

On the Lambek Calculus with the Kleene Star and the Exponential

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Abstract

We introduce an infinitary extension of the Lambek calculus with the Kleene star and the linear logic exponential modality. For this system we establish cut elimination and some other properties and also prove that, due to complexity reasons, it cannot be replaced by a system with finite proofs.

Keywords: Lambek calculus, Kleene star, exponential modality, linear logic, infinitary logic

1 The Lambek Calculus

The Lambek calculus was introduced in 1958 by J. Lambek [5] for formal description of natural language syntax. Nowadays it is considered as part of linear and substructural logic framework [1]. In this paper we consider \mathbf{L}_1 , the variant of the Lambek calculus that includes the unit constant and allows empty left-hand sides of sequents.

Formulae of \mathbf{L}_1 are built from a countable set of *variables* $\text{Var} = \{p, q, r, \dots\}$ and the *unit constant* $\mathbf{1}$ using three binary connectives: \backslash (*left division*), $/$ (*right division*), and \cdot (*product*). Formulae are denoted by capital Latin letters; capital Greek letters denote finite (possibly empty) ordered sequences of formulae. *Sequents* of \mathbf{L}_1 are of the form $\Pi \rightarrow A$. Π and A are called the left-hand side and right-hand side (also the *antecedent* and the *succedent*) of this sequent.

Axioms of \mathbf{L}_1 are sequents of the form $A \rightarrow A$ and additionally the sequent $\rightarrow \mathbf{1}$ (with an empty left-hand side). The rules of \mathbf{L}_1 are as follows:

$$\begin{array}{ccc} \frac{\Pi \rightarrow A \quad \Gamma, B, \Delta \rightarrow C}{\Gamma, (B/A), \Pi, \Delta \rightarrow C} (/ \rightarrow) & \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B/A} (\rightarrow /) & \\ \frac{\Pi \rightarrow A \quad \Gamma, B, \Delta \rightarrow C}{\Gamma, \Pi, (A \backslash B), \Delta \rightarrow C} (\backslash \rightarrow) & \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} (\rightarrow \backslash) & \\ \frac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, (A \cdot B), \Delta \rightarrow C} (\cdot \rightarrow) & \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \cdot B} (\rightarrow \cdot) & \frac{\Gamma, \Delta \rightarrow C}{\Gamma, \mathbf{1}, \Delta \rightarrow C} (\mathbf{1} \rightarrow) \end{array}$$

The (cut) rule is eliminable. Due to the linear nature of \mathbf{L}_1 , the cut-elimination procedure is very simple and goes exactly as for the original Lambek calculus [5].

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2 Extending \mathbf{L}_1 with Two Unary Connectives

We consider the extension of \mathbf{L}_1 with two unary connectives.

The first one is the *exponential*, denoted by $!$. This connective follows the spirit of the exponential in linear logic, the weakening rule, however, is not suitable for linguistic applications of $!$ [7], therefore we consider both variants with and without the weakening rule.

The rules for $!$ are as follows:

$$\frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C} (! \rightarrow) \quad \frac{!A_1, \dots, !A_n \rightarrow B}{!A_1, \dots, !A_n \rightarrow !B} (\rightarrow !)$$

$$\frac{\Gamma, !A, !A, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C} (\text{contr})$$

$$\frac{\Gamma, !A, \Delta, \Phi \rightarrow C}{\Gamma, \Delta, !A, \Phi \rightarrow C} (\text{perm}) \quad \frac{\Gamma, \Delta, !A, \Phi \rightarrow C}{\Gamma, !A, \Delta, \Phi \rightarrow C} (\text{perm}) \quad \frac{\Gamma, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C} (\text{weak})$$

As said above, the (weak) rule is optional. We consider both variants.

The second connective we consider in this paper is the Kleene star, $*$, written in the postfix form (A^*) . The rules introducing the Kleene star to the succedent are quite straightforward:

$$\frac{\Gamma_1 \rightarrow A \quad \dots \quad \Gamma_n \rightarrow A}{\Gamma_1, \dots, \Gamma_n \rightarrow A^*} (\rightarrow^*)_n \quad \text{for all } n \geq 0$$

(In particular, for $n = 0$ we have axioms of the form $\rightarrow A^*$.)

For the right-hand side, the situation is more interesting:

$$\frac{\Gamma, A^n, \Delta \rightarrow C \quad \text{for all } n \geq 0}{\Gamma, A^*, \Delta \rightarrow C} (* \rightarrow)_\omega$$

This is an ω -rule, i.e., it has an infinite number of premises. Therefore the proofs in this system, involving this rule, would be trees with infinite branching (but every particular branch has finite length). Later we show that this is inevitable: proof systems with finite derivations (that make use of some form of induction) are necessarily weaker than the system presented above.

This infinitary axiomatization of the Kleene star are essentially the same as that in *infinitary action logic* [2]. However, the essential difference of our presentation from [2] is that here we do not have *additive* connectives, \wedge and \vee , but include the exponential modality, $!$, instead. The choice of these connectives is motivated linguistically (see [7] for more details: in the calculus $\mathbf{Db}!$? in that paper $!$ is the exponential without the weakening rule, and $?$ is a restricted version of the Kleene star). For this system we get a stronger lower bound for complexity (Π_2^0 -hard, whereas in [2] the system is Π_1^0 -hard). The complexity question for the system with neither \wedge and \vee , nor $!$ (i.e., the infinitary Lambek calculus with the Kleene star) remains open.

We denote the variants of our system with and without (weak) by $*_\omega!_w\mathbf{L}_1$ and $*_\omega!\mathbf{L}_1$ respectively.

3 Cut Elimination

Theorem 1. *Any sequent derivable in $*_\omega!\mathbf{L}_1$ or $*_\omega!_w\mathbf{L}_1$ can be derived without using the cut rule.*

To prove this, we use the standard linear logic technique [6, Appendix A], but, thanks to the ω -rule, the induction becomes transfinite. This transfinite part here generally follows the cut elimination strategy for a fragment of the Lambek calculus with iteration in [8] and the unpublished paper [9].

For every formula we define an ordinal that we shall call the size of the formula, $\|A\|$. It is defined recursively: $\|p\| = 1$ for $p \in \text{Var}$, $\|A \setminus B\| = \|B / A\| = \|A \cdot B\| = \|A\| \oplus \|B\| + 1$, $\|!A\| = \|A\| + 1$, $\|A^*\| = \omega\|A\|$. Here $\alpha \oplus \beta$ for two ordinals $\alpha, \beta < \omega^\omega$ is defined as follows: if

$\alpha = \omega^k c_k + \dots + \omega c_1 + c_0$ and $\beta = \omega^m d_m + \dots + \omega d_1 + d_0$, then $\alpha \oplus \beta = \omega^n (c_n + d_n) + \dots + \omega(c_1 + d_1) + c_0 + d_0$, where $n = \max\{k, m\}$ (for $i > k$ and $j > m$ let $c_i = d_j = 0$).

Next, the depth of a derivation is also measured by an ordinal (on the ω -rule step we take the supremum plus one).

Now the schema of the cut elimination procedure is as follows. As in [6] (and actually following Gentzen's ideas with his "mix" rule), we eliminate two rules, (cut) itself and the rule

$$\frac{\Pi \rightarrow !A \quad \Gamma_0, !A, \Gamma_1, !A, \dots, \Gamma_{i-1}, !A, \Gamma_i, \dots, !A, \Gamma_n \rightarrow C}{\Gamma_0, \Gamma_1, \dots, \Gamma_{i-1}, \Pi, \Gamma_i, \dots, \Gamma_n \rightarrow C} \text{ (cut!)}$$

that is a combination of (cut) and (contr).

Now we proceed by nested induction. The outer induction parameter is $\|A\|$, where A is the formula being cut. The inner one goes on the pair of the depths of the premises of cut (at least one of them should become smaller, while the other doesn't grow). Finally, every branch of the derivation is finite (though the derivation in whole could be infinite, and include an infinite number of cuts), therefore one can remove all the cuts on a branch one-by-one, and doing so for all branches leaves the tree cut-free.

As we see, the cut-elimination procedure for our system combines two ideas, first from [8][9], second from [6].

In the presence of (cut) the infinite series of $(\rightarrow^*)_n$ rules can be replaced by the $\rightarrow A^*$ axioms and two rules:

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A^*} \quad \frac{\Gamma \rightarrow A^* \quad \Delta \rightarrow A^*}{\Gamma, \Delta \rightarrow A^*}$$

These rules coincide with the rules for the "existential exponential" (denoted by $?$) from [7].

4 Cyclic Proofs

Here we introduce yet another variant of $*_\omega!L_1$ (or $*_{\omega!_w}L_1$), namely, we replace $(\rightarrow^*)_\omega$ with two rules:

$$\frac{\Gamma, \Delta \rightarrow C \quad \Gamma, A, A^*, \Delta \rightarrow C}{\Gamma, A^*, \Delta \rightarrow C} (* \rightarrow)_L \quad \frac{\Gamma, \Delta \rightarrow C \quad \Gamma, A^*, A, \Delta \rightarrow C}{\Gamma, A^*, \Delta \rightarrow C} (* \rightarrow)_R$$

These two rules look finite, but, as a trade-off, we allow infinite branches in the derivation tree. For the cut-free version, all such trees are legal; when we have cut in the system, we impose the restriction that any branch must contain at least one (actually infinitely many) application of $(\rightarrow^*)_L$ or $(\rightarrow^*)_R$ for every formula of the form A^* in the antecedent.

We denote these systems by $*_\infty!L_1$ and $*_{\infty!_w}L_1$.

Lemma 1. $*_\infty!L_1$ and $*_{\infty!_w}L_1$ are equivalent to $*_\omega!L_1$ and $*_{\omega!_w}L_1$ respectively. This holds both for the versions with and without cut.

Consider one example:

$$\frac{p \rightarrow p \quad \frac{(p/p)^*, p \rightarrow p}{(p/p)^*, (p/p), p \rightarrow p} (/ \rightarrow)}{(p/p)^*, p \rightarrow p} (* \rightarrow)_R}{(p/p)^* \rightarrow p/p} (\rightarrow /)$$

Note that we actually don't have to develop the derivation tree further, since the sequent $(p/p)^*, p \rightarrow p$ on top already appears lower in the derivation, and now this tree can be built up to an infinite one in a regular fashion. Such situations are called *cyclic proofs* [10].

As we show below, however, the system with cyclic proofs is strictly weaker than the whole infinitary calculus.

5 Complexity

In this section we sketch a proof that the derivability problems in $*_{\omega}!\mathbf{L}_1$ and $*_{\omega}!_{\mathbf{w}}\mathbf{L}_1$ are at least Π_2^0 -hard. Thus any system with finite proofs (including systems with cyclic proofs discussed in the previous section) is strictly weaker than the systems with the ω -rule (or infinite derivations).

Let \mathcal{K} be a Kleene algebra. \mathcal{K} is called **-continuous* if it satisfies the infinitary condition $pq^*r = \sup_{n \geq 0} pq^n r$ for arbitrary elements p, q, r of the algebra. A mapping $\iota: \text{Var} \rightarrow \mathcal{K}$ is called an *interpretation*. An interpretation can be uniquely propagated to formulae built from variables and the unit constant using \cdot and $*$.

Let E be a finite set of equivalences of the form $A \leftrightarrow B$, where A and B are formulae built from variables and the unit constant using only the \cdot connective. In other words, they represent finite sequences of variables; for the empty sequence we use $\mathbf{1}$. Now let U and V be formulae built from variables and $\mathbf{1}$ using \cdot and $*$. We say that the *Horn clause* $E \Rightarrow (U \rightarrow V)$ is *true* in \mathcal{K} under the interpretation ι if either $\iota(A) \neq \iota(B)$ for some $(A \leftrightarrow B) \in E$ or $\iota(U) \leq \iota(V)$ in \mathcal{K} .

Let $*_{\omega}\mathbf{L}_1 + E$ be the calculus $*_{\omega}\mathbf{L}_1$ extended with additional axioms $A \rightarrow B$ and $B \rightarrow A$ for every $(A \leftrightarrow B) \in E$. In $*_{\omega}\mathbf{L}_1 + E$, cut is included as an official rule of the system and is not completely eliminable. However, the conservativity statement still holds: if we want to derive a sequent without $/$ and \backslash , we can do it without using these connectives inside the derivation.

Lemma 2. *The Horn clause $E \Rightarrow (U \rightarrow V)$ is true in all *-continuous Kleene algebras under all interpretations iff $*_{\omega}\mathbf{L}_1 + E \vdash U \rightarrow V$.*

This lemma is proved using the standard canonical model construction (Lindenbaum algebra for the fragment of $*_{\omega}\mathbf{L}_1 + E$ where we have only two connectives, \cdot and $*$).

Theorem 2 (D. Kozen). *The problem of deciding whether a given Horn clause $E \Rightarrow (U \rightarrow V)$ is true in all *-continuous Kleene algebras under all interpretations is Π_2^0 -hard [4].*

Note that formally in [4] U and V are also allowed to include the disjunction ($+$) connective. However, it doesn't appear in the construction.

Now we embed E into $U \rightarrow V$ using $!$. Here we generally follow [6] and [3]. Let $\mathcal{G}_E = \{(A/B) \mid (A \leftrightarrow B) \in E\} \cup \{(B/A) \mid (A \leftrightarrow B) \in E\}$. If $\mathcal{G}_E = \{C_1, \dots, C_m\}$ (the ordering here is arbitrary, since these formulae will undergo $!$ that allows permutation), let $!\Gamma_E = !C_1, \dots, !C_m$ and $!\Phi_E = !(1/!C_1), \dots, !(1/!C_m)$.

Theorem 3. *The following are equivalent:*

1. $*_{\omega}\mathbf{L}_1 + E \vdash \Pi \rightarrow C$;
2. $*_{\omega}!_{\mathbf{w}}\mathbf{L}_1 \vdash !\Gamma_E, \Pi \rightarrow C$;
3. $*_{\omega}!\mathbf{L}_1 \vdash !\Phi_E, !\Gamma_E, \Pi \rightarrow C$.

Proof. The (i) \Rightarrow (ii) part is easy: extra axioms from E transform into derivable sequents $!(A/B), B \rightarrow A$, and by weakening we add the rest of $!\Gamma_E$; then $!\Gamma_E$ is propagated along the derivation, applying permutation and contraction.

For the (ii) \Rightarrow (i) part, we first notice that all formulae from $!\Gamma_E$ are derivable in $*_{\omega}\mathbf{L}_1 + E$, and then apply the cut rule, we get $*_{\omega}!_{\mathbf{w}}\mathbf{L}_1 + E \vdash \Pi \rightarrow C$, and then by conservativity this sequent is also derivable without using $!$.

Statement (iii) is handled in the same way as (ii). Here we don't have weakening as a rule of our calculus, but the combination $!(1/!C_i), !C_i$ can weaken itself: $\Gamma, !(1/!C_i), !C_i, \Delta \rightarrow C$ is derivable from $\Gamma, \Delta \rightarrow C$. For the backwards direction, notice that all formulae from $!\Phi_E$ are derivable in $*_{\omega}!_{\mathbf{w}}\mathbf{L}_1$ (here we use weakening, but we end up with a sequent without $!$). \square

Corollary 1. *The derivability problems for $*_{\omega}!\mathbf{L}_1$ and $*_{\omega}!_{\mathbf{w}}\mathbf{L}_1$ are Π_2^0 -hard, and, therefore, the sets of derivable sequents in these calculi are not recursively enumerable.*

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