# Cohomology of the Steenrod algebra

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As it was shown in a recent paper by Adams [1] for a number of problems in modern algebraic topology the problem of computing the cohomology $H^*(A) = \sum H^{s,t}(A)$ of the Steenrod algebra $A$ corresponding to a prime $p$ is of a great importance. Using the knowledge of the structure of the Steenrod algebra $A$ Adams obtained some information about the structure of the algebra $H^*(A)$ ([1], Theorem 2.4) which allowed him, for example, to move forward in the problem of existence of mappings with odd Hopf invariant. In particular, Adams showed that for $p = 2$ the group $H^1(A) = \sum_t H^{1,t}(A)$ has a basis which consists of elements $h_i \in H^{1,2^i}(A),\ i = 0, 1, 2, ...$ that satisfy the following relations:

$$h_i h_{i+1} = 0,\ h_i^2 h_{i+2} = h_{i+1}^3,\ h_i h_{i+2}^2 = 0. \tag{1}$$

Moreover all relations between polynomials which depend on $h_i' s$ and have degree $\leq 3$ are consequences of the relations from (1).

It turns out that the previous result of Adams admits the following generalization:

**Theorem 1.** *Besides the relations from (1) the elements $h_i$ also satisfy the following relations*

$$h_i^2 h_{i+3}^2 = 0,\ h_0^{2^k} h_{k+2}^2 = 0,\ h_0^{2^{k+1}} h_{k+2} = 0,\ k = 1, 2, ..., \tag{2}$$

*and, possibly, the relations*

$$h_i h_{i+k}^2 h_{i+k+3} = 0,\ h_i^2 h_{i+k+1}^2 h_{i+k+4} = 0, i = 0, 1, 2, ...,\ k = 3, 4, 5, ... \tag{3}$$

*All relations between polynomials which depend on $h_i$'s and have degree $\leq 5$ follow from the relations (1),(2) and (3). The group $H^2(A)$ is generated by*

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*monomials $h_ih_j$. The group $H^3(A)$ is generated by monomials $h_ih_jh_k$ and certain elements $m_i \in H^{3,11\cdot 2^i}(A)$, $\lambda_{i,k} \in H^{3,2^i(1+2^{k-1}+2^{k+1})}(A)$. Here the elements $m_i$ and $\lambda_{i,4}$ are non-trivial, whereas the elements $\lambda_{i,k}$ for $k \neq 4$ might be trivial. The elements*

$$m_ih_j,\ j \neq i-1, i, i+2, i+3;$$

$$\lambda_{i,4}h_j,\ j \neq i, i+2, i+3, i+4, \tag{4}$$

*of the group $H^4(A)$ are linearly independent, and the subgroup of $H^4(A)$ generated by these elements has trivial intersection with the subgroup generated by the elements $h_ih_jh_kh_l$. The group $H^4(A)$ contains non-trivial elements $\beta_i \in H^{4,3\cdot 2^{i+2}}(A)$, $i = 1, 2, ...$ which do not belong to the subgroup generated by the elements from (4) and the monomials $h_ih_jh_kh_l$. The groups $H^{5,14}(A)$ and $H^{5,16}(A)$ are non-trivial.*

The proof of the theorem above, as well as the proof of the theorem of Adams, relies on certain Serre-Hochschild spectral sequences. It is well-known that when working with spectral sequences it is very useful to use cohomological operations which commute with transgression (the Steenrod squares and the Steenrod reduced powers). It turns out that in the cohomology algebra of the Steenrod algebra modulo 2 it is also possible to define "cohomological operations" $Sq^i$ which posses certain formal properties analogous to the properties of the Steenrod squares. These operations can be defined not only for the Steenrod algebra but also for any Hopf algebra with a certain endomorphism $\alpha$ which reduces dimension in two. In particular, such an endomorphism was constructed by Adams for the algebra $A$ and defined by him algebras $A^i//A^j$ (see [1], Theorems 5.1-5.12). The main difference between the "algebraic" operations $Sq^i$ and the "topological" ones is that the "algebraic" operation $Sq^0$ is not the identity map but coincides with the endomorphism $\alpha^*$ which is induced by the endomorphism $\alpha$. There is the following relation which connects the Steenrod operations and the bigraduation of the algebra $H^*(A)$:

$$Sq^i(H^{k,l}(A)) \subset H^{k+i,2l}(A).$$

Starting with the elements

$$g_{n,i} \in H^{2,2^i(2^{n+1}-1)}(A^1//A^{n+1}),\ i \geq 0, n \geq 2,$$

which were defined by Adams (see [1], p. 210), we define the elements

$$g_{n,i}^{(k)} \in H^{2^k+1,2^{i+k}(2^{n+1}-1)}(A^1//A^{n+1})$$

by the following inductive formula:

$$g_{n,i}^{(0)} = g_{n,i},\ g_{n,i}^{(k+1)} = Sq^{2^k}(g_{n,i}^{(k)}).$$

It turns out that one has the following lemma:

**Lemma 1.** *Non of the elements $g_{n,i}^{(k)}$ belongs to the ideal in $H^*(A^1//A^{n+1})$ generated by the elements $g_{n,j}^{(l)}$ for $j < k$ and $j \geq 0$.*

Theorem 1 which was formulated above is a pretty straightforward corollary of the previous lemma.

Analogously one can study the case $p > 2$. It turns out that in this case one has the following theorem:

**Theorem 2.** *The group $H^1(A)$ has a basis which consists of elements $h_0 \in H^{1,1}(A)$ and $h_i \in h^{1,2p^{i-1}(p-1)}(A)$, $i \geq 1$. The group $H^2(A)$ is generated by monomials $h_ih_j$ and some non-trivial elements*

$$\bar{h}_i \in H^{2,2p^i(p-1)}(A),\ \varkappa_i \in H^{2,2p^{i-1}(p-1)(1+p+p^2)}(A),\ i \geq 1,$$

$$w_i \in H^{2,2p^{i-1}(p-1)(p+2)}(A),\ v_i \in H^{2,2p^{i-1}(p-1)(p+2p)}(A),\ i \geq 1, \tag{5}$$

$$v_0 \in H^{2,4p-3}(A),\ \varkappa_0 \in H^{2,2p^2+2p+3}(A). \tag{6}$$

*The elements $h_i$ and the elements from (5) and (6) satisfy the following relations:*

$$\begin{gathered} h_ih_{i+1} = 0,\ h_0^{p^i}h_{i+1} = 0,\ h_0^{p^i(p-1)}h_{i+1} = 0,\ i \geq 0, \\ \bar{h}_i^{p^k}h_{i+k+2} = h_{i+k+1}\bar{h}_{i+1}^{p^k},\ i \geq 1,\ k \geq 0, \\ \bar{h}_i^{p^{k+1}}\bar{h}_{i+k+2} = \bar{h}_{i+k+1}\bar{h}_{i+1}^{p^k+1},\ i \geq 1,\ k \geq 0; \end{gathered} \tag{7}$$

$$v_{i+1}h_{i+1} = w_{i+1}h_{i+2},\ v_0h_o = 0,\ v_ih_{i+2} = w_{i+1}h_i,\ i \geq 0, \tag{8}$$

*and, possibly, the following relations:*

$$h_0^{p^i(p-1)}\bar{h}_{i+2} = 0,\ i \geq 0. \tag{9}$$

*All relations between polynomials depending on $h_i$'s of degree $\leq 3$ and polynomials depending on $h_i$'s and $\bar{h}_i$'s of degree $\leq 2$ follow from the relations*

*(7) and skew-commutativity. All linear relations between monomials $v_i h_j$, $\varkappa_i h_i$ and $w_i h_k$, where $j \neq i-1, i+1$; $k \neq i, i+2$ and $|i-l| > 3$, follow from the relations (8). The subgroup of $H^3(A)$ generated by these elements has trivial intersection with the subgroup generated by $h_i$'s and $\bar{h}_i$'s. The groups $H^{3,2p^i(p-1)(p+2)}(A), H^{3,2p^i(p-1)(1+2p)}(A)$ and $H^{3,2p^i(p-1)(1+2p+p^2)}(A)$ are non-trivial for all $i = 1, 2, ....$*

The proof of the theorem above is analogous to the proof of Theorem 1. Only instead of operations $Sq^i$ one has to use the operations $St_p^i$ (Steenrod reduced powers). Unlike in the "topological" case the operations $St_p^i$ are trivial when $i \neq 0, 1(mod\ p-1)$.

Using Theorems 1 and 2 and a result of Adams which establishes a connection between cohomology of the Steenrod algebra and the stable homotopy groups of spheres one can prove the following two theorems:

**Theorem 3.** *For any $r$ there exist elements $\alpha \in \pi_q(S^3)$ and $\beta \in \pi_{m+q}(S^q)$, where $q > 3,\ m > 0$, such that*

$$E^{rm+l}\beta \circ E^{(r-1)m+l}\beta \circ ... \circ E^l\beta \circ E^l\alpha \neq 0$$

*for any $l \geq 0$.*

In other words there exist arbitrary "long" compositions of mappings of spheres which are essential.

**Theorem 4.** *For $q \leq 14$ the 2-primary components of the stable homotopy groups of spheres are as follows:*

| *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* | *13* | *14* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_8$ | *0* | *0* | $\mathbb{Z}_2$ | $\mathbb{Z}_{16}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_8$ | *0* | *0* | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |

*Moreover, one can choose a basis*

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_6,\ \alpha_7,\ \alpha_8,\ \beta_8,\ \alpha_9,\ \beta_9,\ \gamma_9,\ \alpha_{10},\ \alpha_{11},\ \alpha_{14},\ \beta_{14},$$

*of these 2-primary components such that*

$$E\alpha_1 \circ \alpha_1 = \alpha_2,\ E^2\alpha_1 \circ \alpha_2 = 4\alpha_3,\ E^3\alpha_3 \circ \alpha_3 = \alpha_6,\ E\alpha_7 \circ \alpha_1 = \alpha_8,$$
$$E\beta_8 \circ \alpha_1 = \alpha_9,\ E^6\alpha_3 \circ \alpha_6 = \beta_9,\ E\gamma_9 \circ \alpha_1 = \alpha_{10},\ E\alpha_{10} \circ \alpha_1 = 4\alpha_{11},$$
$$E^7\alpha_7 \circ \alpha_7 = \alpha_{14},\ E^3\alpha_{11} \circ \alpha_3 = \beta_{14}.$$

Theorem 4 by different methods was proved by Toda [2].

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# References


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[2] H. Toda, *On the double suspension of* $E^2$, J. Inst. Polytechn. Osaka City Univ. Ser. A, 7 (1956), pp. 103-145