Categorial Grammars Based on Variants of the Lambek Calculus

Candidate of Science (Ph. D.) Thesis

(translated from Russian)

by Stepan Lvovich Kuznetsov,

Department of Mathematical Logic and Theory of Algorithms,
Faculty of Mathematics and Mechanics, Moscow State University

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Supervisor: Prof., Dr. Sc. M. R. Pentus.
Official opponents:

Dr. Sc. I. G. Lysionok (Steklov Mathematical Institute, RAS);

C. Sc. (Ph. D.) A. V. Kudinov (Institute for Information Transmission Problems, RAS);

Tver State University.

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Introduction

Historical Background

The Lambek calculus L was introduced by J. Lambek [16] for defining natural language syntax using categorial grammars [22][21]. This calculus uses syntactic types built from primitive ones using three binary connectives: multiplication, left and right division.

Chomsky [7] suggested another family of grammars, called the Chomsky hierarchy. The most well-known class grammars from this hierarchy is the class of context-free grammars. Context-free grammars are widely used for parsing artificial languages (e.g., programming languages [1]), but for natural languages categorial grammars have significant advantages. In particular, they enjoy the lexicalisation property: all the information about syntax is kept in the categorial dictionary, and the analyser needs only that part of the dictionary, which is relevant to the text being parsed. Categorial grammars also allow to do semantic analysis, e.g., using Montague semantics [6].

We focus on comparing classes of languages generated by different classes of grammars just as sets of words, without syntactic structure. In this (so called *weak*) sense Lambek grammars are equivalent to context-free ones: the class of languages generated by grammars, based on L, coincides with the class of all context-free languages without the empty word [30]. Similar questions can be posed for grammars based on variants of the Lambek calculus (its fragments and extensions). It is known [5] that all context-free languages without the empty word can be generated by grammars based on the fragment of the Lambek calculus with only one division. Kanazawa [15] considered languages generated by grammars based on the Lambek calculus with additive conjunction and disjunction. This class of languages strictly contains finite intersections of context-free languages and lies inside the class of all context-sensitive languages. Moortgat [20] enriched the Lambek calculus with two modalities; as shown by Jäger [14], this extension does not enlarge the class of languages. Dikovsky and Dekhtyar [8] considered categorial dependency grammars (CDGs) based on a fragment of the Lambek calculus without multiplication, but with additional connectives for nonlocal dependencies, with a restriction: all the denominators are primitive types. Languages generated by CDGs form a special class that strictly contains the class context-free languages and is closed under finite unions, intersections with regular languages, and taking image or preimage of a homomorphism. As shown by Buszkowski [4], any recursively enumerable language can be generated by a grammar based on an extension of the Lambek calculus with a finite set of axioms.

The Lambek calculus is complete with respect to models interpreting Lambek types as formal languages over an alphabet (connectives of L correspond to multiplication, left and right divisions of languages) [31]. Such models are called L-models.

Derivability problems for L and its fragments $L(\backslash, /)$ and $L(\cdot, \backslash)$ are NP-complete as shown by Pentus [25] for L and by Savateev [32] for $L(\backslash, /)$ and $L(\cdot, \backslash)$. On the other hand, if the complexity of types is bounded (this is the usual case in practice), then there are polynomial ($O(n^5)$) time algorithms. These algorithms were independently presented by Pentus [26] and Fowler [9]. Fowler used such algorithms for parsing English sentences from CCGBank [10]. Derivability problem for L($\backslash$) is decideable in polynomial time [32].

Main Results

1. The class of languages generated by $\mathrm{L}_{\mathbf{1}}$-grammars coincides with the class of all context-free languages.
2. The derivability problem for $\mathrm{L}_{\mathbf{1}}(\backslash)$ is decidable in polynomial time.
3. The class of languages generated by $\mathrm{L}(\backslash; p_1)$-grammars coincides with the class of all context-free languages without the empty word.
4. The derivability problem for $\mathrm{L}(\cdot, \backslash; p_1)$ is NP-complete.
5. We present a calculus L^{R} for the unary operation of language reversal and prove its completeness with respect to models on subsets of free semigroups (language models); the class of languages generated by L^{R}-grammars coincides with the class of all context-free languages without the empty word.

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Chapter 1

The Lambek Calculus and Categorial Grammars

1.1 The Calculus L and Its Fragments

We consider the *Lambek calculus* L, introduced in [16]. The countable set $\mathrm{Pr} \leftrightharpoons \{p_1, p_2, p_3, \dots\}$ is called the set of *primitive types* (here and further "$\leftrightharpoons$" means "equals by definition"). *Types* of the Lambek calculus are built from primitive types using three binary connectives: $\backslash$ (left division), $/$ (right division), and $\cdot$ (multiplication); we shall denote the set of all types by Tp. Formally Tp is defined as the smallest (by inclusion) set that satisfies the following two conditions:

1. $\mathrm{Pr} \subset \mathrm{Tp}$;
2. if $A, B \in \mathrm{Tp}$, then $(A \backslash B), (B / A), (A \cdot B) \in \mathrm{Tp}$.

Capital letters ($A, B, \dots$) range over types. Capital Greek letters range over finite (possibly empty) sequences of types; Λ stands for the empty sequence; A^k stands for a sequence of A written k times. Expressions of the form $\Gamma \to C$ are called *sequents* of the Lambek calculus; Γ is the *antecedent* and C *succedent* of $\Gamma \to C$. Sequents with empty antecedents are written as $\to C$.

The calculus L is defined by axioms $p_i \to p_i$ (called (ax)) for every i and the following rules of inference:

$$\frac{A\,\Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash),\ \text{where } \Pi \neq \Lambda; \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Pi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to);$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /),\ \text{where } \Pi \neq \Lambda; \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Pi\, \Delta \to C}\ (/ \to);$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot); \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to);$$

$$\frac{\Pi \to A \quad \Gamma\, A\, \Delta \to C}{\Gamma\, \Pi\, \Delta \to C}\ (\text{cut}).$$

Example 1.1. $\mathrm{L} \vdash (p_1 \backslash p_2)\ ((p_3 \backslash p_2) \backslash p_4) \to ((p_3 \backslash p_1) \backslash p_4)$:

$$\dfrac{\dfrac{\dfrac{p_3 \to p_3 \qquad \dfrac{p_1 \to p_1 \qquad p_2 \to p_2}{p_1\ (p_1 \backslash p_2) \to p_2}}{p_3\ (p_3 \backslash p_1)\ (p_1 \backslash p_2) \to p_2}}{(p_3 \backslash p_1)\ (p_1 \backslash p_2) \to (p_3 \backslash p_2)} \qquad p_4 \to p_4}{\dfrac{(p_3 \backslash p_1)\ (p_1 \backslash p_2)\ ((p_3 \backslash p_2) \backslash p_4) \to p_4}{(p_1 \backslash p_2)\ ((p_3 \backslash p_2) \backslash p_4) \to ((p_3 \backslash p_1) \backslash p_4)}}$$

It is easy to see that sequents with empty antecedents cannot be derivable in L.

The (cut) rule is eliminable:

Theorem 1 (J. Lambek, 1958)**.** *If a sequent is derivable in* L*, then it is derivable without using the* (cut) *rule.* [16]

By Tp(\) we denote the set of types not containing $\cdot$ and /. The calculus defined by axioms (ax) and rules ($\backslash \to$) and ($\to \backslash$) is called L(\). (One can dually consider / instead of \; we shall always use \.) The calculi L($\cdot$, \) and L(\, /) are defined in a similar way.

By $\mathrm{Tp}(p_1, \ldots, p_N)$ we denote the set of types containing only primitive types from the set $\{p_1, \ldots, p_N\}$. The corresponding fragment of the Lambek calculus is denoted by $\mathrm{L}(p_1, \ldots, p_N)$; its fragments with restricted sets of connectives are called $\mathrm{L}(\backslash; p_1, \ldots, p_N)$, $\mathrm{L}(\cdot, \backslash; p_1, \ldots, p_N)$, $\mathrm{L}(\backslash, /; p_1, \ldots, p_N)$. The most interesting case for us is $N = 1$. For convenience we sometime write simply p instead of p_1.

1.2 The Calculi L^* and $\mathrm{L}_{\mathbf{1}}$

The calculus L^* *(the Lambek calculus allowing empty antecedents)* is obtained from L by dropping the restriction $\Pi \neq \Lambda$ on the rules ($\to \backslash$) and ($\to /$).

By $\mathrm{Tp}_{\mathbf{1}}$ we denote the set of types generated from primitive types and the constant $\mathbf{1}$ (unit) using the connectives \, /, and $\cdot$. The calculus $\mathrm{L}_{\mathbf{1}}$ (*the Lambek calculus with the unit,* introduced in [17]) is obtained from L^* by adding an extra axiom $\to \mathbf{1}$ (denoted by ($\to \mathbf{1}$)) and an extra rule

$$\frac{\Gamma\Delta \to C}{\Gamma\mathbf{1}\Delta \to C}\ (\mathbf{1} \to).$$

Fragments of $\mathrm{L}_{\mathbf{1}}$ and L^* with restricted sets of primitive types and/or connectives are defined exactly for L.

The cut elimination theorem [17] is true for $\mathrm{L}_{\mathbf{1}}$ and L^*.

1.3 Conservativity. Substitution and Equivalence

Here and further $\mathscr{L}$ stands for one of the variants of the Lambek calculus: L, L^*, L(\), $\mathrm{L}^*(\backslash)$, $\mathrm{L}(\backslash; p_1, \ldots, p_N)$, $\mathrm{L}^*(\backslash; p_1, \ldots, p_N)$, $\mathrm{L}_{\mathbf{1}}$, or L^{R} (defined in Chapter 4). We denote the corresponding set of types (Tp for L and L^*, Tp(\) for L(\) etc) by $\mathrm{Tp}_{\mathscr{L}}$.

Definition. A calculus $\mathscr{L}_1$ is called a *fragment* of a calculus $\mathscr{L}_2$ if $\mathrm{Tp}_{\mathscr{L}_1} \subseteq \mathrm{Tp}_{\mathscr{L}_2}$ and any sequent built from types from $\mathrm{Tp}_{\mathscr{L}_1}$ is derivable in $\mathscr{L}_1$ if and only if it is derivable in $\mathscr{L}_2$. In this case $\mathscr{L}_2$ is called a *conservative extension* of $\mathscr{L}_1$.

Due to the cut elimination theorem L(\) is a fragment of L, $\mathrm{L}(\backslash; p_1, \ldots, p_N)$ is a fragment of L(\) etc. On the other hand, L^* is not a fragment of L. For example, the sequent $(p_1 \backslash p_1) \backslash p_2 \to p_2$ is derivable in L^*, but not in L.

By $\mathscr{L}(p_1, \ldots, p_N)$ we denote the fragment of $\mathscr{L}$ with a restricted set of primitive types (e. g., $\mathrm{L}(\cdot, \backslash; p_1, p_2, p_3) = \mathrm{L}(\cdot, \backslash)(p_1, p_2, p_3)$).

The substitution of type A for z ($z \in \mathrm{Pr} \cup \{\mathbf{1}\}$) into $\mathscr{A}$ is denoted by $\mathscr{A}[z := A]$. Here $\mathscr{A}$ can be any syntactic object: a type, a sequence of types, a sequent, or (see further) a categorial grammar. The expression $\mathscr{A}[z_1 := A_1, z_2 := A_2, \ldots]$ (or $\mathscr{A}[z_i := A_i]$) means that all substitutions are performed simultaneously. In all calculi considered the *substitution rule* is admissible:

Proposition 1.1. *If* $\mathscr{L} \vdash \Pi \to C$, *then* $\mathscr{L} \vdash (\Pi \to C)[q_1 := A_1, q_2 := A_2, \ldots]$, *where* $q_i \in \Pr$.

Proof. All of the rules of our calculi are given by schemata, therefore they keep valid after substituting arbitrary types for primitive types (here it is essential, that q_i is a primitive type, but not $\mathbf{1}$). It is sufficient to check that the axioms become derivable sequents, in other words, to derive $A \to A$ for every $A \in \mathrm{Tp}_{\mathscr{L}}$. This is done by induction on the length of type A. The base case is trivial, and induction step is established by the following derivations:

$$\dfrac{\dfrac{A_1 \to A_1 \quad A_2 \to A_2}{A_1\, A_2 \to A_1 \cdot A_2}}{A_1 \cdot A_2 \to A_1 \cdot A_2} \qquad \dfrac{\dfrac{A_1 \to A_1 \quad A_2 \to A_2}{A_1\, (A_1 \setminus A_2) \to A_2}}{A_1 \setminus A_2 \to A_1 \setminus A_2} \qquad \dfrac{\dfrac{A_1 \to A_1 \quad A_2 \to A_2}{(A_2 \mathbin{/} A_1)\, A_1 \to A_2}}{A_2 \mathbin{/} A_1 \to A_2 \mathbin{/} A_1}$$

□

If for every $\Pi \to C$ the inverse implication also holds, the substitution is called *faithful.*

Definition. Types A and B are called *equivalent* in $\mathscr{L}$ ($A \leftrightarrow_{\mathscr{L}} B$ or just $A \leftrightarrow B$) if $\mathscr{L} \vdash A \to B$ and $\mathscr{L} \vdash B \to A$.

The relation $\leftrightarrow_{\mathscr{L}}$ is an equivalence relation. Moreover, it is a congruence w. r. t. the Lambek calculus connectives:

Proposition 1.2. *The relation* $\leftrightarrow_{\mathscr{L}}$ *is reflexive, symmetric and transitive. If* $A_1 \leftrightarrow A_2$ *and* $B_1 \leftrightarrow B_2$, *then* $A_1 \cdot B_1 \leftrightarrow A_2 \cdot B_2$, $A_1 \setminus B_1 \leftrightarrow A_2 \setminus B_2$, $B_1 \mathbin{/} A_1 \leftrightarrow B_2 \mathbin{/} A_2$.

Proof. Reflexivity follows from the derivability of $A \to A$, transitivity is due to (cut), symmetricity is obvious.

The remaining statements are proved using, respectively, the rules $(\cdot \to)$ and $(\to \cdot)$, $(\to \setminus)$ and $(\setminus \to)$, $(\to /)$ and $(/ \to)$. □

Using (cut) we prove that if replacing a subtype with an equivalent one does not affect derivability.

1.4 Lambek Categorial Grammars

We call an *alphabet* an arbitrary nonempty finite set. The set of all finite sequences (including the empty one) of elements of an alphabet Σ *(words over* Σ*)* is denoted by Σ^*. The empty word is denoted by ϵ. The set of all nonempty word is denoted by Σ^+. Subsets of Σ^* are called *formal languages* (or simply *languages*) over Σ.

Languages are usually infinite as sets. Various types of formal grammars are used to define some of them in a finite way. The Lambek calculus and its variants are the base for *Lambek categorial grammars.*

Definition. A *grammar based on the calculus* $\mathscr{L}$ (an $\mathscr{L}$*-grammar*) is a triple $\mathcal{G} = \langle \Sigma, H, \rhd \rangle$, where Σ is an alphabet, $H \in \mathrm{Tp}_{\mathscr{L}}$, and $\rhd \subset \mathrm{Tp}_{\mathscr{L}} \times \Sigma$ is a finite binary correspondence. The language generated by $\mathcal{G}$ is $\mathfrak{L}(\mathcal{G}) \risingdotseq \{a_1 \ldots a_k \in \Sigma^* \mid \exists B_1, \ldots, B_k \colon B_i \rhd a_i \ \ \mathscr{L} \vdash B_1 \ldots B_k \to H\}$. Such languages are called $\mathscr{L}$*-languages.*

The notion of *substitution into $\mathscr{L}$-grammar* is defined in a natural way: if $\mathcal{G} = \langle \Sigma, H, \triangleright \rangle$ is an $\mathscr{L}$-grammar, $z \in \Pr \cup \{\mathbf{1}\}$ and $A \in \mathrm{Tp}_{\mathscr{L}}$, then $\mathcal{G}[z := A] \leftrightharpoons \langle \Sigma, H[z := A], \triangleright[z := A] \rangle$, where $\triangleright[z := A] \leftrightharpoons \{\langle B[z := A], a \rangle \mid B \triangleright a\}$.

Proposition 1.3. *If $\mathscr{L}$ is a fragment of $\mathscr{L}'$, then any $\mathscr{L}$-language is an $\mathscr{L}'$-language.*

Proof. Any $\mathscr{L}$-grammar can be considered as $\mathscr{L}'$-grammar. The language will remain the same due to conservativity. □

Vice versa, if an $\mathscr{L}'$-grammar contains only types from $\mathrm{Tp}_{\mathscr{L}}$, it can be considered as an $\mathscr{L}$-grammar generating the same language.

Besides categorial grammars, based on the Lambek calculus and its variants, languages can be also defined by a family of formalisms called the *Chomsky hierarchy* [7]. We consider Chomsky grammars of type 2, also called context-free grammars.

Definition. A *context-free grammar* is a quadruple $G = \langle N, \Sigma, P, S \rangle$, where N and Σ are two disjoint alphabets, $P \subset N \times (N \cup \Sigma)^*$, P is finite, and $S \in N$. We define a binary relation $\Rightarrow_G$ as follows: for all $\omega, \psi \in (N \cup \Sigma)^*$ we have $\omega \Rightarrow_G \psi$ if and only if $\omega = \eta A \theta$, $\psi = \eta \beta \theta$, and $\langle A, \beta \rangle \in P$ for some $A \in N$, $\beta, \eta, \theta \in (N \cup \Sigma)^*$. The binary relation $\Rightarrow_G^*$ is the reflexive transitive closure of $\Rightarrow_G$. The language $\mathfrak{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$ is the *language generated by* G. Such languages are called context-free.

We study the relations between classes of languages generated, on one hand, by context-free grammars, and, on the other hand, by $\mathscr{L}$-grammars for various $\mathscr{L}$. The answers for $\mathscr{L} = \mathrm{L}$ and $\mathscr{L} = \mathrm{L}(\backslash)$ are given by the following theorems:

Theorem 2 (C. Gaifman, 1960; W. Buszkowski, 1985)**.** *Every context-free language without the empty word is an* $\mathrm{L}(\backslash)$*-language.* [2, 5]

Theorem 3 (M. Pentus, 1993)**.** *Every* L*-language is context-free and does not contain the empty word.* [30]

Proposition 1.3 now yields that the following three classes of languages coincide: L-languages, $\mathrm{L}(\backslash)$-languages, and context-free languages without the empty word.

These theorems have variants for the calculi allowing empty antecedents. Theorem 3 remains true:

Theorem 4 (M. Pentus, 1993)**.** *Every* L^**-language is context-free.* [30]

To prove the inverse statement we shall need a refinement of Theorem 2. Essentially, it follows from the Greibach normal form theorem for context-free grammars.

Definition. Let us call a type a *G-type* (after Greibach and Gaifman) if it is of the form p_i, $p_j \backslash p_i$, or $p_k \backslash (p_j \backslash p_i)$, and $k, j \neq 1$. Let us call a sequent $\Pi \to C$ a *G-sequent* if all types in Π are G-types and $C = p_1$.

Theorem 5. *Any context-free language without the empty word is generated by some* $\mathrm{L}(\backslash)$*-grammar* $\mathcal{G} = \langle \Sigma, H, \triangleright \rangle$*, where* $H = p_1$ *and all types used in* $\triangleright$ *are G-types.*

Evidently, when we check whether a word is in a language generated by such sort of grammar, all sequents we try to derive are G-sequents.

Proposition 1.4. *If* $\Pi \to p_1$ *is a G-sequent, then*

$$\mathrm{L}^* \vdash \Pi \to p_1 \iff \mathrm{L} \vdash \Pi \to p_1.$$

Proof. The succedents of all sequents in a derivation of a G-sequent (both in L and in L*) are primitive types. Therefore, the rules ($\to \backslash$) and ($\to /$) cannot be applied, and other rules and axioms are the same in L and L*. □

Now let M be a context-free language. If $\epsilon \notin M$, then let $\mathcal{G}$ be the L($\backslash$)-grammar for M given by Theorem 5. Due to Proposition 1.4 the language $\mathfrak{L}(\mathcal{G})$ will not change if we consider $\mathcal{G}$ as an L*($\backslash$)-grammar. The case when $\epsilon \in M$ is handled in Section 1.8.

Definition. Categorial grammars in which $\triangleleft$ is a partial function from Σ to $\mathrm{Tp}_{\mathscr{L}}$ (in other words, for every $a \in \Sigma$ there exists not more than one type $B \in \mathrm{Tp}_{\mathscr{L}}$, such that $B \triangleright a$), are called *grammars with single type assignment.* (The term *"deterministic grammars"*, used in in [33], is not as appropriate, since these grammars still have some nondeterminicity: the sequent $B_1 \dots B_n \to H$ can have several essentially different derivations in $\mathscr{L}$.)

Theorem 6 (A. Safiullin, 2007)**.** *Every context-free language without the empty word can be generated by an* L*-grammar with single type assignment.* [33]

In [33] it is also stated that this theorem remains true for L($\backslash, /$). The question for other fragments of L (namely, for L($\backslash$) or even for L($\cdot, \backslash$)) and for $\mathrm{L}_{\mathbf{1}}$ and its fragments remains open.

1.5 Complexity Results for the Lambek Calculus and its Fragments

In this section we give a brief survey of known results concerning complexity of derivation problems for the Lambek calculus and its fragments. A more substantial discussion can be found in [27].

Due to the cut elimination theorem every derivable sequent has a derivation of polynomial (w. r. t. the length of the sequent) size. Therefore, derivability problems for L, $\mathrm{L}_{\mathbf{1}}$ and their fragments are in the NP class.

Theorem 7 (M. Pentus, 2003)**.** *Derivability problems for* L *and* L* *(and, therefore, also for* $\mathrm{L}_{\mathbf{1}}$*) are NP-complete.* [25]

A stronger result was obtained by Savateev:

Theorem 8 (Yu. Savateev, 2008–2009)**.** *Derivability problems for* L($\backslash, /$)*,* L*($\backslash, /$)*,* L($\cdot, \backslash$)*, and* L*($\cdot, \backslash$) *are NP-complete.*

The case of only one connective is different:

Theorem 9 (Yu. Savateev, 2006)**.** *There exists an algorithm that checks derivability of a sequent in* L($\backslash$) *(*L*($\backslash$)*) in* $O(n^3)$ *steps, where* n *is the length of the sequent (the number of connective and primitive type occurrences).*

Moreover, there eixsts (Yu. Savateev, 2008) an algorithm that checks whether a word belongs to the language generated by a given L($\backslash$)-grammar (L*($\backslash$)-grammar) in polynomial time: the number of steps is not greater than a polynom of the sum of the length of the word and the size of the grammar.

Savateev's results can be found in his C. Sc. (Ph. D.) thesis [32].

1.6 Multiplicative Cyclic Linear Logic

We consider an auxiliary calculus $\mathrm{MCLL}_{\mathbf{1},\bot}$, the *multiplicative linear logic*. $\mathrm{MCLL}_{\mathbf{1},\bot}$ is a fragment of Girard's [11] linear logic and was first mentioned in [28]. Its connection with the Lambek calculus is established in [23].

Elements of a countable set $\mathrm{Var} = \{p_1, p_2, p_3, \dots\}$ are called *variables*; $\mathrm{At} \leftleftharpoons \mathrm{Var} \cup \{\bar{q} \mid q \in \mathrm{Var}\} \cup \{\mathbf{1}, \bot\}$ is the set of *atoms*. Formulae of $\mathrm{MCLL}_{\mathbf{1},\bot}$ are built from atoms using two binary connectives: $\parr$ (multiplicative disjunction, "*par*") and $\otimes$ (multiplicative conjunction, "*tensor*"). We denote the set of all formulae by $\mathrm{Fm}_{\mathbf{1},\bot}$. Capital Latin letters range over formulae. Capital Greek letters denote finite sequences of formulae; Λ stands for the empty sequence. Sequents of $\mathrm{MCLL}_{\mathbf{1},\bot}$ are of the form $\to \Gamma$.

Definition. The *linear negation* is introduced externally as an inductively defined mapping $(\cdot)^\bot \colon \mathrm{Fm} \to \mathrm{Fm}$: $p_i^\bot \leftleftharpoons \bar{p}_i$, $\bar{p}_i^\bot \leftleftharpoons p_i$, $(A \parr B)^\bot \leftleftharpoons B^\bot \otimes A^\bot$, $(A \otimes B)^\bot \leftleftharpoons B^\bot \parr A^\bot$.

Axioms of $\mathrm{MCLL}_{\mathbf{1},\bot}$ are sequents of the form $\to p_i\, \bar{p}_i$ and the sequent $\to \mathbf{1}$. Rules of inference:

$$\frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta}\ (\to \parr); \qquad \frac{\to \Gamma\, A \quad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta}\ (\to \otimes);$$

$$\frac{\to \Gamma\, \Delta}{\to \Gamma\, \bot\, \Delta}\ (\to \bot); \qquad \frac{\to \Gamma\, \Delta}{\to \Delta\, \Gamma}\ (\mathrm{rot}).$$

We also consider *two-sided* $\mathrm{MCLL}_{\mathbf{1},\bot}$-sequents: $A_1 A_2 \dots A_n \to B_1 \dots B_m$ means $\to A_n^\bot \dots A_2^\bot A_1^\bot B_1 \dots B_m$.

The fragment of $\mathrm{MCLL}_{\mathbf{1},\bot}$ without constants (defined by axioms $\to p_i \bar{p}_i$ and rules $(\to \parr)$, $(\to \otimes)$, and (rot)) is called MCLL.

Definition. The *standard translation* $\widehat{A} \in \mathrm{Fm}$ of $A \in \mathrm{Tp}_{\mathbf{1}}$ is defined as follows:

1. $\widehat{p_i} \leftleftharpoons p_i$ ($p_i \in \mathrm{Pr}$);
2. $\widehat{\mathbf{1}} \leftleftharpoons \mathbf{1}$;
3. $\widehat{A \cdot B} \leftleftharpoons \widehat{A} \otimes \widehat{B}$;
4. $\widehat{A \setminus B} \leftleftharpoons \widehat{A}^\bot \parr \widehat{B}$;
5. $\widehat{B \,/\, A} \leftleftharpoons \widehat{B} \parr \widehat{A}^\bot$.

If $\Pi = B_1 \dots B_m$, then $\widehat{\Pi} \leftleftharpoons \widehat{B}_1 \dots \widehat{B}_m$. The standard translation of a sequent $\Pi \to C$ is the sequent $\widehat{\Pi} \to \widehat{C}$, in other notation, $\to \widehat{B}_m^\bot \dots \widehat{B}_1^\bot C$.

In terms of this translation $\mathrm{MCLL}_{\mathbf{1},\bot}$ is a conservative extension of $\mathrm{L}_{\mathbf{1}}$:

Theorem 10. *Let* $\Pi = B_1 \dots B_m$ *and* $B_1, \dots, B_m, C \in \mathrm{Tp}_{\mathbf{1}}$. *Then*

$$\mathrm{L}_{\mathbf{1}} \vdash \Pi \to C \iff \mathrm{MCLL}_{\mathbf{1},\bot} \vdash \widehat{\Pi} \to \widehat{C}.$$

Due to conservativity of $\mathrm{MCLL}_{\mathbf{1},\bot}$ over MCLL and $\mathrm{L}_{\mathbf{1}}$ over L^* we get a similar theorem for L^*:

Theorem 11. *Let* $\Pi = B_1 \dots B_m \;\; B_1, \dots, B_m, C \in \mathrm{Tp}$. *Then*

$$\mathrm{L}^* \vdash \Pi \to C \iff \mathrm{MCLL} \vdash \widehat{\Pi} \to \widehat{C}.$$

The calculus L corresponds to a modified version of MCLL, called MCLL′. It is obtained from MCLL by adding the restriction $\Gamma\Delta \neq \Lambda$ on the $(\to \parr)$ rule.

Theorem 12. *Let* $\Pi = B_1 \dots B_m \quad B_1, \dots, B_m, C \in \mathrm{Tp}$. *Then*

$$\mathrm{L} \vdash \Pi \to C \iff \mathrm{MCLL}' \vdash \widehat{\Pi} \to \widehat{C}.$$

Theorems 10 and 12 are proved in [23].

Example 1.2. In this example we write $A^{\frown}$ instead of $\widehat{A}$. By definition $\big((p_3 \setminus p_1) \setminus p_4\big)^{\frown} = (\bar{p}_1 \otimes p_3) \parr p_4$, $\big((p_3 \setminus p_2) \setminus p_4\big)^{\frown\perp} = \bar{p}_4 \otimes (\bar{p}_3 \parr p_2)$, and $\big(p_1 \setminus p_2\big)^{\frown\perp} = \bar{p}_2 \otimes p_1$. Therefore, by Example 1.1 and Theorem 12 we get $\mathrm{MCLL}' \vdash \to (\bar{p}_4 \otimes (\bar{p}_3 \parr p_2))\ (\bar{p}_2 \otimes p_1)\ ((\bar{p}_1 \otimes p_3) \parr p_4)$. Here is the explicit derivation:

$$\dfrac{\to p_4\ \bar{p}_4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\to p_2\ \bar{p}_2 \qquad \to p_1\ \bar{p}_1}{\to p_2\ (\bar{p}_2 \otimes p_1)\ \bar{p}_1} \qquad \to p_3\ \bar{p}_3}{\to p_2\ (\bar{p}_2 \otimes p_1)\ (\bar{p}_1 \otimes p_3)\ \bar{p}_3}}{\to \bar{p}_3\ p_2\ (\bar{p}_2 \otimes p_1)\ (\bar{p}_1 \otimes p_3)}}{\to (\bar{p}_3 \parr p_2)\ (\bar{p}_2 \otimes p_1)\ (\bar{p}_1 \otimes p_3)}}{\dfrac{\dfrac{\to p_4\ (\bar{p}_4 \otimes (\bar{p}_3 \parr p_2))\ (\bar{p}_2 \otimes p_1)\ (\bar{p}_1 \otimes p_3)}{\to (\bar{p}_4 \otimes (\bar{p}_3 \parr p_2))\ (\bar{p}_2 \otimes p_1)\ (\bar{p}_1 \otimes p_3)\ p_4}}{\to (\bar{p}_4 \otimes (\bar{p}_3 \parr p_2))\ (\bar{p}_2 \otimes p_1)\ ((\bar{p}_1 \otimes p_3) \parr p_4)}}$$

Substitution $\to \Gamma[z := A]$, where $z \in \mathrm{Var} \cup \{\mathbf{1}\}$, is defined, respecting the negation, in the following way: if $z = p_i$, then for every p_i we substitute A and for every $\bar{p}_i$ we substitute $A^{\perp}$. If $z = \mathbf{1}$, then A is substituted for each occurrence of $\mathbf{1}$, and $A^{\perp}$ is substituted for each occurrence of $\perp$. The substitution rule is admissible in both $\mathrm{MCLL}_{\mathbf{1},\perp}$ and MCLL':

Proposition 1.5. *If* $\to \Gamma$ *is derivable in* $\mathrm{MCLL}_{\mathbf{1},\perp}$ *(resp., in* MCLL'*), then* $\to \Gamma[q_1 := A_1, q_2 := A_2, \dots]$*, where* $q_i \in \mathrm{Var}$, $A_i \in \mathrm{Fm}$*, is also derivable in* $\mathrm{MCLL}_{\mathbf{1},\perp}$ *(resp.,* MCLL'*).*

The proof is similar as for the Lambek calculus (Proposition 1.1).

1.7 Proof Nets

We introduce *proof nets* (graph-theoretical criteria of derivability) for MCLL and MCLL′. The idea of proof nets belongs to mathematical folklore. We consider the variant of proof nets by Pentus [24] and its adaptation for MCLL′. This variant of proof nets is also close to the one proposed by de Groote [12]. Due to Theorems 11 and 12 these criteria can also be used to study derivability in L* and L (and this is our goal).

Let $\to \Gamma$ be a sequent of MCLL or MCLL′. First we build a relation structure $\mathbf{\Omega}_\Gamma = \langle \Omega_\Gamma, <_\Gamma, \prec_\Gamma \rangle$. Let $\Gamma = B_1 \dots B_m$. We put $\diamond$ signs before B_1 and between B_i and B_{i+1} ($i = 1, \dots, m-1$) ($\diamond$ is a new formal symbol, $\diamond \notin \mathrm{Fm}$): $\diamond B_1 \diamond B_2 \diamond \cdots \diamond B_m$. In this string we number all symbols except brackets (an atom is considered one symbol) and denote the set of pairs $\langle \text{symbol}, \text{number} \rangle$ by Ω_Γ. Elements of Ω_Γ are called *occurrences* of the corresponding symbols in Γ and denoted by lowercase Greek letters. For $\alpha = \langle s_1, k_1 \rangle$, $\beta = \langle s_2, k_2 \rangle \in \Omega_\Gamma$ we define $\alpha <_\Gamma \beta \iff k_1 < k_2$.

For a subformula of Γ the corresponding subset of Ω_Γ is called the occurrence of this subformula ($B_1, B_2, \dots, B_m$ are also considered subformulae). If X is a subformula occurrence, then we denote by $\mathrm{l}(X)$ the occurrence of a connective or $\diamond$ immediately to the left of X and by $\mathrm{r}(X)$ the occurrence of a connective or $\diamond$ immediately to the right of X (if there is no such occurrence, $\mathrm{r}(X)$ is defined cyclically as the leftmost occurrence of $\diamond$).

The transitive closure of the union of parse trees of the formulae $B_1, \ldots, B_m$ (where the vertices are occurrences, that is, elements of Ω_Γ) is denoted by $\prec_\Gamma$. In other words, $\alpha \prec_\Gamma \beta$ if and only if α and β are the occurrences of main connectives of two formulae (or atoms), such that the first one is a subformula of the second one.

We denote the set of all occurrences of $\parr$ by $\Omega_\Gamma^{\parr}$, of $\otimes$ by $\Omega_\Gamma^{\otimes}$, of $\diamond$ by $\Omega_\Gamma^{\diamond}$, of $p_1, p_2, \ldots$ by $\Omega_\Gamma^{\mathrm{At}^+}$, of $\bar{p}_1, \bar{p}_2, \ldots$ by $\Omega_\Gamma^{\mathrm{At}^-}$; $\Omega_\Gamma^{\parr\diamond} \leftrightharpoons \Omega_\Gamma^{\parr} \cup \Omega_\Gamma^{\diamond}$, $\Omega_\Gamma^{\otimes\parr\diamond} \leftrightharpoons \Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\parr\diamond}$, $\Omega_\Gamma^{\mathrm{At}} \leftrightharpoons \Omega_\Gamma^{\mathrm{At}^+} \cup \Omega_\Gamma^{\mathrm{At}^-}$. For $X \subseteq \Omega_\Gamma$ we define the *count check* $\#(X) \leftrightharpoons |X \cap \Omega_\Gamma^{\mathrm{At}^+}| - |X \cap \Omega_\Gamma^{\mathrm{At}^-}|$.

For $\alpha, \beta \in \Omega_\Gamma$ we define $(\alpha, \beta) \leftrightharpoons \{\gamma \mid \alpha <_\Gamma \gamma <_\Gamma \beta\}$, $\mathrm{In}(\alpha, \beta) \leftrightharpoons (\alpha, \beta) \cup (\beta, \alpha)$, and $\mathrm{Out}(\alpha, \beta) \leftrightharpoons \Omega_\Gamma - (\mathrm{In}(\alpha, \beta) \cup \{\alpha, \beta\})$ (the minus sign here means set-theoretic difference).

Definition. An oriented graph $\langle \Omega_\Gamma, \mathcal{C} \rangle$, where $\mathcal{C} \subseteq \Omega_\Gamma \times \Omega_\Gamma$, is called $<_\Gamma$*-planar* if for any $\langle \alpha, \beta \rangle \in \mathcal{C}$ and $\langle \gamma, \delta \rangle \in \mathcal{C}$ such that $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$ we have $\gamma \in \mathrm{In}(\alpha, \beta) \iff \delta \in \mathrm{In}(\alpha, \beta)$. Geometrically this means that the graph can be drawn in the upper semiplane without intersections if its vertices lie on the semiplane's border in the $<_\Gamma$ order.

Definition. A *proof net* is a triple $\mathfrak{N} = \langle \boldsymbol{\Omega}_\Gamma, \mathcal{A}, \mathcal{E} \rangle$, where $\mathcal{A} \subset \Omega_\Gamma \times \Omega_\Gamma$ and $\mathcal{E} \subset \Omega_\Gamma \times \Omega_\Gamma$, that satisfies the following axioms:

1. $|\Omega_\Gamma^{\parr\diamond}| - |\Omega_\Gamma^{\otimes}| = 2$;
2. $\mathcal{A}$ is a total function from $\Omega_\Gamma^{\otimes}$ to $\Omega_\Gamma^{\parr\diamond}$;
3. $\mathcal{E}$ is a bijective function from $\Omega_\Gamma^{\mathrm{At}^+}$ to $\Omega_\Gamma^{\mathrm{At}^-}$, and if α is an occurrence of p_i then $\mathcal{E}(\alpha)$ is an occurrence of $\bar{p}_i$;
4. the graph $\langle \Omega_\Gamma, \mathcal{A} \cup \mathcal{E} \rangle$ is $<_\Gamma$-planar;
5. the graph $\langle \Omega_\Gamma, \mathcal{A} \cup \prec_\Gamma \rangle$ is acyclic.

Definition. A *strong proof net* is a proof net that additionally satisfies the following axiom:

6. for any subformula occurrence $X \subset \Omega_\Gamma$ we have $\widetilde{\mathcal{A}}(\mathrm{l}(X)) \neq \widetilde{\mathcal{A}}(\mathrm{r}(X))$, where the mapping $\widetilde{\mathcal{A}} \colon \Omega_\Gamma^{\otimes\parr\diamond} \to \Omega_\Gamma^{\parr\diamond}$ is defined as follows:

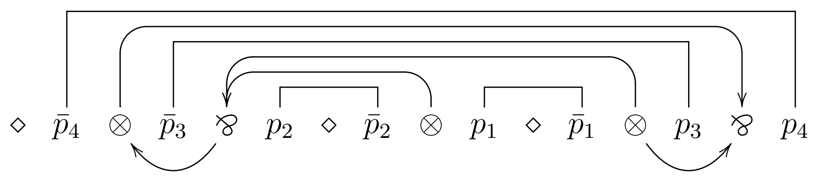
$$\widetilde{\mathcal{A}}(\alpha) \leftrightharpoons \begin{cases} \alpha & \text{if } \alpha \in \Omega_\Gamma^{\parr\diamond}, \\ \mathcal{A}(\alpha) & \text{if } \alpha \in \Omega_\Gamma^{\otimes}. \end{cases}$$

Theorem 13 (M. Pentus, 1996)**.** *A sequent* $\to \Gamma$ *is derivable in* MCLL *if and only if there exists a proof net for* Γ*.*

Theorem 14. *A sequent* $\to \Gamma$ *is derivable in* MCLL$'$ *if and only if there exists a strong proof net for* Γ*.*

Theorem 13 is proved in [24]. Here we modify this proof in order to prove Theorem 14.

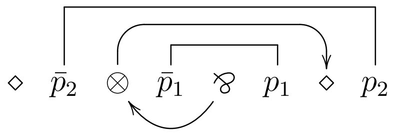
Example 1.3. The following figure shows a proof net for the sequent Example 1.2:



Here (and further in the figures illustrating fragments of strong proof nets) the graphs $\mathcal{A}$ and $\mathcal{E}$ are drawn in the upper semiplane, and the relation $\prec_\Gamma$ (restricted to $\Omega_\Gamma^{\otimes\parr\diamond}$) is drawn in the lower semiplane.

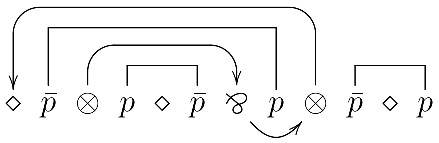
In this example $\Omega_\Gamma = \{\langle\diamond,1\rangle,\ \langle\bar{p}_4,2\rangle,\ \langle\otimes,3\rangle,\ \langle\bar{p}_3,4\rangle,\ \langle\parr,5\rangle,\ \langle p_2,6\rangle,\ \langle\diamond,7\rangle,\ \langle\bar{p}_2,8\rangle,\ \langle\otimes,9\rangle,\ \langle p_1,10\rangle,\ \langle\diamond,11\rangle,\ \langle\bar{p}_1,12\rangle,\ \langle\otimes,13\rangle,\ \langle p_3,14\rangle,\ \langle\parr,15\rangle,\ \langle p_4,16\rangle\}$.

Example 1.4. The sequent $\to \big(\bar{p}_2 \otimes (\bar{p}_1 \parr p_1)\big)\ p_2$ (which is the translation of $(p_1 \setminus p_1) \setminus p_2 \to p_2$, and the latter is derivable in L*, but not in L) is derivable in MCLL, but not in MCLL$'$. The proof net below is not strong:

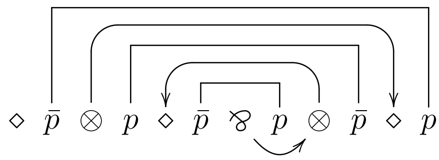


Indeed, let X be the occurrence of the subformula $\bar{p}_1 \parr p_1$. Then $\mathrm{l}(X) = \langle\otimes,3\rangle$, $\mathrm{r}(X) = \langle\diamond,7\rangle$ and $\widetilde{\mathcal{A}}(\mathrm{l}(X)) = \widetilde{\mathcal{A}}(\mathrm{r}(X)) = \langle\diamond,7\rangle$, and this violates Axiom 6.

Example 1.5. There also exist sequents that have both a strong proof net and a proof net that is not strong. One example is the sequent $\big(p\,/(p \setminus p)\big)\ (p \setminus p) \to p$. Its standard translation is $\to (\bar{p} \otimes p)\ \big((\bar{p} \parr p) \otimes \bar{p}\big)\ p$, and this sequent has two proof nets:



and



The first one is strong, and the second one is not. These two proof nets correspond to two Lambek derivations of the original sequent: $\big(p\,/(p \setminus p)\big)\ (p \setminus p) \to p$:

$$\dfrac{\dfrac{\dfrac{p \to p \quad p \to p}{p\ (p \setminus p) \to p}}{p \setminus p \to p \setminus p} \qquad p \to p}{\big(p\,/(p \setminus p)\big)\ (p \setminus p) \to p} \qquad \text{and} \qquad \dfrac{\dfrac{p \to p}{\to p \setminus p} \qquad \dfrac{p \to p \quad p \to p}{p\ (p \setminus p) \to p}}{\big(p\,/(p \setminus p)\big)\ (p \setminus p) \to p}$$

The second derivation is valid only in L*.

Lemma 1.6. *If there exists a strong proof net* $\mathfrak{N} = \langle \boldsymbol{\Omega}_\Gamma, \mathcal{A}, \mathcal{E}\rangle$, *then* Γ *contains at least two formulae.*

Proof. Suppose the contrary: $\Gamma = A_1$. Let X be the occurrence of A_1 (as a subformula of Γ). We have $\mathrm{l}(X) = \mathrm{r}(X) = \langle\diamond,0\rangle$, whence $\widetilde{\mathcal{A}}(\mathrm{l}(X)) = \mathrm{l}(X) = \mathrm{r}(X) = \widetilde{\mathcal{A}}(\mathrm{r}(X))$. Contradiction with axiom 6. □

Proof of Theorem 14. The "only if" part is proved by constructing the strong proof net inductively from the derivation of $\to \Gamma$ (exactly as in the proof of Theorem 13 [24]).

To prove the "if" part we proceed by induction on the number of connective occurrences in Ω_Γ.

The induction base is trivial: in the case $\Omega_\Gamma^{\parr} \cup \Omega_\Gamma^{\otimes} = \emptyset$ our sequent can be either $\to p_i \bar{p}_i$ (axiom) or $\to \bar{p}_i p_i$ (derivable from $\to p_i \bar{p}_i$ by one application of (rot)).

Induction step. We define a new binary relation $\ll$ as the restriction of $\prec_\Gamma \cup \mathcal{A}$ to $\Omega_\Gamma^{\parr} \cup \Omega_\Gamma^{\otimes}$. Due to the acyclicity of $\prec_\Gamma \cup \mathcal{A}$ there exists an element $\gamma \in \Omega_\Gamma^{\parr} \cup \Omega_\Gamma^{\otimes}$ that is maximal with respect to $\ll$. If $\gamma \in \Omega_\Gamma^{\parr}$, then we replace it by $\diamond$ (thus getting a strong proof net with fewer connective occurrences), use the induction hypothesis, and apply the rule $(\to \parr)$. The restriction of this rule is satisfied due to Lemma 1.6.

Now let $\gamma \in \Omega_\Gamma^{\otimes}$, $\beta = \mathcal{A}(\gamma)$. We have $\beta \in \Omega_\Gamma^{\diamond}$, because γ is maximal with respect to $\ll$. We can assume that $\beta = \langle \diamond, 1 \rangle$ (that is, β is the leftmost occurrence of $\diamond$): in the other case we apply (rot) and do a cyclic transformation of the net.

We have $\Gamma = \Phi\,(A \otimes B)\,\Psi$, and the arc $\langle \gamma, \beta \rangle \in \mathcal{A}$ leads from the occurrence of $\otimes$ to the leftmost occurrence of $\diamond$. This arc divides the upper semiplane into two parts. We take the upper part as the strong proof net for $\to B\Psi$ and the lower part for $\to \Phi A$. Axioms 2, 3, 4, 5, and 6 are checked trivially. Axiom 1 is proved in [24], Lemma 7.10.

Now, using the induction hypothesis for $\to \Phi A$ and $\to B\Psi$, we conclude that these sequents are derivable in MCLL$'$. Therefore MCLL$' \vdash\to \Phi\,(A \otimes B)\,\Psi$ by application of $(\to \otimes)$. $\square$

We shall consider $\mathcal{E}$ a non-oriented graph on Ω_Γ; the edges of $\mathcal{E}$ will be called *links*. Intuitively, links connect occurrences of atoms that come from one axiom leaf in the derivation tree. For a link $\mathbf{C}$ with vertices α and β we define $\mathrm{In}(\mathbf{C}) \leftrightharpoons \mathrm{In}(\alpha, \beta)$ and $\mathrm{Out}(\mathbf{C}) \leftrightharpoons \mathrm{Out}(\alpha, \beta)$. The graph $\langle \Omega_\Gamma, \mathcal{E} \rangle$ is $<_\Gamma$-planar, whence $\#(\mathrm{In}(\mathbf{C})) = \#(\mathrm{Out}(\mathbf{C})) = 0$. Links divide the upper semiplane into *regions*. For each link $\mathbf{C}$ we define its *inner* and *outer* regions as the regions that have $\mathbf{C}$ as a part of their borders and intersect with $\mathrm{In}(\mathbf{C})$ and $\mathrm{Out}(\mathbf{C})$ respectively. Evidently, each region must contain at least one occurrence of $\invamp$ or $\diamond$ (because $\mathcal{A} \cup \mathcal{E}$ is $<_\Gamma$-planar). On the other hand, it is easy to see that $|\Omega_\Gamma^{\invamp\diamond}|$ is equal to the number of regions, so each region contains only one occurrence of $\invamp$ or $\diamond$.

Let X and Y be two subformula occurrences such that $X \cap Y = \emptyset$. We define the *fragment from X to Y* as $\{\alpha \in \Omega_\Gamma \mid X <_\Gamma \{\alpha\} <_\Gamma Y\}$ if $X <_\Gamma Y$ and as $\{\alpha \in \Omega_\Gamma \mid \{\alpha\} <_\Gamma Y$ or $X <_\Gamma \{\alpha\}\}$ if $Y <_\Gamma X$ (other cases are impossible).

If $\mathbf{C}$ is a link and $\mathscr{K}$ is a subset of Ω_Γ, we define $\mathscr{D}(\mathbf{C}, \mathscr{K}) = \mathrm{In}(\mathbf{C})$, if $\mathscr{K} \subseteq \mathrm{In}(\mathbf{C})$ and $\mathscr{D}(\mathbf{C}, \mathscr{K}) = \mathrm{Out}(\mathbf{C})$ otherwise (if $\mathscr{K}$ is a fragment from one subformula occurrence to another, and it doesn't contain vertices of $\mathbf{C}$, then in this case we have $\mathscr{K} \subseteq \mathrm{Out}(\mathbf{C})$, thus getting $\mathscr{K} \subseteq \mathscr{D}(\mathbf{C}, \mathscr{K})$ always). It is easy to see that $\#(\mathscr{D}(\mathbf{C}, \mathscr{K})) = 0$.

Further we shall sometimes omit the word "occurrence".

1.8 $\mathrm{L}^*(\backslash)$-grammars for Context-Free Languages with the Empty Word

Theorem 15. *Every context-free language is generated by some $\mathrm{L}^*(\backslash)$-grammar.*

To finish the proof of this theorem (see Section 1.4) is is sufficient to handle the case when the language contains the empty word. Let M be context-free and $\epsilon \in M$. Using Theorem 5, we build an $\mathrm{L}(\backslash)$-grammar $\mathcal{G}$ for the language $M - \{\epsilon\}$.

Consider the type $D \leftrightharpoons \big((r \backslash r) \backslash \big((s \backslash s) \backslash q\big)\big) \backslash q$, where q, r, and s are primitive types not occurring in grammar $\mathcal{G}$.

Lemma 1.7.

1. $\mathrm{L}^*(\backslash) \vdash\, \to D$.
2. *If a sequent $\Gamma \to p_1$, where $\Pi \neq \Lambda$, is a G-sequent and does not contain primitive types q, r, s, then the following equivalence holds:*

$$\mathrm{L}(\backslash) \vdash \Pi \to p_1 \iff \mathrm{L}^*(\backslash) \vdash \Pi[p_1 := D] \to D.$$

Proof. The first statement is established by the following derivation:

$$\dfrac{\dfrac{\dfrac{r \to r}{\to r \backslash r} \qquad \dfrac{\dfrac{s \to s}{\to s \backslash s} \quad q \to q}{(s \backslash s) \backslash q \to q}}{(r \backslash r) \backslash \big((s \backslash s) \backslash q\big) \to q}}{\to \big((r \backslash r) \backslash \big((s \backslash s) \backslash q\big)\big) \backslash q}$$

The left-to-right implication in the second statement follows from the fact that any L($\backslash$)-derivable sequent is L*($\backslash$)-derivable and the substitution rule.

The non-trivial part is the right-to-left implication in the second statement. We have $\mathrm{L}^*(\backslash) \vdash \Pi[p_1 := D] \to D$ and we need to prove that $\mathrm{L}(\backslash) \vdash \Pi \to p_1$. Due to Proposition 1.4, since $\Pi \to p_1$ is a G-sequent, it is sufficient to prove that $\mathrm{L}^*(\backslash) \vdash \Pi \to p_1$.

Consider the translation to MCLL. It is easy to see that

$$\widehat{D} = \big(\bar{q} \otimes (\bar{s} \parr s) \otimes (\bar{r} \parr r)\big) \parr q \qquad \text{and} \qquad \widehat{D}^{\perp} = \bar{q} \otimes \big((\bar{r} \otimes r) \parr (\bar{s} \otimes s) \parr q\big).$$

Let $\Pi = B_1 \ldots B_n$ and let $\tilde{\Pi} = \widehat{B_n}^{\perp} \ldots \widehat{B_1}^{\perp}$. Now we have $\mathrm{MCLL} \vdash\to \tilde{\Pi}[p_1 := \widehat{D}]\, \widehat{D}$ and we need to prove that $\mathrm{MCLL} \vdash\to \tilde{\Pi}\, p_1$.

In the sequent $\to \tilde{\Pi}[p_1 := \widehat{D}]\, \widehat{D}$ variables q, r, and s appear only in the substituted $\widehat{D}$ and $\widehat{D}^{\perp}$ subformulae.

Consider the proof net for $\to \tilde{\Pi}[p_1 := \widehat{D}]\, \widehat{D}$. The occurrence of $\bar{s}$ from $\widehat{D}^{\perp}$ cannot be connected to an occurrence of s from some $\widehat{D}^{\perp}$ (neither the same, nor some other), since in this case there would be two $\parr$s in one area:

$$\ldots \quad \bar{q} \otimes \bar{r} \otimes r \parr \bar{s} \otimes s \parr q \quad \ldots \quad \bar{q} \otimes \bar{r} \otimes r \parr \bar{s} \otimes s \parr q \quad \ldots$$

$$\ldots \quad \bar{q} \otimes \bar{r} \otimes r \parr \bar{s} \otimes s \parr q \quad \ldots \quad \bar{q} \otimes \bar{r} \otimes r \parr \bar{s} \otimes s \parr q \quad \ldots$$

$$\ldots \quad \bar{q} \otimes \bar{r} \otimes r \parr \bar{s} \otimes s \parr q \quad \ldots$$

Hence this occurrence of $\bar{s}$ is connected to an s from a $\widehat{D}$ occurrence.

The sequent $\Pi \to p_1$ is a G-sequent, whence there is only one positive occurrence of p_1. Therefore $\to \tilde{\Pi}[p_1 := \widehat{D}]\, \widehat{D}$ contains only one occurrence of $\widehat{D}$ (namely, the last formula) and, due to the previous argument, at most one occurrence of $\widehat{D}^{\perp}$. Consider two cases.

Case 1: there are no occurrences of $\widehat{D}^{\perp}$. Then q and $\bar{q}$ from $\widehat{D}$ are connected, and $\tilde{\Pi} = \Lambda$ (in the other case there would be two $\diamond$s in one area). Contradiction: $\Pi \neq \Lambda$.

Case 2: there is one occurrence of $\widehat{D}^{\perp}$. Then occurrences of q, r, s, $\bar{q}$, $\bar{r}$, and $\bar{s}$ from $\widehat{D}$ and $\widehat{D}^{\perp}$ are connected pairwise, and we can replace $\widehat{D}$ and $\widehat{D}^{\perp}$ by p_1 and $\bar{p}_1$, thus getting a proof net for $\to \tilde{\Pi}\, p_1$. □

From this lemma it easily follows that substituting D for p_1 in $\mathcal{G}$ yields an L*($\backslash$)-grammar generating the language $\mathfrak{L}(\mathcal{G}) \cup \{\epsilon\} = M$.

Chapter 2

The Lambek Calculus with the Unit

2.1 An Alternative Axiomatisation of $\mathrm{L}_{\mathbf{1}}$

We shall use an alternative axiomatisation of $\mathrm{L}_{\mathbf{1}}$, equivalent to the original one. The rule $(\mathbf{1} \to)$ can be considered a special case of the weakening rule. Every $\mathrm{L}_{\mathbf{1}}$-derivation can be rebuilt in such a way that all applications of this rule will immediately follow the axioms. Formally this is done in the following way. Consider the calculus obtained from $\mathrm{L}_{\mathbf{1}}$ by removing $(\mathbf{1} \to)$ and adding two new series of axioms: $\mathbf{1}^k \to \mathbf{1}$ ($k \geqslant 0$) and $\mathbf{1}^k p_i \mathbf{1}^m \to p_i$ ($k, m \geqslant 0$, $i \geqslant 1$); denote them by $(\to \mathbf{1})_{\mathrm{w}}$ and $(\mathrm{ax})_{\mathrm{w}}$ respectively. The new calculus will be temporarily called $\widetilde{\mathrm{L}}_{\mathbf{1}}$.

Proposition 2.1. *For any sequent $\Pi \to B$, where $\Pi = A_1 \dots A_n$, $A_1, \dots, A_n, B \in \mathrm{Tp}_{\mathbf{1}}$, the following holds:*

$$\mathrm{L}_{\mathbf{1}} \vdash \Pi \to B \iff \widetilde{\mathrm{L}}_{\mathbf{1}} \vdash \Pi \to B.$$

Proof. To prove the right-to-left implication it is sufficient to establish derivability of the new axioms $(\to \mathbf{1})_{\mathrm{w}}$ $(\mathrm{ax})_{\mathrm{w}}$ in $\mathrm{L}_{\mathbf{1}}$. This is done by induction on k and $k+m$ respectively: the base case ($k = 0$ and $k + m = 0$) corresponds to the axioms $(\to \mathbf{1})$ and (ax) of $\mathrm{L}_{\mathbf{1}}$, and the induction step is an application of $(\mathbf{1} \to)$.

The left-to-right implication is established by induction on the length of the $\mathrm{L}_{\mathbf{1}}$-derivation: if the rule $(\mathbf{1} \to)$ was applied (possibly, several times) after an application of another rule, then the derivation can be rebuilt in such a way that the applications of $(\mathbf{1} \to)$ go, vice versa, before the application of the other rule. Applying $(\mathbf{1} \to)$ (several times) to the axioms (ax) and $(\to \mathbf{1})$ yields $(\mathrm{ax})_{\mathrm{w}}$ and $(\to \mathbf{1})_{\mathrm{w}}$ respectively. □

Further we shall use this new axiomatisation for $\mathrm{L}_{\mathbf{1}}$.

2.2 Reduction of $\mathrm{L}_{\mathbf{1}}$ to L^*. $\mathrm{L}_{\mathbf{1}}$-grammars

We present a substitution that reduces derivability in $\mathrm{L}_{\mathbf{1}}$ to derivability in L^*.

Theorem 16. *For any sequent $\Pi \to C$ built from types that belong to $\mathrm{Tp}_{\mathbf{1}}$ and for any primitive type q not occurring in $\Pi \to C$, the following equivalence holds:*

$$\mathrm{L}_{\mathbf{1}} \vdash \Pi \to C \iff \mathrm{L}^* \vdash \big(\Pi \to C\big)[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}][\mathbf{1} := q \setminus q].$$

Here and further the shorthand "$p_i := (\mathbf{1} \cdot p_i) \cdot p_i$" means that the substitution is performed for *every* i.

Lemma 2.2. *For every $k \geqslant 0$ the following holds:* $\mathrm{L}^*(\backslash) \vdash (q \backslash q)^k \to q \backslash q$.

Proof.

$$\dfrac{\dfrac{q \to q \qquad \dfrac{q \to q \qquad \dfrac{\dfrac{q \to q \qquad \dfrac{q \to q \quad q \to q}{q\,(q \backslash q) \to q}}{\vdots}}{q\,(q \backslash q) \ldots (q \backslash q) \to q}}{q\,(q \backslash q)\,(q \backslash q) \ldots (q \backslash q) \to q}}{q\,(q \backslash q)\,(q \backslash q)\,(q \backslash q) \ldots (q \backslash q) \to q}}{(q \backslash q)\,(q \backslash q)\,(q \backslash q) \ldots (q \backslash q) \to q \backslash q}$$

□

Consider an auxiliary calculus $\mathrm{L}_{\mathbf{1}}^{-}$, which is obtained from L^* by adding axioms $(\to \mathbf{1})_{\mathrm{w}}$. It is clear that every $\mathrm{L}_{\mathbf{1}}^{-}$-derivable sequent is $\mathrm{L}_{\mathbf{1}}$-derivable.

Lemma 2.3. *For every sequent $\Pi \to C$ built from types that belong to $\mathrm{Tp}_{\mathbf{1}}$ the following equivalences hold:*

$$\mathrm{L}_{\mathbf{1}} \vdash \Pi \to C \iff \mathrm{L}_{\mathbf{1}} \vdash \big(\Pi \to C\big)[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}] \iff$$
$$\iff \mathrm{L}_{\mathbf{1}}^{-} \vdash \big(\Pi \to C\big)[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}].$$

Proof. The first equivalence follows from the fact that $p_i \leftrightarrow_{\mathrm{L}_{\mathbf{1}}} (\mathbf{1} \cdot p_i) \cdot \mathbf{1}$.

In the second equivalence the right-to-left implication is obvious. Let us prove the other one: we shall deduce the third statement from the first one (which is equivalent to the second one). We substitute $(\mathbf{1} \cdot p_i) \cdot \mathbf{1}$ in the $\mathrm{L}_{\mathbf{1}}$-derivation of $\Pi \to C$. It is easy to see that this substitution conserves the $(\to \mathbf{1})_{\mathrm{w}}$ axioms and all rules. Now it is sufficient to check that the result of such a substitution in $(\mathrm{ax})_{\mathrm{w}}$ is derivable in $\mathrm{L}_{\mathbf{1}}^{-}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{1}^{k+1} \to \mathbf{1} \quad p_i \to p_i}{\mathbf{1}^{k+1}\, p_i \to \mathbf{1} \cdot p_i} \qquad \mathbf{1}^{m+1} \to \mathbf{1}}{\mathbf{1}^k\, \mathbf{1}\, p_i\, \mathbf{1}^{m+1} \to (\mathbf{1} \cdot p_i) \cdot \mathbf{1}}}{\mathbf{1}^k\, (\mathbf{1} \cdot p_i)\, \mathbf{1}\, \mathbf{1}^m \to (\mathbf{1} \cdot p_i) \cdot \mathbf{1}}}{\mathbf{1}^k\, \big((\mathbf{1} \cdot p_i) \cdot \mathbf{1}\big)\, \mathbf{1}^m \to (\mathbf{1} \cdot p_i) \cdot \mathbf{1}}$$

□

The next two lemmas essentially repeat the argument from [19] about the closed (without variables but with constants) fragment of multiplicative cyclic linear logic.

Lemma 2.4. *If* $\mathrm{L}_{\mathbf{1}}^{-} \vdash \Pi \to C \;\; q \in \mathrm{Pr}$, *then* $\mathrm{L}^* \vdash \big(\Pi \to C\big)[\mathbf{1} := q \backslash q]$.

Proof. Perform the substitution in the $\mathrm{L}_{\mathbf{1}}^{-}$-derivation of $\Pi \to C$. Axioms $(\mathrm{ax})_{\mathrm{w}}$ and rules of inference will remain untouched. Axioms $(\to \mathbf{1})_{\mathrm{w}}$ will transform into sequents $(q \backslash q)^k \to q \backslash q$, which are derivable in L^* by Lemma 2.2. □

Lemma 2.5. *If* $\mathrm{L}^* \vdash \big(\Pi \to C\big)[\mathbf{1} := q \backslash q]$ *and q is a primitive type that does not occur in $\Pi \to C$, then* $\mathrm{L}_{\mathbf{1}} \vdash \Pi \to C$.

Proof. Let $L^* \vdash \big(\Pi \to C\big)[\mathbf{1} := q \setminus q]$. Consider the sequent $\big(\Pi \to C\big)[\mathbf{1} := q \setminus q][q := \mathbf{1}]$. On the one hand, it is derivable in $L_{\mathbf{1}}$, since $\big(\Pi \to C\big)[\mathbf{1} := q \setminus q]$ is derivable in $L_{\mathbf{1}}$ (due to the conservativity of $L_{\mathbf{1}}$ over L^*) and the substitution rule is valid in $L_{\mathbf{1}}$. On the other hand, the sequent involved is actually $(\Pi \to C)[\mathbf{1} := \mathbf{1} \setminus \mathbf{1}]$, because occurrences of q could appear only inside the types $q \setminus q$ that are substituted for $\mathbf{1}$. Therefore the derivability of this sequent in $L_{\mathbf{1}}$ is equivalent to the derivability of $\Pi \to C$ (since $\mathbf{1} \leftrightarrow \mathbf{1} \setminus \mathbf{1}$). □

Proof of Theorem 16.

$$\begin{aligned} L_{\mathbf{1}} \vdash \Pi \to C \Longrightarrow L_{\mathbf{1}}^- \vdash \big(\Pi \to C\big)[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}] \Longrightarrow \\ \Longrightarrow L^* \vdash \big(\Pi \to C\big)[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}][\mathbf{1} := q \setminus q] \Longrightarrow \\ \Longrightarrow L_{\mathbf{1}} \vdash \big(\Pi \to C\big)[p_1 := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}] \Longrightarrow L_{\mathbf{1}} \vdash \Pi \to C. \end{aligned}$$

Here the first and the fourth implications hold due to Lemma 2.3, the second one holds due to Lemma 2.4, and the third one holds due to Lemma 2.5. □

Theorem 16 easily yields a corollary concerning $L_{\mathbf{1}}$-grammars. (In [30] it is stated that the theorem below can be proved by the same means as for L^*, but this is not true.)

Theorem 17. *The class of languages generated by* $L_{\mathbf{1}}$*-grammars coincides with the class of all context-free languages.*

Proof. Let $\mathcal{G}$ be an $L_{\mathbf{1}}$-grammar. Then, due to Theorem 16, the L^*-grammar $\mathcal{G}[p_i := (\mathbf{1} \cdot p_i) \cdot \mathbf{1}][\mathbf{1} := q \setminus q]$ generates the same language as $\mathcal{G}$. On the other hand, this language is context-free by Theorem 4.

The inverse inclusion is due to conservativity of $L_{\mathbf{1}}$ over L^* and Theorem 15. □

2.3 Reduction of $L_{\mathbf{1}}(\setminus)$ to $L^*(\setminus)$. A Polynomial Algorithm for the Derivability Problem in $L_{\mathbf{1}}(\setminus)$

Now we shall need a refinement of Theorem 16, a substitution with only one division.

Theorem 18. *For any sequent $\Pi \to C$ built from types belonging to $\mathrm{Tp}_{\mathbf{1}}(\setminus)$ and for any primitive type q not occurring in $\Pi \to C$, the following equivalence holds:*

$$L_{\mathbf{1}}(\setminus) \vdash \Pi \to C \iff L^*(\setminus) \vdash \big(\Pi \to C\big)[\mathbf{1} := q \setminus q,\ p_i := (q \setminus q) \setminus (p_i \setminus q)].$$

This theorem yields polynomial decidability of the derivability problem in $L_{\mathbf{1}}(\setminus)$ (derivability problems for $L_{\mathbf{1}}$, $L_{\mathbf{1}}(\setminus, /)$, and $L_{\mathbf{1}}(\cdot, \setminus)$ are NP-complete due to conservativity and Theorem 8):

Theorem 19. *There exists an algorithm that checks derivability of a sequent in $L_{\mathbf{1}}(\setminus)$ in $O(n^3)$ steps, where n is the length of the sequent.*

Proof. The algorithm is as follows: first we build the sequent $\big(\Pi \to C\big)[\mathbf{1} := q \setminus q, p_i := (q \setminus q) \setminus (p_i \setminus q)]$, where q is a new primitive type. The length of the sequent grows not more than seven times. Then we use Savateev's algorithm (Theorem 9) to check derivability of this sequent in $L^*(\setminus)$. □

Before proving Theorem 18, we prove several lemmas.

Lemma 2.6. *For any $k, m \geqslant 0$, $p_i \in \Pr$ the following holds:*

$$\mathrm{L}^*(\backslash) \vdash (q \backslash q)^k \big((q \backslash q) \backslash (p_i \backslash q)\big)\, (q \backslash q)^m \to (q \backslash q) \backslash (p_i \backslash q).$$

Proof.

$$\dfrac{\dfrac{(q \backslash q)^{k+1} \to q \backslash q \qquad \dfrac{\dfrac{p_i \to p_i \quad q\,(q \backslash q)^m \to q}{p_i\,(p_i \backslash q)\,(q \backslash q)^m \to q}}{(p_i \backslash q)\,(q \backslash q)^m \to p_i \backslash q}}{(q \backslash q)^{k+1} \big((q \backslash q) \backslash (p_i \backslash q)\big)\,(q \backslash q)^m \to p_i \backslash q}}{(q \backslash q)^{k} \big((q \backslash q) \backslash (p_i \backslash q)\big)\,(q \backslash q)^m \to (q \backslash q) \backslash (p_i \backslash q)}$$

Sequents $(q \backslash q)^{k+1} \to q \backslash q$ and $q\,(q \backslash q)^m \to q$ were derived in the proof of Lemma 2.2. □

Lemma 2.7. *If* $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash\ \to \Gamma[p_i := \bar{p}_i]$, *then* $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash\ \to \Gamma$.

Proof. After substituting $\bar{p}_i$ for p_i in the $\mathrm{MCLL}_{\mathbf{1},\bot}$-derivation of a sequent, the rules of inference and axiom $\to \mathbf{1}$ remain valid, and axioms $\to p_i\,\bar{p}_i$ become sequents $\to \bar{p}_i\,p_i$, derivable by one application of (rot). Finally, notice that $\Gamma[p_i := \bar{p}_i][p_i := \bar{p}_i] = \Gamma$. □

Lemma 2.8. *The following equivalences hold in* $\mathrm{MCLL}_{\mathbf{1},\bot}$*:* $\mathbf{1} \parr \bot \leftrightarrow \mathbf{1}$ *and* $(\mathbf{1} \otimes \bot) \parr (\bar{p}_i \parr \bot) \leftrightarrow \bar{p}_i$.

Proof. Since $A \leftrightarrow_{\mathrm{MCLL}_{\mathbf{1},\bot}} B$ means that $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash\ \to A^\perp B$ $\ \mathrm{MCLL}_{\mathbf{1},\bot} \vdash\ \to B^\perp A$, and $(\mathbf{1} \parr \bot)^\perp = \mathbf{1} \otimes \bot$, $\mathbf{1}^\perp = \bot$, $\bar{p}_i^\perp = p_i$, $\big((\mathbf{1} \otimes \bot) \parr (\bar{p}_i \parr \bot)\big)^\perp = (\mathbf{1} \otimes p_i) \otimes (\mathbf{1} \parr \bot)$, we need to derive the following sequents in $\mathrm{MCLL}_{\mathbf{1},\bot}$: $\to (\mathbf{1} \otimes \bot)\,\mathbf{1}$, $\to \bot\,(\mathbf{1} \parr \bot)$, $\to \big((\mathbf{1} \otimes p_i) \otimes (\mathbf{1} \parr \bot)\big)\,\bar{p}_i$ $\ \to p_i\,\big((\mathbf{1} \otimes \bot) \parr (\bar{p}_i \parr \bot)\big)$. The derivations are as follows:

$$\dfrac{\to \mathbf{1} \qquad \dfrac{\to \mathbf{1}}{\to \bot\,\mathbf{1}}}{\to (\mathbf{1} \otimes \bot)\,\mathbf{1}} \qquad\qquad \dfrac{\dfrac{\dfrac{\to \mathbf{1}}{\to \mathbf{1}\,\bot}}{\to \bot\,\mathbf{1}\,\bot}}{\to \bot\,(\mathbf{1} \parr \bot)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\to \mathbf{1} \quad \to p_i\,\bar{p}_i}{\to (\mathbf{1} \otimes p_i)\,\bar{p}_i}}{\to \bar{p}_i\,(\mathbf{1} \otimes p_i)} \qquad \dfrac{\dfrac{\to \mathbf{1}}{\to \mathbf{1}\,\bot}}{\to \mathbf{1} \parr \bot}}{\to \bar{p}_i\,\big((\mathbf{1} \otimes p_i) \otimes (\mathbf{1} \parr \bot)\big)}}{\to \big((\mathbf{1} \otimes p_i) \otimes (\mathbf{1} \parr \bot)\big)\,\bar{p}_i} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\to \mathbf{1} \quad \dfrac{\dfrac{\to p_i\,\bar{p}_i}{\to \bar{p}_i\,p_i}}{\to \bot\,\bar{p}_i\,p_i}}{\to (\mathbf{1} \otimes \bot)\,\bar{p}_i\,p_i}}{\to p_i\,(\mathbf{1} \otimes \bot)\,\bar{p}_i}}{\to p_i\,(\mathbf{1} \otimes \bot)\,\bar{p}_i\,\bot}}{\to p_i\,(\mathbf{1} \otimes \bot)\,(\bar{p}_i \parr \bot)}}{\to p_i\,\big((\mathbf{1} \otimes \bot) \parr (\bar{p}_i \parr \bot)\big)}$$

□

Proof of Theorem 18. Prove the left-to-right implication. Consider an $\mathrm{L}_{\mathbf{1}}(\backslash)$-derivation of $\Pi \to C$. Perform the substitution $\mathbf{1} := q \backslash q$, $p_i := (q \backslash q) \backslash (p_i \backslash q)$ in this derivation. The rules of inference remain valid, axioms $(\to \mathbf{1})_{\mathrm{w}}$ transform to sequents $(q \backslash q)^k \to q \backslash q$, derivable by Lemma 2.2, and axioms $(\mathrm{ax})_{\mathrm{w}}$ transform to sequents $(q \backslash q)^k \to q \backslash q$, derivable by Lemma 2.6. Therefore, the sequent $\big(\Pi \to C\big)[\mathbf{1} := q \backslash q, p_i := (q \backslash q) \backslash (p_i \backslash q)]$ is derivable in $\mathrm{L}^*(\backslash)$.

Now prove the right-to-left implication. Let $\mathrm{L}^*(\backslash) \vdash \big(\Pi \to C\big)[\mathbf{1} := q \backslash q, p_i := (q \backslash q) \backslash (p_i \backslash q)]$. Then $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash \big(\widehat{\Pi} \to \widehat{C}\big)[\mathbf{1} := \bar{q} \parr q, p_i := (\bar{q} \otimes q) \parr (\bar{p}_i \parr q)]$. Substitute $\bot$ for q. This yields to a sequent derivable in $\mathrm{MCLL}_{\mathbf{1},\bot}$, and, since q does not occur in $\widehat{\Pi} \to \widehat{C}$, this sequent is actually $\big(\widehat{\Pi} \to \widehat{C}\big)[\mathbf{1} := \mathbf{1} \parr \bot, p_i := (\mathbf{1} \otimes \bot) \parr (\bar{p}_i \parr \bot)]$. Due to Lemma 2.8, $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash \big(\widehat{\Pi} \to \widehat{C}\big)[p_i := \bar{p}_i]$. Finally, by Lemma 2.7 $\mathrm{MCLL}_{\mathbf{1},\bot} \vdash \widehat{\Pi} \to \widehat{C}$, whence $\mathrm{L}_{\mathbf{1}}(\backslash) \vdash \Pi \to C$. Q. E. D. $\square$

Chapter 3

The Lambek Calculus with One Primitive Type

3.1 Reduction of $\mathrm{L}(\backslash; p_1, \ldots, p_N)$ to $\mathrm{L}(\backslash; p)$

Let N be an arbitrary natural number. We build a faithful substitution that reduces derivability in $\mathrm{L}(\backslash; p_1, \ldots, p_N)$ to derivability in $\mathrm{L}(\backslash; p)$.

Let $p^m \leftrightharpoons \underbrace{p \cdot \ldots \cdot p}_{m \text{ times}}$.

Now construct the types $A_1, \ldots, A_N$:

$$A_k \leftrightharpoons \big(p^{k+1} \cdot \big(((p \cdot p) \setminus p) \setminus p\big) \cdot p^{N-k+1}\big) \setminus p, \qquad k = 1, \ldots, N.$$

Note that in $\mathrm{L}(\backslash)$ there is no multiplication; the $\cdot$ connective here (multiplication in the denominator) is just a shortcut used to minimise the number of brackets: $(D_1 \cdot \ldots \cdot D_m) \setminus C \leftrightharpoons D_m \setminus (D_{m-1} \setminus (D_{m-2} \setminus \ldots \setminus (D_1 \setminus C) \ldots))$. This notation is consistent with the "real" multiplication: $(D_1 \cdot \ldots \cdot D_m) \setminus C \leftrightarrow_{\mathrm{L}(\cdot,\backslash)} D_m \setminus (D_{m-1} \setminus (D_{m-2} \setminus \ldots \setminus (D_1 \setminus C) \ldots))$.

Theorem 20. *For any sequent $\Pi \to C$, where $\Pi = B_1 \ldots B_m$, $m \geqslant 1$, and $B_1, \ldots, B_m, C \in \mathrm{Tp}(\backslash; p_1, \ldots, p_N)$, the following holds:*

$$\mathrm{L}(\backslash; p_1, \ldots, p_N) \vdash \Pi \to C \iff \mathrm{L}(\backslash; p) \vdash (\Pi \to C)[p_1 := A_1, \ldots, p_N := A_N].$$

Theorem 21. *For every sequent $\Pi \to C$, where $\Pi = B_1 \ldots B_m$, $m \geqslant 0$, and $B_1, \ldots, B_m, C \in \mathrm{Tp}(\backslash; p_1, \ldots, p_N)$, the following holds:*

$$\mathrm{L}^*(\backslash; p_1, \ldots, p_N) \vdash \Pi \to C \iff \mathrm{L}^*(\backslash; p) \vdash (\Pi \to C)[p_1 := A_1, \ldots, p_N := A_N].$$

Due to conservativity of MCLL' over $\mathrm{L}(\backslash)$ (and, resp., $\mathrm{MCLL}'(p)$ over $\mathrm{L}(\backslash; p)$ and $\mathrm{MCLL}'(p_1, \ldots, p_N)$ over $\mathrm{L}(\backslash; p_1, \ldots, p_N)$) and MCLL over $\mathrm{L}^*(\backslash)$ (resp., $\mathrm{MCLL}(p)$ over $\mathrm{L}^*(\backslash; p)$ and $\mathrm{MCLL}(p_1, \ldots, p_N)$ over $\mathrm{L}^*(\backslash; p_1, \ldots, p_N)$) these two theorems are corollaries from the following two stronger statements, which we are going to prove in the next section:

Theorem 22. *For any sequent $\to \Gamma$, where $\Gamma = B_1 \ldots B_m$, $m \geqslant 2$, and $B_1, \ldots, B_m \in \mathrm{Fm}(p_1, \ldots, p_N)$, the following holds:*

$$\mathrm{MCLL}'(p_1, \ldots, p_N) \vdash \to \Gamma \iff \mathrm{MCLL}'(p) \vdash \to \Gamma[p_1 := \widehat{A}_1, \ldots, p_N := \widehat{A}_N].$$

Theorem 23. *For every sequent* $\to \Gamma$*, where* $\Gamma = B_1 \dots B_m$*,* $m \geqslant 1$*, and* $B_1, \dots, B_m \in \mathrm{Fm}(p_1, \dots, p_N)$*, the following holds:*

$$\mathrm{MCLL}(p_1, \dots, p_N) \vdash \to \Gamma \iff \mathrm{MCLL}(p) \vdash \to \Gamma[p_1 := \widehat{A}_1, \dots, p_N := \widehat{A}_N].$$

A faithful substitution that reduces derivability in $\mathrm{MCLL}(p_1, \dots, p_N)$ to derivability in $\mathrm{MCLL}(p)$ was earlier presented by Métayer [19]. The construction of Métayer can be easily modified to obtain a substitution that reduces derivability in $\mathrm{L}^*(\backslash, /; p_1, \dots, p_N)$ to derivability in $\mathrm{L}^*(\backslash, /; p)$ (we substitute $A_k \leftrightharpoons p^k \mathbin{\backslash} p \mathbin{/} p^{N+1-k}$ for p_k; here we use both divisions); a similar result can be obtained for L using strong proof nets.

Yet another substitution reducing derivability in $\mathrm{L}(\backslash)$ to derivability in $\mathrm{L}(\backslash; p)$ was independently presented by Hendriks in [13]. Hendriks' construction uses only one division, but the types used in it have exponential size w. r. t. N. Therefore this construction cannot be used to prove NP-completeness of $\mathrm{L}(\cdot, \backslash; p)$, $\mathrm{L}(\backslash, /; p)$ and their variants allowing empty antecedents. Also, in our opinion, Hendriks' proof is more complicated then the proof presented here.

3.2 Faithfullness of the Substitution (Proof)

We prove Theorems 20 and 21 in parallel. The left-to-right implications are trivial.

Let $R_k \leftrightharpoons \widehat{A}_k$. It is easy to see that

$$R_k = \underbrace{\bar{p} \parr \dots \parr \bar{p}}_{N-k+1} \parr \big(\bar{p} \otimes (\bar{p} \parr \bar{p} \parr p_*)\big) \parr \underbrace{\bar{p} \parr \dots \parr \bar{p}}_{k+1} \parr p_+;$$

$$R_k^\perp = \bar{p}_+ \otimes \underbrace{p \otimes \dots \otimes p}_{k+1} \otimes \big((\bar{p}_* \otimes p \otimes p) \parr p\big) \otimes \underbrace{p \otimes \dots \otimes p}_{N-k+1}.$$

(We assume that $\parr$s associate to the right and $\otimes$s associate to the left.) The subscripts $*$ and $+$ here mark concrete occurrences of p and $\bar{p}$ for further reference. Let us call R_k a *positive* formula and $R_k^\perp$ a *negative* formula.

If X is an occurrence of R_k, then $\#(X) = -(N+3)$, and if Y is an occurrence of $R_k^\perp$, then $\#(Y) = N+3$. We shall use these count checks later.

The sequent $\to \Gamma[p_1 := R_1, \dots, p_N := R_N]$ is derivable in $\mathrm{MCLL}'(p)$ (resp., in $\mathrm{MCLL}(p)$). Hence there exists a strong proof net (resp., just a proof net) $\mathfrak{N}$ for this sequent. We shall modify $\mathfrak{N}$ to get a proof net $\mathfrak{N}'$ for $\to \Gamma$. If the original proof net were strong, $\mathfrak{N}'$ will also be strong.

First we shall prove some lemmata about $\mathfrak{N}$.

Lemma 3.1. *The number of positive formula occurrences is equal to the number of negative formula occurrences.*

Proof. Suppose there are m_1 positive formula occurrences and m_2 negative formula occurrences. Then we have $0 = \#(\Omega_\Gamma) = (m_2 - m_1)(N+3)$, whence $m_2 = m_1$. □

Lemma 3.2. *Any occurrence of* p_* *from* R_k *is connected to an occurrence of* $\bar{p}_*$ *from some* $R_{k'}^\perp$ *(possibly,* $k \neq k'$*).*

Proof. Suppose the contrary. Let $\mathbf{C}$ be a link from some p_* that does not lead to $\bar{p}_*$. We consider 3 cases:

Case 1: $\mathbf{C}$ leads to an occurrence of $\bar{p}$ from the same R_k. Since $\#(\mathrm{In}(\mathbf{C})) = 0$, this is the neighbour occurrence of $\bar{p}$. But then there are two $\parr$ in the outer region of $\mathbf{C}$. Contradiction.

Case 2: $\mathbf{C}$ leads to an occurrence of $\bar{p}$ from another $R_{k'}$. There are $\parr$ connectives on both sides of p_* and on at least one side of any $\bar{p}$ from $R_{k'}$, therefore either in the inner or in the outer region of $\mathbf{C}$ there are two $\parr$ connectives. Contradiction.

Case 3: $\mathbf{C}$ leads to $\bar{p}_+$ (from some $R_{k'}^{\perp}$):

$$\bar{p}\parr\ldots\parr\bar{p}\parr\bar{p}\otimes\bar{p}\parr\bar{p}\parr p_*\parr\underbrace{\bar{p}\parr\ldots\parr\bar{p}}_{k+1}\parr p_+ \ \boxed{\mathscr{K}}\ \bar{p}_+\otimes p\otimes\ldots\otimes p\otimes\bar{p}_*\otimes p\otimes p\parr p\otimes p\otimes\ldots\otimes p$$

Let $\mathscr{K}$ be the fragment from R_k to $R_{k'}^{\perp}$. The fragment $\mathscr{K}$ consists of several occurrences of positive and negative formulae and connectives between them. Let m_1 be the number of positive formulae there and m_2 be the number of the negative ones. Then we get $0 = \#(\mathscr{D}(\mathbf{C}, \mathscr{K})) = 1 - (k+1) + \#(\mathscr{K}) = -k + (m_2 - m_1)(N+3)$, therefore $k = (m_2 - m_1)(N+3)$. This is absurd, because $1 \le k \le N$. ☐

Lemma 3.3. *Any occurrence of $\bar{p}_*$ from $R_k^{\perp}$ is connected to an occurrence of p_* from some $R_{k'}$.*

Proof. Any occurrence of p_* is connected to an occurrence of $\bar{p}_*$, and the numbers of p_* and $\bar{p}_*$ occurrences coincide. ☐

Lemma 3.4. *If an occurrence of p_* from R_k is connected to an occurrence of $\bar{p}_*$ from $R_{k'}^{\perp}$, then $k = k'$.*

Proof.

$$\bar{p}\parr\ldots\parr\bar{p}\parr\bar{p}\otimes\bar{p}\parr\bar{p}\parr p_*\parr\underbrace{\bar{p}\parr\ldots\parr\bar{p}}_{k+1}\parr p_+ \ \boxed{\mathscr{K}}\ \bar{p}_+\otimes\underbrace{p\otimes\ldots\otimes p}_{k'+1}\otimes\bar{p}_*\otimes p\otimes p\parr p\otimes p\otimes\ldots\otimes p$$

Let $\mathscr{K}$ be the fragment from R_k to $R_{k'}^{\perp}$. Occurrences of positive formulae inside $\mathscr{K}$ are in one-to-one correspondence with occurrences of negative formulae there (by the links connecting p_* and $\bar{p}_*$), thus $\#(\mathscr{K}) = 0$. Therefore $0 = \#(\mathscr{D}(\mathbf{C}, \mathscr{K})) = k' - k + \#(\mathscr{K})$. Hence $k = k'$. ☐

Lemma 3.5. *Any occurrence of p_+ from R_k is connected to an occurrence of $\bar{p}_+$ from some $R_{k'}^{\perp}$ (possibly, $k \neq k'$).*

Proof. We consider several cases:

Case 1: the occurrence of p_+ is connected to an occurrence of $\bar{p}$ from the same R_k by a link $\mathbf{C}$. Since $\#(\mathrm{In}(\mathbf{C})) = 0$, it is the rightmost occurrence of $\bar{p}$:

$$\bar{p}\parr\ldots\parr\bar{p}\parr\bar{p}\otimes\bar{p}\parr\bar{p}\parr p_*\parr\bar{p}\parr\ldots\parr\bar{p}\parr p_+\otimes$$

But then immediately to the right of R_k there is an occurrence τ of $\otimes$ (otherwise there would be two occurrences of $\parr$ in one region) connected by an $\mathcal{A}$-arc with the occurrence π of $\parr$ on the left side of $\mathbf{C}$ (due to $<_\Gamma$-planarity of $\mathcal{A} \cup \mathcal{E}$). On the other hand, $\pi \prec_\Gamma \tau$ (since τ is situated immediately next to R_k, and therefore R_k is a subformula of the formula where τ is the occurrence of the main connective). Contradiction with the acyclicity of $\mathcal{A} \cup \prec_\Gamma$.

Case 2: p_+ is connected to the occurrence of $\bar{p}$ in $R_{k'}$, which is the third to the left from p_*, by a link $\mathbf{C}$:

$$\bar p \parr \ldots \parr \bar p \parr \bar p \otimes \bar p \parr \bar p \parr p_* \parr \bar p \parr \ldots \parr \bar p \parr p_+ \boxed{\mathscr{K}} \underbrace{\bar p \parr \ldots \parr \bar p}_{N-k'+1} \parr \bar p \otimes \bar p \parr \bar p \parr p_* \parr \bar p \parr \ldots \parr \bar p \parr p_+$$

Let $\mathscr{K}$ be the fragment from R_k to $R_{k'}$. The same argument as in Lemma 3.4 shows that $\#(\mathscr{K}) = 0$. But then $\#(\mathscr{D}(\mathbf{C}, \mathscr{K})) = -(N - k' + 1) + \#(\mathscr{K}) \neq 0$. Contradiction.

Case 3: p_+ is connected by a link $\mathbf{C}$ to an occurrence of $\bar{p}$ from another $R_{k'}$, but not the third to the left from p_*. In this case either in the inner (if $R_{k'}$ lies to the left from R_k) or in the outer region of $\mathbf{C}$ there are two occurrences of $\parr$. Contradiction.

Case 4: p_+ is connected to $\bar{p}_*$ from some $R_{k'}^\perp$. Contradiction with Lemma 3.3: $\bar{p}_*$ is connected to an occurrence of p_*, but not p_+.

So the only possible situation is the **5th case:** p_+ is connected to an occurrence of $\bar{p}_+$ from some $R_{k'}^\perp$. □

Lemma 3.6. *Any occurrence of $\bar{p}_+$ from $R_k^\perp$ is connected to an occurrence of p_+ from some $R_{k'}$.*

Proof. The same argument as in Lemma 3.3. □

Let us call an occurrence of a connective *old* if it is not inside an occurrence of a positive or negative formula (thus this occurrence comes from the original sequent $\to \Gamma$).

Lemma 3.7. *If an occurrence τ of $\otimes$ is old, then $\mathcal{A}(\tau)$ is also old.*

Proof. Suppose the contrary. Let $\mathcal{A}(\tau)$ be not old and let $\mathcal{A}(\tau) <_\Gamma \tau$ (in the other case we proceed symmetrically with respect to the arc $\langle \tau, \mathcal{A}(\tau) \rangle$). Consider several cases:

Case 1: $\mathcal{A}(\tau)$ lies inside some $R_k^\perp$:

$$\bar p_+ \otimes p \otimes \ldots \otimes p \otimes \bar p_* \otimes p \otimes p \parr p \otimes p \otimes \ldots \otimes p \boxed{\mathscr{K}} \otimes$$

Let $\mathscr{D} = \mathrm{In}(\tau, \mathcal{A}(\tau))$ and $\mathscr{K}$ be the fragment of Ω_Γ between $R_k^\perp$ and τ. We have $\#(\mathscr{K}) = 0$ and $\#(\mathscr{D}) = 0$. Contradiction.

Case 2: $\mathcal{A}(\tau)$ is an occurrence of $\parr$ inside R_k, but not the second from the right side. From the two sets $\mathrm{In}(\tau, \mathcal{A}(\tau))$ and $\mathrm{Out}(\tau, \mathcal{A}(\tau))$ we take the one not containing p_* from this R_k and call it $\mathscr{D}$. For $\mathscr{K}$ we take the subset of Ω_Γ containing all elements of $\mathscr{D}$, except those from R_k. Now we proceed exactly as in case 1.

Case 3: $\mathcal{A}(\tau)$ is the second from the right side occurrence of $\parr$ in R_k:

$$\bar p \parr \ldots \parr \bar p \parr \bar p \otimes \bar p \parr \bar p \parr p_* \parr \bar p \parr \ldots \parr \bar p \parr p_+ \boxed{\mathscr{K}} \otimes$$

We define $\mathscr{D}$ and $\mathscr{K}$ as in case 1. The numbers of p_* and $\bar{p}_*$ in $\mathscr{K}$ are equal. Therefore, the number of p_+ and $\bar{p}_+$ in $\mathscr{K}$ are also equal. On the other hand, the same is true for $\mathscr{D}$. Contradiction: the number of $\bar{p}_+$ in $\mathscr{D}$ is the same as in $\mathscr{K}$, but the number of p_+ is greater by one. □

Now we construct a proof net $\mathfrak{N}'$ for the original sequent $\to \Gamma$: $\mathfrak{N}' \leftrightharpoons \langle \mathbf{\Omega}_\Gamma, \mathcal{A}', \mathcal{E}' \rangle$. Here $\mathcal{A}'$ contains the arcs of $\mathcal{A}$ that start at old $\otimes$ occurrences, and edges of $\mathcal{E}'$ connect those occurrences of p_k and $\bar{p}_k$ for which the occurrences of p_* and $\bar{p}_*$ from the corresponding R_k and $R_k^\perp$ are connected by edges of $\mathcal{E}$. It easily follows from the lemmata above that $\mathfrak{N}'$ is a proof net for $\to \Gamma$, therefore MCLL $\vdash \to \Gamma$. Moreover, if $\mathfrak{N}$ were strong, then $\mathfrak{N}'$ is also strong, and $\to \Gamma$ is derivable in MCLL$'$.

3.3 $\mathrm{L}(\backslash; p)$-grammars and $\mathrm{L}^*(\backslash; p)$-grammars

Theorem 24. *Any context-free language without the empty word is generated by an* $\mathrm{L}(\backslash; p_1)$*-grammar. Any context-free language is generated by an* $\mathrm{L}^*(\backslash; p_1)$*-grammar.*

Proof. Due to Theorems 2 and 15 it is sufficient to show that any $\mathrm{L}(\backslash)$-language (resp., $\mathrm{L}^*(\backslash)$-language) is generated by an $\mathrm{L}(\backslash; p_1)$-grammar (resp., $\mathrm{L}^*(\backslash; p_1)$-grammar). Let M be a language generated by an $\mathrm{L}(\backslash)$-grammar (resp., an $\mathrm{L}^*(\backslash)$-grammar) $\mathcal{G}$. The grammar is finite; let N be the maximal subscript of a primitive type used in $\mathcal{G}$. Then $\mathcal{G}$ is an $\mathrm{L}(\backslash; p_1, \ldots, p_N)$-grammar (an $\mathrm{L}^*(\backslash; p_1, \ldots, p_N)$-grammar). Consider the grammar $\mathcal{G}[p_1 := A_1, \ldots, p_N := A_N]$. Due to Theorem 20 (resp., Theorem 21) this grammar generates the same language M. □

Thus, the class of $\mathrm{L}(\backslash; p_1)$-languages coincides with the class of context-free languages without the empty words and the class of $\mathrm{L}^*(\backslash; p_1)$-languages coincides with the class of all context-free languages.

This construction (or the less complicated Métayer's one) and Theorem 6 yield that any context-free language without the empty word is generated by an $\mathrm{L}(\backslash, /; p_1)$-grammar with single type assignment.

3.4 NP-completeness of Derivability Problems for $\mathrm{L}(\cdot, \backslash; p)$, $\mathrm{L}(\backslash, /; p)$, and $\mathrm{L}^*(\backslash, /; p)$

Theorem 25. *Derivability problems for* $\mathrm{L}(\cdot, \backslash; p)$*,* $\mathrm{L}^*(\cdot, \backslash; p)$*,* $\mathrm{L}(\backslash, /; p)$*, and* $\mathrm{L}^*(\backslash, /; p)$ *(and, therefore, in their conservative extensions* $\mathrm{L}(p)$*,* $\mathrm{L}^*(p)$*,* $\mathrm{L}_{\mathbf{1}}(\cdot, \backslash; p)$*,* $\mathrm{L}_{\mathbf{1}}(\backslash, /; p)$*, and* $\mathrm{L}_{\mathbf{1}}(p)$*) are NP-complete.*

Proof. Let $\mathscr{L}$ be one of the calculi $\mathrm{L}(\cdot, \backslash)$, $\mathrm{L}^*(\cdot, \backslash)$, $\mathrm{L}(\backslash, /)$, or $\mathrm{L}^*(\backslash, /)$. We construct a polynomially computable function ϕ that reduces derivability in $\mathscr{L}$ to derivability in $\mathscr{L}(p)$. Let $\Pi \to C$ be an arbitrary sequent and let $N_{\Pi \to C}$ be the maximal subscript of a primitive type occurring in $\Pi \to C$. For $N = N_{\Pi \to C}$ build the types A_1, A_2, …, A_N as shown in Section 3.1. Let $\phi(\Pi \to C) \leftrightharpoons \big(\Pi \to C\big)[p_1 := A_1, p_2 := A_2, \ldots, p_N := A_N]$. Due to conservativity Theorems 22 and 23 yield $\mathscr{L} \vdash \Pi \to C \iff \mathscr{L}(p) \vdash \phi(\Pi \to C)$. Thus, ϕ is indeed the reducing function and, since the derivability problem for $\mathscr{L}$ is NP-complete (Theorem 8), so is the derivability problem for $\mathscr{L}(p)$.

For calculi with two divisions one can also use the Métayer construction. □

Derivability problems for $\mathrm{L}(\backslash; p)$, $\mathrm{L}^*(\backslash; p)$, and $\mathrm{L}_{\mathbf{1}}(\backslash; p)$ are, of course, decidable in polynomial time due to conservativity and Theorems 9 and 19.

3.5 The Uniform Substitution

Though for Theorems 24 and 25 it was sufficient to have a faithful substitution acting on a finite set of primitive types, it is interesting whether there exists a uniform substitution, that reduces derivability in $\mathrm{L}(\backslash)$ to derivability in $\mathrm{L}(\backslash; p_1)$ (resp., $\mathrm{L}^*(\backslash)$ to $\mathrm{L}^*(\backslash; p_1)$) independently from the number of primitive types used in the sequent. We build such a substitution using the existing construction. First note that the proof allows *parameters* to be added to the sequent. Namely, the following statement is true: for any sequent $\Pi \to C$, where $\Pi = B_1 \dots B_m \quad B_1, \dots, B_m, C \in \mathrm{Tp}(\backslash; p_1, \dots, p_N, p_{N+1}, \dots, p_{N+M})$, we have

$$\mathrm{L}(\backslash) \vdash \Pi \to C \iff \mathrm{L}(\backslash) \vdash \big(\Pi \to C\big)[p_1 := A_1, \dots, p_N := A_N];$$

$$\mathrm{L}^*(\backslash) \vdash \Pi \to C \iff \mathrm{L}^*(\backslash) \vdash \big(\Pi \to C\big)[p_1 := A_1, \dots, p_N := A_N].$$

Here $p_{N+1}, \dots, p_{N+M}$ are parameters, that are not affected by the substitution. During the transformation of the (strong) proof net links, connecting their occurrences, are just copied from $\mathfrak{N}$ to $\mathfrak{N}'$.

Consider $N = 2$. Our construction gives the following types A_1 and A_2 (we continue using the notation "multiplication in the denominator"):

$$A_1 = \big(p \cdot p \cdot \big(((p \cdot p) \backslash p) \backslash p\big) \cdot p \cdot p\big) \backslash p,$$
$$A_2 = \big(p \cdot p \cdot p \cdot \big(((p \cdot p) \backslash p) \backslash p\big) \cdot p\big) \backslash p.$$

Define two functions from $\mathrm{Tp}(\backslash)$ to $\mathrm{Tp}(\backslash)$: $\mathcal{L}\colon E \mapsto A_1[p_1 := E]$ and $\mathcal{R}\colon E \mapsto A_2[p_1 := E]$. By renaming primitive types, we get the following:

Lemma 3.8. *For any sequent $\Pi \to C$, where $\Pi = B_1 \dots B_m$, and $B_1, \dots, B_m, C \in \mathrm{Tp}(\backslash; p_1, \dots, p_{n-1}, p_n, p_{n+1})$, we have*

$$\mathrm{L}(\backslash) \vdash \Pi \to C \iff \mathrm{L}(\backslash) \vdash \big(\Pi \to C\big)[p_n = \mathcal{L}(p_n), p_{n+1} := \mathcal{R}(p_n)];$$

$$\mathrm{L}^*(\backslash) \vdash \Pi \to C \iff \mathrm{L}^*(\backslash) \vdash \big(\Pi \to C\big)[p_n = \mathcal{L}(p_n), p_{n+1} := \mathcal{R}(p_n)].$$

(Here p_1, ..., p_{n-1} act as parameters.)

Now by induction we get the following statement:

Lemma 3.9. *For any $n \geqslant 1$ and for any sequent $\Pi \to C$, where $\Pi = B_1 \dots B_m$ $B_1, \dots, B_m, C \in \mathrm{Tp}(\backslash; p_1, \dots, p_n)$, the following holds:*

$$\mathrm{L}(\backslash) \vdash \Pi \to C \iff \mathrm{L}(\backslash; p_1) \vdash \big(\Pi \to C\big)[p_1 := \mathcal{L}(p_1), p_2 := \mathcal{L}(\mathcal{R}(p_1)), \dots, p_{n-1} := \mathcal{L}(\mathcal{R}^{n-2}(p_1)), p_n := \mathcal{R}^{n-1}(p_1)];$$

$$\mathrm{L}^*(\backslash) \vdash \Pi \to C \iff \mathrm{L}^*(\backslash; p_1) \vdash \big(\Pi \to C\big)[p_1 := \mathcal{L}(p_1), p_2 := \mathcal{L}(\mathcal{R}(p_1)), \dots, p_{n-1} := \mathcal{L}(\mathcal{R}^{n-2}(p_1)), p_n := \mathcal{R}^{n-1}(p_1)].$$

Proof. Proceed by induction on n. The induction base is trivial. Suppose our lemma is true for n and prove it for $n+1$. By Lemma 3.8 we have

$$\mathrm{L}(\backslash) \vdash \Pi \to C \iff \mathrm{L}(\backslash) \vdash \big(\Pi \to C\big)[p_n := \mathcal{L}(p_n), p_{n+1} := \mathcal{R}(p_n)].$$

The sequent from the right side contains only primitive types $p_1, \ldots, p_{n-1}, p_n$. Therefore, by induction hypothesis its derivability is equivalent to

$$\begin{aligned} \mathrm{L}(\backslash) \vdash \big(\Pi \to C\big)[p_n := \mathcal{L}(p_n), p_{n+1} := \mathcal{R}(p_n)][p_1 := \mathcal{L}(p_1),\\ p_2 = \mathcal{L}(\mathcal{R}(p)), \ldots, p_{n-1} := \mathcal{L}(\mathcal{R}^{n-2}(p)), p_n := \mathcal{R}^{n-1}(p)], \end{aligned}$$

and the latter sequent is graphically equal to $\big(\Pi \to C\big)[p_1 := \mathcal{L}(p), p_2 = \mathcal{L}(\mathcal{R}(p)), \ldots, p_{n-1} := \mathcal{L}(\mathcal{R}^{n-2}(p)), p_n := \mathcal{L}(\mathcal{R}^{n-1}(p)), p_{n+1} := \mathcal{R}^n(p)]$.

The $\mathrm{L}^*(\backslash)$ is handled in the same way. □

Theorem 26. *For any sequent* $\Pi \to C$*, where* $\Pi = B_1 \ldots B_m$*,* $B_1, \ldots, B_m, C \in \mathrm{Tp}(\backslash)$*, the following equivalences hold:*

$$\begin{aligned} \mathrm{L}(\backslash) \vdash \Pi \to C \iff \mathrm{L}(\backslash; p_1) \vdash \big(\Pi \to C\big)[p_1 := \mathcal{L}(p), p_2 := \mathcal{L}(\mathcal{R}(p)),\\ p_3 := \mathcal{L}(\mathcal{R}(\mathcal{R}(p))), \ldots, p_k := \mathcal{L}(\mathcal{R}^{k-1}(p)), \ldots]; \end{aligned}$$

$$\begin{aligned} \mathrm{L}^*(\backslash) \vdash \Pi \to C \iff \mathrm{L}^*(\backslash; p_1) \vdash \big(\Pi \to C\big)[p_1 := \mathcal{L}(p), p_2 := \mathcal{L}(\mathcal{R}(p)),\\ p_3 := \mathcal{L}(\mathcal{R}(\mathcal{R}(p))), \ldots, p_k := \mathcal{L}(\mathcal{R}^{k-1}(p)), \ldots]. \end{aligned}$$

Proof. Use the previous lemma with $n = N + 1$, where N is the greatest subscript of a primitive type occurring in $\Pi \to C$. □

A disadvantage of this construction is the fact that the sequent length grows exponentially: for p_k we substitute the type $\mathcal{L}(\mathcal{R}^{k-1}(p))$ that contains 9^k occurrences of p. Hence this substitution cannot be used to prove NP-completeness of fragments of the Lambek calculus with one primitive type and at least two connectives.

Chapter 4

The Lambek Calculus with the Reversal Operation

4.1 L-models for L

In this section and the following three sections the alphabet Σ can be finite *or countable.* Three connectives of the Lambek calculus naturally correspond to three operations on languages without the empty word ($M, N \subseteq \Sigma^+$): $M \cdot N \leftrightharpoons \{uv \mid u \in M, v \in N\}$, $M \setminus N \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, vu \in N\}$, $N \,/\, M \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, uv \in N\}$.

Definition. An L-model is a pair $\mathcal{M} = \langle \Sigma, w \rangle$, where Σ is an alphabet, and w is a mapping of Lambek types into formal languages over Σ without the empty word, such that for any $A, B \in \mathrm{Tp}$ the following holds: $w(A \cdot B) = w(A) \cdot w(B)$, $w(A \setminus B) = w(A) \setminus w(B)$ $w(B \,/\, A) = w(B) \,/\, w(A)$.

This mapping can be defined on primitive types in an arbitrary way, and then it is uniquely propagated to all types.

Since Σ^+ with the concatenation operation is the free semigroup generated by Σ, L-models are also called *models on subsets of free semigroups.*

Definition. A sequent $F \to G$ is considered *true* in a model $\mathcal{M} = \langle \Sigma, w \rangle$ ($\mathcal{M} \vDash F \to G$) if $w(F) \subseteq w(G)$.

L-models give sound and complete semantics for L, due to the following theorem:

Theorem 27 (M. Pentus, 1995). *A sequent $F \to G$ is provable in* L *if and only if it is true in all L-models.*

This theorem is proved in [31]; its special case for the product-free fragment (where we keep only types without multiplication) is much easier and appears in [3]. (The notion of truth in an L-model and this theorem can be easily generalized to sequents with more than one type on the left, since $\mathrm{L} \vdash F_1 F_2 \dots F_n \to G$ if and only if $\mathrm{L} \vdash F_1 \cdot F_2 \cdot \ldots \cdot F_n \to G$.)

4.2 The L^{R} Calculus

Now let us consider an extra operation on languages, the *reversal.* For $u = a_1 a_2 \dots a_n$ ($a_1, \dots, a_n \in \Sigma$, $n \geq 1$) let $u^{\mathrm{R}} \leftrightharpoons a_n \dots a_2 a_1$, and for $M \subseteq \Sigma^+$ let $M^{\mathrm{R}} \leftrightharpoons \{u^{\mathrm{R}} \mid u \in M\}$. Let us enrich the language of the Lambek calculus with a new unary connective $^{\mathrm{R}}$

(written in the postfix form, A^{R}). We shall denote the extended set of types by Tp^{R}. If $\Gamma = A_1 A_2 \dots A_n$, then $\Gamma^{\mathrm{R}} \leftrightharpoons A_n^{\mathrm{R}} \dots A_2^{\mathrm{R}} A_1^{\mathrm{R}}$.

The notion of L-model is also easily adapted to the new language by adding an additional constraint on w: $w(A^{\mathrm{R}}) = w(A)^{\mathrm{R}}$.

The calculus L^{R} is obtained from L by adding three new rules for $^{\mathrm{R}}$:

$$\frac{\Gamma \to C}{\Gamma^{\mathrm{R}} \to C^{\mathrm{R}}}\ ({}^{\mathrm{R}} \to {}^{\mathrm{R}}) \qquad \frac{\Gamma A^{\mathrm{RR}} \Delta \to C}{\Gamma A \Delta \to C}\ ({}^{\mathrm{RR}} \to)_{\mathrm{E}} \qquad \frac{\Gamma \to C^{\mathrm{RR}}}{\Gamma \to C}\ (\to {}^{\mathrm{RR}})_{\mathrm{E}}$$

It is easy to see that L^{R} is sound with respect to L-models.

Proposition 4.1. *The calculus* L^{R} *is a conservative extension of* L.

Proof. The "if" part is obvious. The "only if" part follows from L-completeness of L and L-soundness of L^{R}: if $F \to G$ is provable in L^{R}, then it is true in all L-models, and, therefore, is provable in L. □

L-completeness for the product-free fragment of L^{R} is proved in [29] by a modification of Buszkowski's argument [3] (in [29] the reversal connective is called *involution* and denoted by $\breve{}$ instead of $^{\mathrm{R}}$; the calculus is formulated in a different, but equivalent way). In [29] one can also find a proof of L-completeness of the division-free fragment (where only $\cdot$ and $^{\mathrm{R}}$ connectives are kept). We shall prove L-completeness of the whole calculus.

Theorem 28. *A sequent* $F \to G$ *(*$F, G \in \mathrm{Tp}^{\mathrm{R}}$*) is provable in* L^{R} *if and only if it is true in all L-models.*

A variant of this calculus that allows empty antecedents (an extension with the $^{\mathrm{R}}$ connective of L^*, the variant of L without the restriction $\Pi \neq \Lambda$ on the $(\to \backslash)$ and $(\to /)$ rules) is presented in [17]. The calculus L^* itself is complete with respect to L-models allowing empty words in the languages (free monoid models) [24], but L-completeness of its extension with the $^{\mathrm{R}}$ connective is still an open problem.

4.3 Normal Form for L^{R} types

The relation $\leftrightarrow_{\mathrm{L}^{\mathrm{R}}}$ is an equivalence relation. Moreover, it is a congruence relation with respect to all connectives (due to the $({}^{\mathrm{R}} \to {}^{\mathrm{R}})$ rule for $^{\mathrm{R}}$ and Proposition 1.2 for others). The following lemma is checked explicitly by presenting the corresponding derivations in L^{R}:

Lemma 4.2. *The following equivalences hold in* L^{R}*:*

1. $(A \cdot B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \cdot A^{\mathrm{R}}$;
2. $(A \backslash B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} / A^{\mathrm{R}}$;
3. $(B / A)^{\mathrm{R}} \leftrightarrow A^{\mathrm{R}} \backslash B^{\mathrm{R}}$;
4. $A^{\mathrm{RR}} \leftrightarrow A$.

Definition. For $A \in \mathrm{Tp}^{\mathrm{R}}$ we define $tr(A)$ by induction on the number of connectives in A:

1. $tr(p_i) \leftrightharpoons p_i$;
2. $tr(p_i^{\mathrm{R}}) \leftrightharpoons p_i^{\mathrm{R}}$;
3. $tr(A \cdot B) \leftrightharpoons tr(A) \cdot tr(B)$;
4. $tr(A \backslash B) \leftrightharpoons tr(A) \backslash tr(B)$;

5. $tr(B / A) \leftrightharpoons tr(B) / tr(A)$;
6. $tr((A \cdot B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \cdot tr(A^{\mathrm{R}})$;
7. $tr((A \setminus B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) / tr(A^{\mathrm{R}})$;
8. $tr((B / A)^{\mathrm{R}}) \leftrightharpoons tr(A^{\mathrm{R}}) \setminus tr(B^{\mathrm{R}})$;
9. $tr(A^{\mathrm{RR}}) \leftrightharpoons tr(A)$.

The following statement is proved by induction using Lemma 4.2:

Proposition 4.3. *Any $A \in \mathrm{Tp}^{\mathrm{R}}$ is equivalent to $tr(A)$.*

We call $tr(A)$ the *normal form* of A. In the normal form, the $^{\mathrm{R}}$ connective can appear only on occurrences of primitive types.

4.4 L-completeness of L^{R} (Proof)

Now we are going to prove Theorem 28 (the "if" part) by contraposition. Let $\mathrm{L}^{\mathrm{R}} \not\vdash F_0 \to G_0$. We need to construct a countermodel for $F_0 \to G_0$, i.e., a model in which this sequent is not true.

Let $\mathrm{Pr}' \leftrightharpoons \mathrm{Pr} \cup \{p^{\mathrm{R}} \mid p \in \mathrm{Pr}\}$, and let L' be the Lambek calculus with Pr' taken as the set of primitive types instead of Pr. Here $^{\mathrm{R}}$ is not a connective, and p^{R} is considered just a new primitive type, independent from p. Obviously, if $\mathrm{L}' \vdash F \to G$, then $\mathrm{L}^{\mathrm{R}} \vdash F \to G$.

Let $F \leftrightharpoons tr(F_0)$, $G \leftrightharpoons tr(G_0)$. Then $\mathrm{L}^{\mathrm{R}} \not\vdash F \to G$, whence $\mathrm{L}' \not\vdash F \to G$. The calculus L' is essentially the same as L, therefore Theorem 27 gives us a structure $\mathcal{M} = \langle \Sigma, w \rangle$ such that $w(F) \not\subseteq w(G)$. The structure $\mathcal{M}$ indeed falsifies $F \to G$, but it is not a model in the sense of our new language: some of the conditions $w(p_i^{\mathrm{R}}) = w(p_i)^{\mathrm{R}}$ might be not satisfied.

Let Φ be the set of all subtypes of $F \to G$ (including F and G themselves; the notion of subtype is understood in the sense of L^{R}). The construction of $\mathcal{M}$ (see [31]) guarantees that $w(A) \neq \varnothing$ for all $A \in \Phi$. This is the only specific property of $\mathcal{M}$ we shall need.

We introduce an inductively defined counter $f(A)$, $A \in \Phi$: $f(p_i) \leftrightharpoons 1$, $f(p_i^{\mathrm{R}}) \leftrightharpoons 1$, $f(A \cdot B) \leftrightharpoons f(A) + f(B) + 10$, $f(A \setminus B) \leftrightharpoons f(B)$, $f(B / A) \leftrightharpoons f(B)$. Let $K \leftrightharpoons \max\{f(A) \mid A \in \Phi\}$, $N \leftrightharpoons 2K + 25$ (N should be odd, greater then K, and big enough itself).

Let $\Sigma_1 \leftrightharpoons \Sigma \times \{1, \ldots, N\}$. We shall denote the pair $\langle a, j \rangle \in \Sigma_1$ by $a^{(j)}$. Elements of Σ and Σ_1 will be called *letters* and *symbols* respectively. A symbol can be *even* or *odd* depending on the parity of the superscript. Consider a homomorphism $h \colon \Sigma^+ \to \Sigma_1^+$, defined as follows: $h(a) \leftrightharpoons a^{(1)} a^{(2)} \ldots a^{(N)}$ ($a \in \Sigma$), $h(a_1 \ldots a_n) \leftrightharpoons h(a_1) \ldots h(a_n)$. Let $P \leftrightharpoons h(\Sigma^+) = \{a_1^{(1)} \ldots a_1^{(N)} \ldots a_n^{(1)} \ldots a_n^{(N)} \mid n \geq 1,\ a_i \in \Sigma\}$. Note that h is a bijection between Σ^+ and P.

Lemma 4.4. *For all $M, N \subseteq \Sigma^+$ we have*

1. $h(M \cdot N) = h(M) \cdot h(N)$;

2. if $M \neq \varnothing$, then $h(M \setminus N) = h(M) \setminus h(N)$ and $h(N / M) = h(N) / h(M)$.

Proof.

1. By the definition of a homomorphism.

2. $\boxed{\subseteq}$ Let $u \in h(M \setminus N)$. Then $u = h(u')$ for some $u' \in M \setminus N$. For all $v' \in M$ we have $v'u' \in N$. Take an arbitrary $v \in h(M)$, $v = h(v')$ for some $v' \in M$. Since $u' \in M \setminus N$, $v'u' \in N$, whence $vu = h(v')h(u') = h(v'u') \in h(N)$. Therefore $u \in h(M) \setminus h(N)$.
   $\boxed{\supseteq}$ Let $u \in h(M) \setminus h(N)$. First we claim that $u \in P$. Suppose the contrary: $u \notin P$. Take $v' \in M$ (M is nonempty by assumption). Since $v = h(v') \in P$, $vu \notin P$. On the other hand, $vu \in h(N) \subseteq P$. Contradiction. Now, since $u \in P$, $u = h(u')$ for some $u' \in \Sigma^+$. For an arbitrary $v' \in M$ and $v \leftrightharpoons h(v')$ we have $h(v'u') = vu \in h(N)$, whence $v'u' \in N$, whence $u' \in M \setminus N$. Therefore, $u = h(u') \in h(M \setminus N)$.
   The / case is handled symmetrically.

$\square$

We construct a new model $\mathcal{M}_1 = \langle \Sigma_1, w_1 \rangle$, where $w_1(z) \leftrightharpoons h(w(z))$ ($z \in \mathrm{Pr}'$). Due to Lemma 4.4, $w_1(A) = h(w_1(A))$ for all $A \in \Phi$, whence $w_1(F) = h(w(F)) \not\subseteq h(w(G)) = w_1(G)$ ($\mathcal{M}_1$ is also a countermodel in the language without $^{\mathrm{R}}$).

Now we introduce several auxiliary subsets of Σ_1^+ (by $\mathrm{Subw}(M)$ we denote the set of all nonempty subwords of words from M, i.e. $\mathrm{Subw}(M) \leftrightharpoons \{u \in \Sigma_1^+ \mid (\exists v_1, v_2 \in \Sigma_1^*)\, v_1 u v_2 \in M\}$):
$T_1 \leftrightharpoons \{u \in \Sigma_1^+ \mid u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})\}$;
$T_2 \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) \mid \text{the first or the last symbol of } u \text{ is even}\}$;
$E \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) - (P \cup P^{\mathrm{R}}) \mid \text{both the first symbol and the last symbol of } u \text{ are odd}\}$.

The sets P, P^{R}, T_1, T_2, and E form a partition of Σ_1^+ into nonintersecting parts. For example, $a^{(1)}b^{(10)}a^{(2)} \in T_1$, $a^{(N)}b^{(1)} \ldots b^{(N-1)} \in T_2$, $a^{(7)}a^{(6)}a^{(5)} \in E$ ($a, b \in \Sigma$).

Let $T \leftrightharpoons T_1 \cup T_2$, $T_i(k) \leftrightharpoons \{u \in T_i \mid |u| \geq k\}$ ($i = 1, 2$, $|u|$ is the length of u), $T(k) \leftrightharpoons T_1(k) \cup T_2(k) = \{u \in T \mid |u| \geq k\}$.

Note that if the first or the last symbol of u is even, then it belongs to T, no matter whether it belongs to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$.

The index k (possibly with subscripts) here and further ranges from 1 to K. For all k we have $T(k) \supseteq T(K)$.

Lemma 4.5.
1. *$P \cdot P \subseteq P$, $P^{\mathrm{R}} \cdot P^{\mathrm{R}} \subseteq P^{\mathrm{R}}$;*
2. *$T^{\mathrm{R}} = T$, $T(k)^{\mathrm{R}} = T(k)$;*
3. *$P \cdot P^{\mathrm{R}} \subseteq T(K)$, $P^{\mathrm{R}} \cdot P \subseteq T(K)$;*
4. *$P \cdot T \subseteq T(K)$, $T \cdot P \subseteq T(K)$;*
5. *$P^{\mathrm{R}} \cdot T \subseteq T(K)$, $T \cdot P^{\mathrm{R}} \subseteq T(K)$;*
6. *$T \cdot T \subseteq T$;*

Proof.
1. Obvious.
2. Directly follows from our definitions.
3. Any element of $P \cdot P^{\mathrm{R}}$ or $P^{\mathrm{R}} \cdot P$ does not belong to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ and its length is at least $2N > K$. Therefore it belongs to $T_1(K) \subseteq T(K)$.
4. Let $u \in P$ and $v \in T$. If $v \in T_1$, then uv is also in T_1. Let $v \in T_2$. If the last symbol of v is even, then $uv \in T$. If the last symbol of v is odd, then $uv \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$, whence $uv \in T_1 \subseteq T$. Since $|uv| > |u| \geq N > K$, $uv \in T(K)$.
   The claim $T \cdot P \subseteq T$ is handled symmetrically.
5. $P^{\mathrm{R}} \cdot T = P^{\mathrm{R}} \cdot T^{\mathrm{R}} = (T \cdot P)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$. $T \cdot P^{\mathrm{R}} = T^{\mathrm{R}} \cdot P^{\mathrm{R}} = (P \cdot T)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$.

6. Let $u, v \in T$. If at least one of these two words belongs to T_1, then $uv \in T_1$. Let $u, v \in T_2$. If the first symbol of u or the last symbol of v is even, then $uv \in T$. In the other case u ends with an even symbol, and v starts with an even symbol. But then we have two consecutive even symbols in uv, therefore $uv \in T_1$.

□

Let us call words of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$, $a^{(N-1)}a^{(N)}b^{(1)}$, and $a^{(N)}b^{(1)}b^{(2)}$ ($a, b \in \Sigma$, $1 \le i \le N-2$) *valid triples of type I* and their reversals (namely, $a^{(i+2)}a^{(i+1)}a^{(i)}$, $b^{(1)}a^{(N)}a^{(N-1)}$, and $b^{(2)}b^{(1)}a^{(N)}$) *valid triples of type II.* Note that valid triples of type I (resp., of type II) are the only possible three-symbol subwords of words from P (resp., P^{R}).

Lemma 4.6. *A word u of length at least three is a subword of a word from $P \cup P^{\mathrm{R}}$ if and only if any three-symbol subword of u is a valid triple of type I or II.*

Proof. The nontrivial part is "if". We proceed by induction on $|u|$. Induction base ($|u| = 3$) is trivial. Let u be a word of length $m+1$ satisfying the condition and let $u = u'x$ ($x \in \Sigma_1$). By induction hypothesis ($|u'| = m$), $u' \in \mathrm{Subw}(P \cup P^{\mathrm{R}})$. Let $u' \in \mathrm{Subw}(P)$ (the other case is handled symmetrically); u' is a subword of some word $v \in P$. Consider the last three symbols of u. Since the first two of them also belong to u', this three-symbol word is a valid triple of type I, not type II. If it is of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$ or $a^{(N)}b^{(1)}b^{(2)}$, then x coincides with the symbol next to the occurrence of u' in v, and therefore $u = u'x$ is also a subword of v. If it is of the form $a^{(N-1)}a^{(N)}b^{(1)}$, then, provided $v = v_1u'v_2$, v_1u' is also an element of P, and so is the word $v_1u'b^{(1)}b^{(2)}\dots b^{(N)}$, which contains $u = u'b^{(1)}$ as a subword. Thus, in all cases $u \in \mathrm{Subw}(P)$. □

Now we construct one more model $\mathcal{M}_2 = \langle \Sigma_1, w_2 \rangle$, where $w_2(p_i) \leftrightharpoons w_1(p_i) \cup w_1(p_i^{\mathrm{R}})^{\mathrm{R}} \cup T$, $w_2(p_i^{\mathrm{R}}) \leftrightharpoons w_1(p_i)^{\mathrm{R}} \cup w_1(p_i^{\mathrm{R}}) \cup T$. This model is a model even in the sense of the enriched language. To finish the proof, we need to check that $\mathcal{M}_2 \not\models F \to G$.

Lemma 4.7. *For any $A \in \Phi$ the following holds:*

1. $w_2(A) \subseteq P \cup P^{\mathrm{R}} \cup T$;
2. $w_2(A) \supseteq T(f(A))$;
3. $w_2(A) \cap P = w_1(A)$ *(in particular, $w_2(A) \cap P \neq \varnothing$);*
4. $w_2(A) \cap P^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$ *(in particular, $w_2(A) \cap P^{\mathrm{R}} \neq \varnothing$).*

Proof. We prove all the statements simultaneously by induction on type A. The induction base is trivial. Further we shall refer to the i-th statement of the induction hypothesis ($i = 1, 2, 3, 4$) as "IH-i".

1. Consider three possible cases.

a) $A = B \cdot C$. Then $w_2(A) = w_2(B) \cdot w_2(C) \subseteq (P \cup P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P \cup P^{\mathrm{R}} \cup T$ (Lemma 4.5).

b) $A = B \setminus C$. Suppose the contrary: in $w_2(A)$ there exists an element $u \in E$. Then $vu \in w_2(C)$ for any $v \in w_2(B)$. We consider several subcases and show that each of those leads to a contradiction.

i) $u \in \mathrm{Subw}(P)$, and the superscript of the first symbol of u is not 1. Let the first symbol of u be $a^{(i)}$. Note that i is odd and $i > 2$. Take $v = a^{(3)} \dots a^{(N)}a^{(1)} \dots a^{(i-1)}$. The word v has length at least $N \ge K$ and ends with an even symbol, therefore $v \in T(K) \subseteq T(f(B)) \subseteq w_2(B)$ (IH-2). On the other hand, $vu \in \mathrm{Subw}(P)$ and the first symbol and the last symbol of vu are odd. Therefore, $vu \in E$ and $vu \in w_2(C)$, but $w_2(C) \cap E = \varnothing$ (IH-1). Contradiction.

ii) $u \in \mathrm{Subw}(P)$, and the first symbol of u is $a^{(1)}$ (then the superscript of the last symbol of u is not N, because otherwise $u \in P$). Take $v \in w_2(B) \cap P$ (this set is nonempty due to IH-3). The first and the last symbol of vu is odd, and $vu \in \mathrm{Subw}(P) - P$, therefore $vu \in E$. Contradiction.

iii) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the superscript of the first symbol of u is not N (the first symbol of u is $a^{(i)}$, i is odd). Take $v = a^{(N-2)} \dots a^{(1)} a^{(N)} \dots a^{(i+1)} \in T(K) \subseteq w_2(B)$. Again, $vu \in E$.

iv) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the first symbol of u is $a^{(N)}$. Take $v \in w_2(B) \cap P^{\mathrm{R}}$ (nonempty due to IH-4). $vu \in E$.

c) $A = C \,/\, B$. Proceed symmetrically.

2. Consider three possible cases.

a) $A = B \cdot C$. Let $k_1 \leftrightharpoons f(B)$, $k_2 \leftrightharpoons f(C)$, $k \leftrightharpoons k_1 + k_2 + 10 = f(A)$. Due to IH-2, $w_2(B) \supseteq T(k_1)$ and $w_2(C) \supseteq T(k_2)$. Take $u \in T(k)$. We have to prove that $u \in w_2(A)$. Consider several subcases.

i) $u \in T_1(k)$. By Lemma 4.6 ($|u| \geq k > 3$ and $u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$) in u there is a three-symbol subword xyz that is not a valid triple of type I or II. Divide the word u into two parts, $u = u_1 u_2$, such that $|u_1| \geq k_1 + 5$, $|u_2| \geq k_2 + 5$. If needed, shift the border between parts by one symbol to the left or to the right, so that the subword xyz lies entirely in one part. Let this part be u_2 (the other case is handled symmetrically). Then $u_2 \in T_1(k_2)$. If u_1 is also in T_1, then the proof is finished. Consider the other case. Note that in any word from $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ among any three consecutive symbols at least one is even. Shift the border to the left by at most 2 symbols to make the last symbol of u_1 even. Then $u_1 \in T(k_1)$, and u_2 remains in $T_1(k_2)$. Thus $u = u_1 u_2 \in T(k_1) \cdot T(k_2) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$.

ii) $u \in T_2(k)$. Let u end with an even symbol (the other case is symmetric). Divide the word u into two parts, $u = u_1 u_2$, $|u_1| \geq k_1 + 5$, $u_2 \geq k_2 + 5$, and shift the border (if needed), so that the last symbol of u_1 is even. Then both u_1 and u_2 end with an even symbol, and therefore $u_1 \in T(k_1)$ and $u_2 \in T(k_2)$.

b) $A = B \setminus C$. Let $k \leftrightharpoons f(C) = f(A)$. By IH-2, $w_2(C) \supseteq T(k)$. Take $u \in T(k)$ and an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$. By Lemma 4.5, statements 4–6, $vu \in (P \cup P^{\mathrm{R}} \cup T) \cdot T \subseteq T$, and since $|vu| > |u| \geq k$, $vu \in T(k) \subseteq w_2(C)$. Thus $u \in w_2(A)$.

c) $A = C \,/\, B$. Symmetrically.

3. Consider three possible cases.

a) $A = B \cdot C$.

$\boxed{\supseteq}$ $u \in w_1(A) = w_1(B) \cdot w_1(C) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-3); $u \in P$.

$\boxed{\subseteq}$ Suppose $u \in P$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$ and $u_2 \in w_2(C)$. First we claim that $u_1 \in P$. Suppose the contrary, $u_1 \notin P$. By IH-1, $u_1 \in P^{\mathrm{R}} \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup T$, and therefore $u = u_1 u_2 \in (P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P^{\mathrm{R}} \cup T$ (Lemma 4.5, statements 1, 3–6). Hence $u \notin P$. Contradiction. Thus, $u_1 \in P$. Similarly, $u_2 \in P$, and by IH-3 we obtain $u_1 \in w_1(B)$ and $u_2 \in w_1(C)$, whence $u = u_1 u_2 \in w_1(A)$.

b) $A = B \setminus C$.

$\boxed{\supseteq}$ Take $u \in w_1(B \setminus C)$. For any $v \in w_1(B)$ we have $vu \in w_1(C)$. We claim that $u \in w_2(B \setminus C)$. Take $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$ (IH-1). If $v \in P$, then $v \in w_1(B)$ (IH-3), and $vu \in w_1(C) \subseteq w_2(C)$ (IH-3). If $v \in P^{\mathrm{R}} \cup T$, then $vu \in (P^{\mathrm{R}} \cup T) \cdot P \subseteq T(K) \subseteq w_2(C)$ (Lemma 4.5, statements 3 and 4, and IH-2). Therefore, $u \in w_2(B) \setminus w_2(C) = w_2(B \setminus C)$; also we have $u \in P$, since $w_1(B \setminus C) \subseteq P$.

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P$, then for any $v \in w_1(B) \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P$, $vu \in P$. By IH-3, $vu \in w_1(C)$. Thus $u \in w_1(B \setminus C)$.

c) $A = C \,/\, B$. Symmetrically.

4. Consider three cases.

a) $A = B \cdot C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ $u \in w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^R)) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-4); $u \in P^{\mathrm{R}}$.

$\boxed{\subseteq}$ Let $u \in P^{\mathrm{R}}$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$, $u_2 \in w_2(C)$. We claim that $u_1, u_2 \in P^{\mathrm{R}}$. Suppose the contrary: $u_1 \notin P^{\mathrm{R}}$. Then $u_1 \in P \cup T$ (IH-1), $u_2 \in P \cup P^{\mathrm{R}} \cup T$, whence $u = u_1 u_2 \in (P \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P \cup T$. Contradiction ($u \in P^{\mathrm{R}}$). Thus, $u_1 \in P^{\mathrm{R}}$, and therefore $u_2 \in P^R$, and, using IH-4, we obtain $u_1 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$, $u_2 \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. Hence $u = u_1 u_2 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{\mathrm{R}})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$.

b) $A = B \setminus C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ Let $u \in w_1(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$, so for every $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ we have $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. We claim that $u \in w_2(B \setminus C)$. Take an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$ (IH-1). If $v \in P^{\mathrm{R}}$, then $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4), whence $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(C)$.
If $v \in P \cup T$, then (since $u \in P^{\mathrm{R}}$) we have $vu \in (P \cup T) \cdot P^{\mathrm{R}} \subseteq T(K) \subseteq w_2(C)$ (Lemma 4.5 and IH-2).

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P^{\mathrm{R}}$, then for any $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P^R$, $vu \in P^{\mathrm{R}}$, therefore $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4). Thus $u \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = w_1(A^{\mathrm{R}})^{\mathrm{R}}$.

c) $A = C \,/\, B$. Symmetrically.

This completes the proof of Lemma 4.7. □

Since $w_1(F) \not\subseteq w_1(G)$, there exists an element u_0 such that $u_0 \in w_1(F)$ and $u_0 \notin w_1(G)$. Since $u_0 \in P$, $u_0 \in w_2(F)$ and $u_0 \notin w_2(G)$. Therefore, $w_2(F) \not\subseteq w_2(G)$. Since $F_0 \leftrightarrow F$, $G_0 \leftrightarrow G$, and L^{R} is L-sound, we see that $w_2(F_0) = w_2(F)$, $w_2(G_0) = w_2(G)$, and $\mathcal{M}_2$ is a countermodel for $F_0 \to G_0$. This completes the proof of Theorem 28.

Note that we have constructed a countermodel (in the sense of the extended language) for any sequent $F \to G$ that is not provable in L' (this could be potentially weaker than $\mathrm{L}^{\mathrm{R}} \not\vdash F \to G$). Thus we get the following statement:

Proposition 4.8. $\mathrm{L}^{\mathrm{R}} \vdash A_1 \dots A_n \to B$ *if and only if* $\mathrm{L}' \vdash tr(A_1) \dots tr(A_n) \to tr(B)$.

4.5 L^{R}: Grammars and Complexity

Theorem 29. *The class of all* L^{R}*-languages coincides with the class of all context-free languages without the empty word.*

Proof. Every context-free language is an L^{R}-language due to Theorem 2 and conservativity of L^{R} over $\mathrm{L}(\setminus)$.

The other inclusion follows from Propositon 4.8: if we replace every type C with $tr(C)$ in an L^{R}-grammar, we obtain an L-grammar (since L' and L differ only in the set of primitive types) generating the same language, and this language is now context-free by Theorem 3. □

Fragments of L^{R} with restricted sets of connectives and/or primitive types are defined in the same way as for L.

Theorem 30. *Derivability problems for* $\mathrm{L}^{\mathrm{R}}(\backslash; p_1)$, L^{R}, *and all the calculi between them (fragments of* L^{R} *and conservative extensions of* $\mathrm{L}^{\mathrm{R}}(\backslash; p_1)$*) are NP-complete.*

Proof. The derivability problem for L^{R} is in the NP class due to Proposition 4.8 and the fact that the derivability problem for L is in NP.

NP-completeness of the derivability problem for $\mathrm{L}^{\mathrm{R}}(\backslash; p_1)$ follows from the equivalence $B \,/\, A \leftrightarrow_{\mathrm{L}^{\mathrm{R}}} (A^{\mathrm{R}} \setminus B^{\mathrm{R}})^{\mathrm{R}}$, that reduces derivability in $\mathrm{L}(\backslash, /; p_1)$ to derivability in $\mathrm{L}^{\mathrm{R}}(\backslash; p_1)$. The derivability problem for $\mathrm{L}(\backslash, /; p_1)$ is NP-complete (Theorem 25). □

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