is provable in* $\mathrm{L}^*$ *if and only if it is true in all L-models with the empty word.*

This theorem is proved in [8] for L and in [9] for $\mathrm{L}^*$; its special case for the product-free fragment (where we keep only types without multiplication) is much easier and appears in [1].

Note that for L and L-models without the empty word it is sufficient to consider only sequents with one type in the antecedent, since $\mathrm{L} \vdash F_1 F_2 \dots F_n \to G$ if and only if $\mathrm{L} \vdash F_1 \cdot F_2 \cdot \ldots \cdot F_n \to G$. For $\mathrm{L}^*$ and L-models with the empty word it is sufficient to consider only sequents with empty antecedent, since $\mathrm{L}^* \vdash F_1 \dots F_{n-1} F_n \to G$ if and only if $\mathrm{L}^* \vdash \;\to F_n \setminus (F_{n-1} \setminus \dots \setminus (F_1 \setminus G) \dots))$.

# 4 L-models with the Reversal Operation

The new $^{\mathrm{R}}$ connective corresponds to the *language reversal* operation. For $u = a_1 a_2 \dots a_n$ ($a_1, \dots, a_n \in \Sigma$, $n \geq 1$) let $u^{\mathrm{R}} \leftrightharpoons a_n \dots a_2 a_1$; $\epsilon^{\mathrm{R}} \leftrightharpoons \epsilon$. For a language $M$ let $M^{\mathrm{R}} \leftrightharpoons \{u^{\mathrm{R}} \mid u \in M\}$. The notion of L-model is easily modified to deal with the new connective by adding additional constraints on $w$: $w(A^{\mathrm{R}}) = w(A)^{\mathrm{R}}$ for every type $A$.

One can easily show that the calculi $\mathrm{L}^{\mathrm{R}}$ and $\mathrm{L}^{*\mathrm{R}}$ are sound with respect to L-models with the reversal operation (without and with the empty word respectively). Now, using this soundness statement and Pentus' completeness theorem (Theorem 1), we can prove Lemma 1 (conservativity of $\mathrm{L}^{\mathrm{R}}$ over L and $\mathrm{L}^{*\mathrm{R}}$ over $\mathrm{L}^*$): if a sequent is provable in $\mathrm{L}^{\mathrm{R}}$ (resp., $\mathrm{L}^{*\mathrm{R}}$) and does not contain the $^{\mathrm{R}}$ connective, then it is true in all L-models without the empty word (resp., with the empty word). Moreover, in these L-models the language reversal operation is never used. Therefore, the sequent involved is provable in L (resp., $L^*$) due to the completeness theorem.

The completeness theorem for $L^R$ is proved in [3] (the product-free case is again easy and is handled in [6] using Buszkowski's argument [1]):

**Theorem 2.** *A sequent is provable in* $L^R$ *if and only if it is true in all L-models with the reversal operation and without the empty word.*

In this paper we present a proof for the $L^{*R}$ version of this theorem:

**Theorem 3.** *A sequent is provable in* $L^{*R}$ *if and only if it is true in all L-models with the reversal operation and with the empty word.*

The proof basically duplicates the proof of Theorem 2 from [3]; changes are made to handle the empty word cases.

The main idea is as follows: if a sequent in normal form is not provable in $L^{*R}$, then it is not provable in $L^{*\prime}$. Therefore, by Theorem 1, there exists a model in which this sequent is not true, but this model does not necessarily satisfy all of the conditions $w(A^R) = w(A)^R$. We want to modify our model by adding $w(A^R)^R$ to $w(A)$. For $L^R$ [3], we can first make the sets $w(A^R)^R$ and $w(A)$ disjoint by replacing every letter $a \in \Sigma$ by a long word $a^{(1)} \dots a^{(N)}$ ($a^{(i)}$ are symbols from a new alphabet); then the new interpretation for $A$ is going to be $w(A) \cup w(A^R)^R \cup T$ with an appropriate "trash heap" set $T$. For $L^{*R}$, we cannot do this directly, because $\epsilon$ will still remain the same word after the substitution of long words for letters. Fortunately, the model given by Theorem 1 enjoys a sort of weak universal property: if a type $A$ is a subtype of our sequent, then $\epsilon \in w(A)$ if *and only if* $L^{*\prime} \vdash \to A$. Hence, if $\epsilon \in w(A)$, then $\epsilon \in w(A^R)$, and vice versa, so the empty word does not do any harm here.

Note that essentially here we need only the fact that our sequent is not derivable in $L^{*\prime}$, but not $L^{*R}$, and from this assumption we prove the existence of a model falsifying it. Hence, the sequent is not provable in $L^{*R}$. Therefore, we have proved Lemma 2.

# 5 L-completeness of $L^{*R}$ (Proof)

Let $L^{*R} \nvdash \to G$ (as mentioned earlier, it is sufficient to consider sequents with empty antecedent). Also let $G$ be in normal form (otherwise replace it by $tr(G)$).

Since $L^{*R} \nvdash \to G$, $L^{*\prime} \nvdash \to G$. The calculus $L^{*\prime}$ is essentially the same as $L^*$, therefore Theorem 1 gives us a structure $\mathcal{M} = \langle \Sigma, w \rangle$ such that $\epsilon \notin w(G)$. The structure $\mathcal{M}$ indeed falsifies $\to G$, but it is not a model in the sense of our new language: some of the conditions $w(p_i^R) = w(p_i)^R$ might be not satisfied.

Let $\Phi$ be the set of all subtypes of $G$ (including $G$ itself; the notion of subtype is understood in the sense of $L^R$).

The construction of $\mathcal{M}$ (see [9]) guarantees that the following two statements hold for every $A \in \Phi$:

1. $w(A) \neq \varnothing$;

2. $\epsilon \in w(A) \iff \mathrm{L}^{*\prime} \vdash \to A$.

We introduce an inductively defined counter $f(A)$, $A \in \Phi$: $f(p_i) \leftrightharpoons 1$, $f(p_i^{\mathrm{R}}) \leftrightharpoons 1$, $f(A \cdot B) \leftrightharpoons f(A) + f(B) + 10$, $f(A \setminus B) \leftrightharpoons f(B)$, $f(B \,/\, A) \leftrightharpoons f(B)$. Let $K \leftrightharpoons \max\{f(A) \mid A \in \Phi\}$, $N \leftrightharpoons 2K + 25$ ($N$ should be odd, greater than $K$, and big enough itself).

Let $\Sigma_1 \leftrightharpoons \Sigma \times \{1, \dots, N\}$. We shall denote the pair $\langle a, j \rangle \in \Sigma_1$ by $a^{(j)}$. Elements of $\Sigma$ and $\Sigma_1$ will be called *letters* and *symbols* respectively. A symbol can be *even* or *odd* depending on the parity of the superscript. Consider a homomorphism $h \colon \Sigma^* \to \Sigma_1^*$, defined as follows: $h(a) \leftrightharpoons a^{(1)}a^{(2)} \dots a^{(N)}$ ($a \in \Sigma$), $h(a_1 \dots a_n) \leftrightharpoons h(a_1) \dots h(a_n)$, $h(\epsilon) = \epsilon$. Let $P \leftrightharpoons h(\Sigma^+)$. Note that $h$ is a bijection between $\Sigma^*$ and $P \cup \{\epsilon\}$ and between $\Sigma^+$ and $P$.

**Lemma 3.** *For all $M, N \subseteq \Sigma^*$ we have*

*1. $h(M \cdot N) = h(M) \cdot h(N)$;*

*2. if $M \neq \varnothing$, then $h(M \setminus N) = h(M) \setminus h(N)$ and $h(N \,/\, M) = h(N) \,/\, h(M)$.*

*Proof.*

1. By the definition of a homomorphism.

2. $\boxed{\subseteq}$ Let $u \in h(M \setminus N)$. Then $u = h(u')$ for some $u' \in M \setminus N$. For all $v' \in M$ we have $v'u' \in N$. Take an arbitrary $v \in h(M)$, $v = h(v')$ for some $v' \in M$. Since $u' \in M \setminus N$, $v'u' \in N$, whence $vu = h(v')h(u') = h(v'u') \in h(N)$. Therefore $u \in h(M) \setminus h(N)$.

   $\boxed{\supseteq}$ Let $u \in h(M) \setminus h(N)$. First we claim that $u \in P \cup \{\epsilon\}$. Suppose the contrary: $u \notin P \cup \{\epsilon\}$. Take $v' \in M$ ($M$ is nonempty by assumption). Since $v = h(v') \in P \cup \{\epsilon\}$, $vu \notin P \cup \{\epsilon\}$. On the other hand, $vu \in h(N) \subseteq P \cup \{\epsilon\}$. Contradiction. Now, since $u \in P \cup \{\epsilon\}$, $u = h(u')$ for some $u' \in \Sigma^*$. For an arbitrary $v' \in M$ and $v \leftrightharpoons h(v')$ we have $h(v'u') = vu \in h(N)$, whence $v'u' \in N$, whence $u' \in M \setminus N$. Therefore, $u = h(u') \in h(M \setminus N)$.

   The / case is handled symmetrically.

□

We construct a new model $\mathcal{M}_1 = \langle \Sigma_1, w_1 \rangle$, where $w_1(z) \leftrightharpoons h(w(z))$ ($z \in \mathrm{Pr}'$). Due to Lemma 3, $w_1(A) = h(w(A))$ for all $A \in \Phi$, and, since $\epsilon \notin w(G)$, we obtain $\epsilon \notin w_1(G) = h(w(G))$. ($\mathcal{M}_1$ is also a countermodel in the language without $^{\mathrm{R}}$). Note that $w_1(A) \subseteq P \cup \{\epsilon\}$ for any type $A$; moreover, if $A \in \Phi$, then $\epsilon \in w_1(A)$ if and only if $\mathrm{L}^{*\prime} \vdash \to A$.

Now we introduce several auxiliary subsets of $\Sigma_1^+$ (by $\mathrm{Subw}(M)$ we denote the set of all nonempty subwords of words from $M$, i.e. $\mathrm{Subw}(M) \leftrightharpoons \{u \in \Sigma_1^+ \mid (\exists v_1, v_2 \in \Sigma_1^*)\, v_1 u v_2 \in M\}$):
$T_1 \leftrightharpoons \{u \in \Sigma_1^+ \mid u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})\}$;
$T_2 \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) \mid \text{the first or the last symbol of } u \text{ is even}\}$;

$E \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) - (P \cup P^{\mathrm{R}}) \mid$ both the first symbol and the last symbol of $u$ are odd$\}$.

The sets $P$, $P^{\mathrm{R}}$, $T_1$, $T_2$, and $E$ form a partition of $\Sigma_1^+$ into nonintersecting parts. The set $\Sigma_1^*$ is now split into six disjoint subsets: $P$, $P^{\mathrm{R}}$, $T_1$, $T_2$, $E$, and $\{\epsilon\}$. For example, $a^{(1)}b^{(10)}a^{(2)} \in T_1$, $a^{(N)}b^{(1)} \dots b^{(N-1)} \in T_2$, $a^{(7)}a^{(6)}a^{(5)} \in E$ $(a, b \in \Sigma)$. Let $T \leftrightharpoons T_1 \cup T_2$, $T_i(k) \leftrightharpoons \{u \in T_i \mid |u| \geq k\}$ $(i = 1, 2$, $|u|$ is the length of $u)$, $T(k) \leftrightharpoons T_1(k) \cup T_2(k) = \{u \in T \mid |u| \geq k\}$. Note that if the first or the last symbol of $u$ is even, then $u \in T$, no matter whether it belongs to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$. The index $k$ (possibly with subscripts) here and further ranges from 1 to $K$. For all $k$ we have $T(k) \supseteq T(K)$.

**Lemma 4.**

1. *$P \cdot P \subseteq P$, $P^{\mathrm{R}} \cdot P^{\mathrm{R}} \subseteq P^{\mathrm{R}}$;*
2. *$T^{\mathrm{R}} = T$, $T(k)^{\mathrm{R}} = T(k)$;*
3. *$P \cdot P^{\mathrm{R}} \subseteq T(K)$, $P^{\mathrm{R}} \cdot P \subseteq T(K)$;*
4. *$P \cdot T \subseteq T(K)$, $T \cdot P \subseteq T(K)$;*
5. *$P^{\mathrm{R}} \cdot T \subseteq T(K)$, $T \cdot P^{\mathrm{R}} \subseteq T(K)$;*
6. *$T \cdot T \subseteq T$.*

*Proof.*

1. Obvious.
2. Directly follows from our definitions.
3. Any element of $P \cdot P^{\mathrm{R}}$ or $P^{\mathrm{R}} \cdot P$ does not belong to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ and its length is at least $2N > K$. Therefore it belongs to $T_1(K) \subseteq T(K)$.
4. Let $u \in P$ and $v \in T$. If $v \in T_1$, then $uv$ is also in $T_1$. Let $v \in T_2$. If the last symbol of $v$ is even, then $uv \in T$. If the last symbol of $v$ is odd, then $uv \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$, whence $uv \in T_1 \subseteq T$. Since $|uv| > |u| \geq N > K$, $uv \in T(K)$.

   The claim $T \cdot P \subseteq T$ is handled symmetrically.
5. $P^{\mathrm{R}} \cdot T = P^{\mathrm{R}} \cdot T^{\mathrm{R}} = (T \cdot P)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$. $T \cdot P^{\mathrm{R}} = T^{\mathrm{R}} \cdot P^{\mathrm{R}} = (P \cdot T)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$.
6. Let $u, v \in T$. If at least one of these two words belongs to $T_1$, then $uv \in T_1$. Let $u, v \in T_2$. If the first symbol of $u$ or the last symbol of $v$ is even, then $uv \in T$. In the other case $u$ ends with an even symbol, and $v$ starts with an even symbol. But then we have two consecutive even symbols in $uv$, therefore $uv \in T_1$.

□

Let us call words of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$, $a^{(N-1)}a^{(N)}b^{(1)}$, and $a^{(N)}b^{(1)}b^{(2)}$ ($a, b \in \Sigma$, $1 \le i \le N-2$) *valid triples of type I* and their reversals (namely, $a^{(i+2)}a^{(i+1)}a^{(i)}$, $b^{(1)}a^{(N)}a^{(N-1)}$, and $b^{(2)}b^{(1)}a^{(N)}$) *valid triples of type II.* Note that valid triples of type I (resp., of type II) are the only possible three-symbol subwords of words from $P$ (resp., $P^{\mathrm{R}}$).

**Lemma 5.** *A word $u$ of length at least three is a subword of a word from $P \cup P^{\mathrm{R}}$ if and only if any three-symbol subword of $u$ is a valid triple of type I or II.*

*Proof.* The nontrivial part is "if". We proceed by induction on $|u|$. Induction base ($|u| = 3$) is trivial. Let $u$ be a word of length $m+1$ satisfying the condition and let $u = u'x$ ($x \in \Sigma_1$). By induction hypothesis ($|u'| = m$), $u' \in \mathrm{Subw}(P \cup P^{\mathrm{R}})$. Let $u' \in \mathrm{Subw}(P)$ (the other case is handled symmetrically); $u'$ is a subword of some word $v \in P$. Consider the last three symbols of $u$. Since the first two of them also belong to $u'$, this three-symbol word is a valid triple of type I, not type II. If it is of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$ or $a^{(N)}b^{(1)}b^{(2)}$, then $x$ coincides with the symbol next to the occurrence of $u'$ in $v$, and therefore $u = u'x$ is also a subword of $v$. If it is of the form $a^{(N-1)}a^{(N)}b^{(1)}$, then, provided $v = v_1u'v_2$, $v_1u'$ is also an element of $P$, and so is the word $v_1u'b^{(1)}b^{(2)}\dots b^{(N)}$, which contains $u = u'b^{(1)}$ as a subword. Thus, in all cases $u \in \mathrm{Subw}(P)$. $\square$

Now we construct one more model $\mathcal{M}_2 = \langle \Sigma_1, w_2 \rangle$, where $w_2(p_i) \leftrightharpoons w_1(p_i) \cup w_1(p_i^{\mathrm{R}})^{\mathrm{R}} \cup T$, $w_2(p_i^{\mathrm{R}}) \leftrightharpoons w_1(p_i)^{\mathrm{R}} \cup w_1(p_i^{\mathrm{R}}) \cup T$. This model is a model even in the sense of the enriched language. To finish the proof, we need to check that $\mathcal{M}_2 \not\models \to G$, e.g. $w_2(G) \not\ni \epsilon$.

**Lemma 6.** *For any $A \in \Phi$ the following holds:*

1. $w_2(A) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$;
2. $w_2(A) \supseteq T(f(A))$;
3. $w_2(A) \cap (P \cup \{\epsilon\}) = w_1(A)$ *(in particular, $w_2(A) \cap (P \cup \{\epsilon\}) \neq \varnothing$);*
4. $w_2(A) \cap (P^{\mathrm{R}} \cup \{\epsilon\}) = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$ *(in particular, $w_2(A) \cap (P^{\mathrm{R}} \cup \{\epsilon\}) \neq \varnothing$);*
5. $\epsilon \in w_2(A) \iff \mathrm{L}^{*\prime} \vdash \to A$.

*Proof.* We prove statements 1–4 simultaneously by induction on type $A$.

The induction base is trivial. Further we shall refer to the $i$-th statement of the induction hypothesis ($i = 1, 2, 3, 4$) as "IH-$i$".

**1.** Consider three possible cases.

**a)** $A = B \cdot C$. Then $w_2(A) = w_2(B) \cdot w_2(C) \subseteq (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (Lemma 4).

**b)** $A = B \setminus C$. Suppose the contrary: in $w_2(A)$ there exists an element $u \in E$. Then $vu \in w_2(C)$ for any $v \in w_2(B)$. We consider several subcases and show that each of those leads to a contradiction.

i) $u \in \mathrm{Subw}(P)$, and the superscript of the first symbol of $u$ (as $\epsilon \notin E$, $u$ contains at least one symbol) is not 1. Let the first symbol of $u$ be $a^{(i)}$. Note that $i$ is odd and $i > 2$. Take $v = a^{(3)} \dots a^{(N)} a^{(1)} \dots a^{(i-1)}$. The word $v$ has length at least $N \geq K$ and ends with an even symbol, therefore $v \in T(K) \subseteq T(f(B)) \subseteq w_2(B)$ (IH-2). On the other hand, $vu \in \mathrm{Subw}(P)$ and the first symbol and the last symbol of $vu$ are odd. Therefore, $vu \in E$ and $vu \in w_2(C)$, but $w_2(C) \cap E = \varnothing$ (IH-1). Contradiction.

ii) $u \in \mathrm{Subw}(P)$, and the first symbol of $u$ is $a^{(1)}$ (then the superscript of the last symbol of $u$ is not $N$, because otherwise $u \in P$). Take $v \in w_2(B) \cap (P \cup \{\epsilon\})$ (this set is nonempty due to IH-3). If $v = \epsilon$, then $vu = u \in E$. Otherwise the first and the last symbol of $vu$ are odd, and $vu \in \mathrm{Subw}(P) - P$, and again we have $vu \in E$. Contradiction.

iii) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the superscript of the first symbol of $u$ is not $N$ (the first symbol of $u$ is $a^{(i)}$, $i$ is odd). Take $v = a^{(N-2)} \dots a^{(1)} a^{(N)} \dots a^{(i+1)} \in T(K) \subseteq w_2(B)$. Again, $vu \in E$.

iv) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the first symbol of $u$ is $a^{(N)}$. Take $v \in w_2(B) \cap (P^{\mathrm{R}} \cup \{\epsilon\})$ (nonempty due to IH-4). $vu \in E$.

**c)** $A = C \,/\, B$. Proceed symmetrically.

**2.** Consider three possible cases.

**a)** $A = B \cdot C$. Let $k_1 \leftrightharpoons f(B)$, $k_2 \leftrightharpoons f(C)$, $k \leftrightharpoons k_1 + k_2 + 10 = f(A)$. Due to IH-2, $w_2(B) \supseteq T(k_1)$ and $w_2(C) \supseteq T(k_2)$. Take $u \in T(k)$. We have to prove that $u \in w_2(A)$. Consider several subcases.

i) $u \in T_1(k)$. By Lemma 5 ($|u| \geq k > 3$ and $u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$) in $u$ there is a three-symbol subword $xyz$ that is not a valid triple of type I or II. Divide the word $u$ into two parts, $u = u_1 u_2$, such that $|u_1| \geq k_1 + 5$, $|u_2| \geq k_2 + 5$. If needed, shift the border between parts by one symbol to the left or to the right, so that the subword $xyz$ lies entirely in one part. Let this part be $u_2$ (the other case is handled symmetrically). Then $u_2 \in T_1(k_2)$. If $u_1$ is also in $T_1$, then the proof is finished. Consider the other case. Note that in any word from $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ among any three consecutive symbols at least one is even. Shift the border to the left by at most 2 symbols to make the last symbol of $u_1$ even. Then $u_1 \in T(k_1)$, and $u_2$ remains in $T_1(k_2)$. Thus $u = u_1 u_2 \in T(k_1) \cdot T(k_2) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$.

ii) $u \in T_2(k)$. Let $u$ end with an even symbol (the other case is symmetric). Divide the word $u$ into two parts, $u = u_1 u_2$, $|u_1| \geq k_1 + 5$, $u_2 \geq k_2 + 5$, and shift the border (if needed), so that the last symbol of $u_1$ is even. Then both $u_1$ and $u_2$ end with an even symbol, and therefore $u_1 \in T(k_1)$ and $u_2 \in T(k_2)$.

**b)** $A = B \setminus C$. Let $k \leftrightharpoons f(C) = f(A)$. By IH-2, $w_2(C) \supseteq T(k)$. Take $u \in T(k)$ and an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$. By Lemma 4, statements 4–6, $vu \in (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \cdot T \subseteq T$, and since $|vu| \geq |u| \geq k$, $vu \in T(k) \subseteq w_2(C)$. Thus $u \in w_2(A)$.

**c)** $A = C \,/\, B$. Symmetrically.

**3.** Consider three possible cases.

**a)** $A = B \cdot C$.

$\boxed{\supseteq}$ $u \in w_1(A) = w_1(B) \cdot w_1(C) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-3); $u \in P \cup \{\epsilon\}$.

$\boxed{\subseteq}$ Suppose $u \in P \cup \{\epsilon\}$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$ and $u_2 \in w_2(C)$. First we claim that $u_1 \in P \cup \{\epsilon\}$. Suppose the contrary. By IH-1, $u_1 \in P^{\mathrm{R}} \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$, and therefore $u = u_1 u_2 \in (P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P^{\mathrm{R}} \cup T$ (Lemma 4, statements 1, 3–6). Hence $u \notin P \cup \{\epsilon\}$. Contradiction. Thus, $u_1 \in P \cup \{\epsilon\}$. Similarly, $u_2 \in P \cup \{\epsilon\}$, and by IH-3 we obtain $u_1 \in w_1(B)$ and $u_2 \in w_1(C)$, whence $u = u_1 u_2 \in w_1(A)$.

**b)** $A = B \setminus C$.

$\boxed{\supseteq}$ Take $u \in w_1(B \setminus C) \subseteq P \cup \{\epsilon\}$. First we consider the case where $u = \epsilon$. Then we have $\mathrm{L}^{*\prime} \vdash\ \to B \setminus C$, whence $u = \epsilon \in w_2(B \setminus C)$. Now let $u \in P$. For any $v \in w_1(B)$ we have $vu \in w_1(C)$. We claim that $u \in w_2(B \setminus C)$. Take $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (IH-1). If $v \in P \cup \{\epsilon\}$, then $v \in w_1(B)$ (IH-3), and $vu \in w_1(C) \subseteq w_2(C)$ (IH-3). If $v \in P^{\mathrm{R}} \cup T$, then $vu \in (P^{\mathrm{R}} \cup T) \cdot P \subseteq T(K) \subseteq w_2(C)$ (Lemma 4, statements 3 and 4, and IH-2). Therefore, $u \in w_2(B) \setminus w_2(C) = w_2(B \setminus C)$.

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P \cup \{\epsilon\}$, then for any $v \in w_1(B) \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P \cup \{\epsilon\}$, $vu \in P \cup \{\epsilon\}$. By IH-3, $vu \in w_1(C)$. Thus $u \in w_1(B \setminus C)$.

**c)** $A = C \,/\, B$. Symmetrically.

**4.** Consider three cases.

**a)** $A = B \cdot C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ $u \in w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{R})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-4); $u \in P^{\mathrm{R}} \cup \{\epsilon\}$.

$\boxed{\subseteq}$ Let $u \in P^{\mathrm{R}} \cup \{\epsilon\}$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$, $u_2 \in w_2(C)$. We claim that $u_1 \in P^{\mathrm{R}} \cup \{\epsilon\}$. Suppose the contrary. By IH-1, $u_1 \in P \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$, whence $u = u_1 u_2 \in (P \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P \cup T$. Contradiction. Thus, $u_1 \in P^{\mathrm{R}} \cup \{\epsilon\}$, and therefore $u_2 \in P^{R} \cup \{\epsilon\}$, and, using IH-4, we obtain $u_1 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$, $u_2 \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. Hence $u = u_1 u_2 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{\mathrm{R}})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$.

**b)** $A = B \setminus C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ Let $u \in w_1(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$, First we consider the case where $u = \epsilon$. Then $\mathrm{L}^{*\prime} \vdash\ \to tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$, whence $\epsilon \in w_2(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})) = w_2(tr(A^{\mathrm{R}}))$. Therefore, $u \in w_2(tr(A^{\mathrm{R}}))^{\mathrm{R}}$. Now let $u \in P^{\mathrm{R}}$. For every $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ we have $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. We claim that $u \in w_2(B \setminus C)$. Take an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (IH-1). If $v \in P^{\mathrm{R}} \cup \{\epsilon\}$, then $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4), whence $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(C)$. If $v \in P \cup T$, then (since $u \in P^{\mathrm{R}}$) we have $vu \in (P \cup T) \cdot P^{\mathrm{R}} \subseteq T(K) \subseteq w_2(C)$ (Lemma 4 and IH-2).

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P^{\mathrm{R}} \cup \{\epsilon\}$, then for any $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P^{R} \cup \{\epsilon\}$, $vu \in P^{\mathrm{R}} \cup \{\epsilon\}$, therefore $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4). Thus $u \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = w_1(A^{\mathrm{R}})^{\mathrm{R}}$.

**c)** $A = C \,/\, B$. Symmetrically.

This completes the proof of statements 1–4 of Lemma 6. Statement 5 follows from statement 3 and immediately yields Theorem 3 ($\mathrm{L}^{*\prime} \nvdash \to G$, whence $\epsilon \notin w_2(G)$). $\square$

# 6 Grammars and Complexity

The Lambek calculus and its variants are used for describing formal languages via Lambek categorial grammars. An $\mathrm{L}^*$-*grammar* is a triple $\mathcal{G} = \langle \Sigma, H, \rhd \rangle$, where $\Sigma$ is a finite alphabet, $H \in \mathrm{Tp}$, and $\rhd$ is a finite correspondence between $\mathrm{Tp}$ and $\Sigma$ ($\rhd \subset \mathrm{Tp} \times \Sigma$). The *language generated by* $\mathcal{G}$ is the set of all nonempty words $a_1 \dots a_n$ over $\Sigma$ for which there exist types $B_1, \dots, B_n$ such that $\mathrm{L}^* \vdash B_1 \dots B_n \to H$ and $B_i \rhd a_i$ for all $i \le n$. We denote this language by $\mathfrak{L}(\mathcal{G})$. The notion of L-*grammar* is defined in a similar way. These classes of grammars are *weakly equivalent* to the classes of context-free grammars with and without $\epsilon$-rules in the following sense:

**Theorem 4.** *A formal language is context-free if and only if it is generated by some* $\mathrm{L}^*$*-grammar. A formal language without the empty word is context-free if and only if it is generated by some* L*-grammar.* [7] [2]

By modifying our definition in a natural way one can introduce the notion of $\mathrm{L}^{*\mathrm{R}}$-grammar and $\mathrm{L}^{\mathrm{R}}$-grammar. These grammars also generate precisely all context-free languages (resp., context-free languages without the empty word):

**Theorem 5.** *A formal language is context-free if and only if it is generated by some* $\mathrm{L}^{*\mathrm{R}}$*-grammar. A formal language without the empty word is context-free if and only if it is generated by some* $\mathrm{L}^{\mathrm{R}}$*-grammar.*

*Proof.* The "only if" part follows directly from Theorem 4 due to the conservativity of $\mathrm{L}^{*\mathrm{R}}$ over $\mathrm{L}^*$ and $\mathrm{L}^{\mathrm{R}}$ over L (Lemma 1).

The "if" part is proved by replacing all types in an $\mathrm{L}^{*\mathrm{R}}$-grammar ($\mathrm{L}^*$-grammar) by their normal forms and applying Lemma 2.

$\square$

Since $A \,/\, B$ is equivalent in $\mathrm{L}^{\mathrm{R}}$ and $\mathrm{L}^{*\mathrm{R}}$ to $(B^{\mathrm{R}} \setminus A^{\mathrm{R}})^{\mathrm{R}}$, and the derivability problem in Lambek calculus with two division operators is NP-complete [10] (this holds both for L and $\mathrm{L}^*$), the derivability problem is NP-complete even for the fragment of $\mathrm{L}^{\mathrm{R}}$ ($\mathrm{L}^{*\mathrm{R}}$) with one division.

**Acknowledgments**

I am grateful to Prof. Mati Pentus for fruitful discussions and constant attention to my work. I am also grateful to Prof. Sergei Adian for inspiring techniques of working with words over an alphabet given in his lectures and papers.

This research was supported by the Russian Foundation for Basic Research (grants 11-01-00281-a and 12-01-00888-a), by the Presidential Council for Support of Leading Research Schools (grant NŠ 5593.2012.1) and by the Scientific and Technological Cooperation Programme Switzerland–Russia (STCP-CH-RU, project "Computational Proof Theory").

# References


[1] Buszkowski, W.: Compatibility of categorial grammar with an associated category system. Zeitschr. für Math. Logik und Grundl. der Math. 28, 229–238 (1982)

[2] Kuznetsov, S.: Lambek grammars with one division and one primitive type. Logic Journal of the IGPL, Vol. 20, No. 1, 207–221 (2012)

[3] Kuznetsov, S.: L-completeness of the Lambek calculus with the reversal operation. In: Béchet, D., Dikovsky, A. (eds.) Logical Aspects of Computational Linguistics 2012. LNCS vol. 7351, pp. 151–160. Springer-Verlag, Berlin, Heidelberg (2012)

[4] Lambek, J.: The mathematics of sentence structure. American Math. Monthly 65(3), 154–170 (1958)

[5] Lambek, J.: From categorial grammar to bilinear logic. In: Došen, K., Schroeder-Heister, P. (eds.) Substructural Logics. Studies in Logic and Computation, vol. 2, pp. 128–139. Clarendon Press, Oxford (1993)

[6] Minina, V. A.: Completeness of the Lambek syntactic calculus with the involution operation (in Russian). Diploma paper, Dept. of Math. Logic and Theory of Algorithms, Moscow State University (2001)

[7] Pentus, M.: Lambek grammars are context free. In: 8th Annual IEEE Symposium on Logic in Computer Science, pp. 429–433. IEEE Computer Society Press, Los Alamitos, California (1993)

[8] Pentus, M.: Models for the Lambek calculus. Annals of Pure and Applied Logic 75(1–2), 179–213 (1995)

[9] Pentus, M.: Free monoid completeness of the Lambek calculus allowing empty premises. In: Larrazabal, J. M., Lascar, D., Mints, G. (eds.) Logic Colloquium '96. LNL vol. 12, pp. 171–209. Springer, Berlin etc. (1998)

[10] Savateev, Y.: Product-free Lambek calculus is NP-complete. In: Artemov, S. N., Nerode, A. (eds.) Logical Foundations of Computer Science. LNCS vol. 5407, pp. 380–394. Springer, Berlin etc. (2009)